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Sur quelques applications géométriques de la théorie spectrale des flots hyperboliques

On some geometrical applications of the spectral theory of hyperbolic flows

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Table des matières

Chapitre 1

Introduction

Cette thèse porte sur certaines séries dynamiques associées à des systèmes *hyperboliques*. Ces derniers participent des systèmes dits *chaotiques* — fortement récurrents et sensibles aux conditions initiales —, dont l'ambassadeur le plus célèbre est peut-être le système à trois corps célestes, étudié par Poincaré à la fin du XIX[e] siècle [Poi90]. Si ces dynamiques sont régies par des lois déterministes, les trajectoires d'évolution semblent complètement imprévisibles, voire aléatoires. Néanmoins, certaines d'entre elles se trouvent être *périodiques* (elles se reproduisent à l'infini, identiques à elles-mêmes) et, dans ce mémoire, c'est principalement sur celles-ci que se portera notre intérêt. L'existence d'orbites périodiques dans un contexte de chaos peut paraître paradoxale ; elles sont pourtant en abondance et la connaissance de leurs périodes permet souvent de récupérer des informations essentielles sur la dynamique qui les a engendrées, notamment *via* l'utilisation de séries dynamiques et autres fonctions zêta. Avant d'exposer en détails les problématiques dont il sera question dans ce manuscript — et pour les motiver quelque peu —, nous discutons brièvement de certains résultats traitant de la théorie spectrale des systèmes hyperboliques.

Flots d'Anosov et comptage des orbites périodiques

En 1898, Hadamard [Had98] a montré que le chaos pouvait surgir dans un contexte géométrique très simple, en exhibant l'instabilité des lignes géodésiques sur les surfaces à courbure négative. Il a montré en outre que chaque classe de déformation libre de lacets contenait une unique géodésique fermée ; la distribution des longueurs de ces courbes privilégiées a depuis lors fait l'objet de nombreux travaux. Pour les surfaces hyperboliques — c'est-à-dire de courbure *constante* égale à -1 — et compactes, Selberg [Sel56] a introduit une fonction zêta qui compte les géodésiques fermées et a relié leurs longueurs aux valeurs propres du laplacien hyperbolique *via* une formule des traces. Huber [Hub61] a montré plus tard un analogue géométrique du théorème des nombres premiers : le nombre de géodésiques fermées dont la longueur est inférieure ou égale à L est équivalent à $\exp(L)/L$ quand L tend vers l'infini. Margulis [Mar69] a ensuite obtenu un résultat similaire pour les surfaces à courbure négative *variable*.

Les flots géodésiques en courbure négative sont en fait des cas particuliers de systèmes dynamiques *hyperboliques*, au sens de la définition donnée par Anosov en 1967 dans un article fondateur [Ano67].

Definition 1.0.1 (Anosov)**.** Soit $\varphi = (\varphi_t)_{t\in\mathbb{R}}$ un flot lisse agissant sur une variété fermée M, et $X = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\varphi_t$ son générateur. Le flot φ sera dit hyperbolique, ou d'Anosov, si pour tout $z \in M$ il existe une décomposition

$$T_zM = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z)$$

dépendant continûment de z, telle que $\mathrm{d}\varphi_t(z)E_\mathrm{b}(z) = E_\mathrm{b}(\varphi_t(z))$ où $b = u, s$, et telle que

$$|\mathrm{d}\varphi_t(z)v| \leqslant C\mathrm{e}^{-\nu t}|v|, \qquad t \geqslant 0, \quad v \in E_s(z),$$
$$|\mathrm{d}\varphi_t(z)v| \leqslant C\mathrm{e}^{-\nu|t|}|v|, \qquad t \leqslant 0, \quad v \in E_u(z),$$

pour des constantes $C, \nu > 0$, où $|\cdot|$ est une norme sur TM.

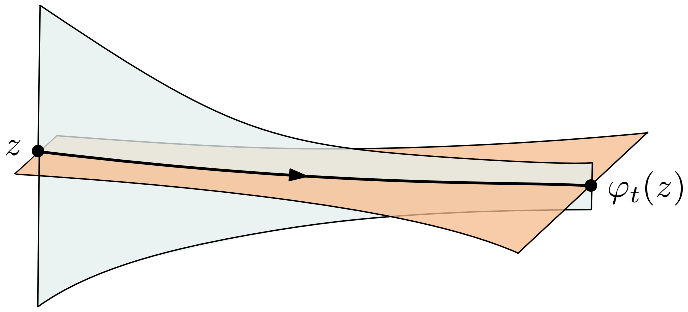

FIGURE 1.1 – Un flot d'Anosov.

La propriété d'hyperbolicité signifie que certaines directions, dites *stables* (les directions de E_s), sont contractées par la dynamique, tandis que d'autres, dites *instables* (les directions de E_u), sont dilatées. Dans ce contexte, le résultat de Margulis évoqué plus haut est toujours valide et s'énonce comme suit.

Théorème 1.0.1. *Soit $\varphi = (\varphi_t)_{t\in\mathbb{R}}$ un flot d'Anosov topologiquement mélangeant. Alors il existe un réel $h > 0$ tel qu'on a l'équivalent*

$$N(\varphi, t) \sim \frac{\mathrm{e}^{ht}}{ht} \tag{1.0.1}$$

quand t tend vers l'infini, où $N(\varphi, t)$ est le nombre d'orbites périodiques primitives du flot φ dont la période est inférieure où égale à t.

Le nombre h est l'*entropie topologique* du flot, c'est une mesure du chaos — pour les flots géodésiques des surfaces hyperboliques compactes, cette entropie vaut 1 conformément au résultat de Huber. Parry et Pollicott [PP83] ont étendu l'équivalent (1.0.1) aux flots *Axiom A* (une classe de flots qui généralise les flots d'Anosov introduite par Smale [Sma67]) après d'importantes contributions de Bowen [Bow72].

Fonction zêta et résonances de Ruelle

Contrairement à Margulis qui a recourt à la théorie ergodique, Parry et Pollicott démontrent le théorème des orbites primitives en usant d'une fonction zêta introduite par Ruelle [Rue76] — une version légèrement modifiée de celle de Selberg — qui compte les orbites périodiques. La fonction zêta de Ruelle est l'homologue dynamique de la fonction zêta de Riemann ; elle est définie par la formule

$$\zeta_\varphi(s) = \prod_\gamma \left(1 - \mathrm{e}^{-s\tau(\gamma)}\right)^{-1}, \quad \mathrm{Re}(s) > h,$$

où le produit porte sur les orbites périodiques *primitives* γ du flot φ, et $\tau(\gamma)$ est la période de γ. En s'appuyant notamment sur le codage symbolique des flots hyperboliques développé par Bowen [Bow73], Parry et Pollicott ont démontré dans [PP83] que ζ s'étend analytiquement à un voisinage ouvert du demi-plan $\{\mathrm{Re}(s) \geqslant h\}$, sauf en $s = h$ où elle a un pôle simple. Ils obtiennent alors l'équivalent (1.0.1) en reproduisant la démonstration du théorème des nombres premiers de Wiener–Ikehara [Wie88] qui repose sur un argument taubérien : la distribution des périodes $\tau(\gamma)$ se lit au travers des singularités analytiques de la fonction ζ_φ.

Smale [Sma67] s'est demandé s'il était possible, pour les flots Axiom A, d'obtenir un prolongement méromorphe à tout le plan complexe pour la fonction ζ_φ, s'exclamant à ce sujet : « I must admit that a positive answer would be a little shocking ! ». Cette question a fait couler beaucoup d'encre et il a fallu presque cinquante ans pour qu'elle soit résolue. D'abord, Ruelle [Rue76] a obtenu un tel prolongement sous la condition que le flot est analytique ainsi que ses distributions stable et instable. Plus tard, Rugh [Rug96] a montré que, pour les flots d'Anosov tri-dimensionnels, l'hypothèse d'analyticité sur les distributions stable et instable (mais pas sur le flot !) pouvait être omise, ce qui a été généralisé en dimension quelconque par Fried [Fri95]. Pour les flots d'Anosov lisses (de classe C^∞), Pollicott a obtenu un prolongement de ζ_φ dans un demi-plan $\{\mathrm{Re}(s) > h - \varepsilon\}$ pour un certain $\varepsilon > 0$ dépendant de φ, résultat étendu aux flots Axiom A par Parry–Pollicott [PP90].

Ces résultats sont typiquement obtenus en codant la dynamique *via* des partitions de Markov et en exprimant la fonction ζ_φ comme un produit alterné de déterminants Fredholm de certains opérateurs agissant sur les fonctions höldériennes d'un sous-décalage de type fini, reliant ainsi les zéros et pôles de ζ_φ au spectre desdits opérateurs. Cette méthode présente cependant le désavantage de ne pas prendre en compte la régularité du flot ; or, le travail de Kitaev [Kit99] suggère que la régularité de la dynamique est intimement liée à la profondeur du demi-plan sur laquelle un prolongement analytique peut être obtenu.

Changeant de paradigme, Blank, Keller et Liverani [BKL02] ont introduit au début des années 2000 des espaces fonctionnels adaptés à un *difféomorphisme* hyperbolique f (la version discrète des flots d'Anosov), sur lesquels l'opérateur de Koopman $u \mapsto u \circ f$ est quasi-compact. La clé est de considérer des distributions dont la régularité est *anisotrope* ; grossièrement, ces distributions sont régulières dans les directions stables et irrégulières dans les directions instables. Ces résultats ont ensuite été affinés par Baladi [Bal05], Gouëzel–Liverani [GL06] et Baladi–Tsujii [BT07], puis par Faure–Roy–Sjöstrand [FRS08] qui ont proposé une approche semi-classique. Liverani [Liv04] (pour les flots de contact) et Butterley–Liverani [BL07] ont adapté ces travaux au cadre continu, construisant des espaces fonctionnels sur lesquels le générateur $X : u \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} u \circ \varphi_t$ d'un flot d'Anosov (φ_t) a une résolvante quasi-compacte. Comme dans le cas discret, Faure–Sjöstrand [FS11] ont ensuite proposé une version micro-locale de ces espaces.

Précisons brièvement ces résultats. Soit φ un flot d'Anosov sur une variété M, et X son générateur. Si s est un nombre complexe, la résolvante $\mathrm{R}_\varphi(s)$ de φ est définie par l'intégrale

$$\mathrm{R}_\varphi(s) = \int_0^\infty \mathrm{e}^{-ts}\varphi_{-t}^*\mathrm{d}t,$$

où φ_{-t}^* est le tiré en arrière par φ_{-t}, agissant sur l'espace $\Omega^\bullet(M)$ des formes différentielles. Dès que la partie réelle de s est assez grande, cette intégrale est convergente et donne lieu à un opérateur $\mathrm{R}_\varphi(s) : \Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$, où $\mathcal{D}'^\bullet(M)$ désigne l'espace des *courants* — le dual de topologique de $\Omega^\bullet(M)$. La terminologie « résolvante » est justifiée par les identités

$$(\mathcal{L}_X + s)\mathrm{R}_\varphi(s) = \mathrm{R}_\varphi(s)(\mathcal{L}_X + s) = \mathrm{Id}_{\Omega^\bullet(M)},$$

où $\mathcal{L}_X$ est la dérivée de Lie dans la direction X.

Théorème 1.0.2 (Butterley–Liverani, Faure–Sjöstrand)**.** *La résolvante* $\mathrm{R}_\varphi(s)$*, définie initialement sur un demi-plan* $\{\mathrm{Re}(s) > C\}$*, admet un prolongement méromorphe en la variable* s*, à tout le plan complexe, comme une famille d'opérateurs* $\Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$*, dont les résidus sont des projecteurs de rang fini. Ses pôles sont appelés* résonances de Ruelle *de* φ*.*

Un spectre de résonances de Ruelle a été obtenu plus tard par Dyatlov–Guillarmou [DG16] pour les systèmes hyperboliques *ouverts* (des trajectoires peuvent s'échapper à l'infini) et plus récemment par Meddane pour les flots Axiom A [Med21], après des contributions de Dang–Rivière sur les flots Morse-Smale [DR20b, DR20c].

Forts de ces techniques modernes, d'abord Giulietti–Liverani–Pollicott [GLP13], puis Dyatlov–Zworski [DZ16] avec une approche semi-classique, ont été en mesure d'obtenir le prolongement analytique de ζ_φ à tout le plan complexe, obtenant ainsi le

Théorème 1.0.3 (Giulietti–Liverani–Pollicott, Dyatlov–Zworski)**.** *Si* φ *est un flot d'Anosov, la fonction* ζ_φ *admet un prolongement méromorphe à tout le plan complexe ; ses pôles et ses zéros sont inclus dans les résonances de Ruelle.*

Dyatlov–Guillarmou ont ensuite étendu ce théorème aux flots Axiom A [DG18] grâce à leur travail sur les systèmes ouverts, répondant ainsi positivement à la question de Smale. Bien sûr, en dehors de celle de Ruelle, beaucoup d'autres fonctions zêta dynamiques existent dans la littérature — notamment pour les dynamiques discrètes — et une introduction plus complète à ce sujet pourra se trouver dans le livre de Baladi [Bal18].

Le théorème 1.0.3 s'obtient en reliant $\zeta_\varphi(s)$ et la résolvante $\mathrm{R}_\varphi(s)$: on peut montrer grâce à la formule des traces de Guillemin [Gui77] que si $\mathrm{Re}(s)$ est assez grande alors

$$\frac{\zeta_\varphi'(s)}{\zeta_\varphi(s)} = \mathrm{e}^{\varepsilon s}\mathrm{tr}_{\mathrm{gr}}^\flat\left(\varphi_{-\varepsilon}^*\mathrm{R}_\varphi(s)\right), \tag{1.0.2}$$

où $\varepsilon > 0$ est un petit nombre et $\mathrm{tr}_{\mathrm{gr}}^\flat$ désigne la *trace bémol graduée* — une extension de la trace graduée L^2 qui est bien définie pour les opérateurs satisfaisant certaines condition de *front d'onde* ; nous renvoyons à l'appendice B.3 pour une définition précise. Grâce à des méthodes semi-classiques (propagation des singularités et estimées radiales), Dyatlov et Zworski ont donné une description précise du front d'onde du noyau de Schwartz de la résolvante et en déduisent que la trace bémol de $\varphi^*_{-\varepsilon}\mathrm{R}_\varphi(s)$ est bien définie ; le théorème 1.0.3 est alors une conséquence de l'égalité (1.0.2) et du théorème 1.0.2.

Nous mentionnons finalement un résultat obtenu par Dyatlov et Zworski [DZ17] sur l'ordre de la singularité de $\zeta_\varphi(s)$ en $s = 0$ pour les flots géodésiques des surfaces.

Théorème 1.0.4 (Dyatlov–Zworski)**.** *Si φ est le flot géodésique d'une surface à courbure négative Σ, alors $\zeta_\varphi(s)$ a un pôle d'ordre $|\chi(\Sigma)|$ en $s = 0$, où $\chi(\Sigma)$ est la caractéristique d'Euler de Σ.*

Ce théorème, connu pour les surfaces hyperboliques depuis le travail de Fried [Fri86b], exhibe un lien entre le comportement de la fonction ζ_φ près de l'origine et la topologie de la variété ambiante. Ce phénomène ne concerne pas seulement les flots géodésiques et nous verrons que certains invariants topologiques peuvent être recouvrés à l'aide des fonctions zêta dynamiques.

Organisation de cette thèse

Dans ce mémoire, nous proposons quelques contributions sur des problématiques d'origine géométrique liées à celles évoquées ci-dessus. Les théorèmes 1.0.1, 1.0.3 et 1.0.4 sont des modèles prototypiques des divers résultats que nous présenterons : comptage d'orbites périodiques, prolongement analytique de fonctions zêta ou de séries dynamiques et nouage d'un lien avec la topologie environnante. Nos résultats seront obtenus en usant systématiquement de la théorie spectrale des flots hyperboliques et en particulier du théorème 1.0.2, ainsi que de son pendant pour les systèmes ouverts. Nous avons choisi de diviser la thèse en trois parties (indépendamment de la trichotomie précédente), comme suit.

Dans la partie I, constituée des chapitres 3 et 4, nous abordons un problème de comptage sous contrainte. Après avoir illustré la problématique sur un modèle jouet au chapitre 3, nous montrons au chapitre 4 un résultat asymptotique dans l'esprit de (1.0.1) pour les géodésiques fermées d'une surface à courbure négative dont on a prescrit les nombres d'intersection avec une famille de courbes simples.

La deuxième partie, formée des chapitres 5 et 6, est plus centrée sur la topologie. Au chapitre 5, nous calulons la valeur à l'origine de certaines séries de Poincaré comptant des arcs géodésiques d'une surface à bord. Puis, dans un cadre assez différent, nous construisons au chapitre 6 un invariant topologique — appelé *torsion dynamique* — défini à l'aide d'une fonction zêta de Ruelle *tordue* par une représentation du groupe fondamental ; nous relions enfin la torsion dynamique à un autre invariant topologique, la *torsion de Turaev*.

La dernière partie est consacrée aux flots de billards associés à une famille finie d'obstacles convexes dans l'espace euclidien et contient les chapitres 7 et 8. D'abord, au chapitre 7, nous étendons au cadre des flots de billards un résultat de comptage sous contrainte obtenu à la première partie. Puis, au chapitre 8, nous montrons que certaines séries de Dirichlet dynamiques liées aux résonances quantiques du système admettent un prolongement méromorphe à tout le plan complexe.

Ces résultats sont exposés plus en détails dans les paragraphes qui suivent.

1.1 Comptage des géodésiques sous contrainte d'intersection

Dans ce paragraphe, nous détaillons les résultats obtenus à la partie I, qui contient notamment l'article *Closed geodesics with prescribed intersection numbers* [Chab].

Soit (Σ, g) une surface riemannienne fermée, orientée et à courbure strictement négative. Soit $\mathcal{P}$ l'ensemble des géodésiques fermées primitives, c'est-à-dire l'ensemble des géodésiques fermées qui ne sont pas multiple d'une géodésique plus courte. Pour tout $L > 0$, notons

$$N(L) = \sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L\}$$

le nombre de ces géodésiques qui sont de longueur inférieure ou égale à L. Rappelons le résultat de Margulis : quand L tend vers l'infini, on a l'équivalent

$$N(L) \sim \frac{\mathrm{e}^{hL}}{hL} \tag{1.1.1}$$

où h est l'entropie topologique du flot géodésique. D'autres résultats de comptage similaires existent pour les surfaces de Riemann non compactes, cf. Sarnak [Sar80], Guillopé [Gui86], ou Lalley [Lal89] ; nous renvoyons à Paulin–Pollicott–Schapira [PPS12] pour des références précises sur les résultats de comptage existant dans des contextes plus généraux.

Avec l'équivalent (1.1.1) en tête, nous nous poserons dans les lignes qui suivent la

Question 1.1.1. Peut-on compter des géodésiques fermées primitives sujettes à certaines contraintes (topologiques ou géométriques) ?

1.1.1 Contraintes homologiques

Une première contrainte que l'on peut vouloir imposer est de nature homologique : étant donnée une classe d'homologie fixée, peut-on compter les géodésiques fermées qui appartiennent à cette classe ? Lalley [Lal88] et Pollicott [Pol91] ont obtenu indépendamment le résultat suivant.

Théorème 1.1.1 (Lalley, Pollicott). *Il existe une constante $c > 0$ telle que pour toute classe d'homologie $\xi \in H_1(\Sigma, \mathbb{Z})$, on a l'équivalent*

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ [\gamma] = \xi\} \sim c\frac{\mathrm{e}^{hL}}{L^{\mathrm{g}+1}}, \tag{1.1.2}$$

quand $L \to \infty$, où g est le genre de la surface.

Des résultats similaires avaient déjà été obtenu pour les surfaces hyperboliques (les surfaces à courbure *constante*, égale à -1) par Phillips–Sarnak [PS87] et Katsuda–Sunada [KS88]. Sans toutefois les énoncer, nous mentionnons que des résultats bien plus précis — par exemple valides pour une classe plus générale de flots hyperboliques, avec des développements asymptotiques comprenant plus de termes, ou encore autorisant la classe d'homologie ξ à dépendre de L — ont été obtenus plus tard par Sharp [Sha93], Babillot–Ledrappier [BL98], Anantharaman [Ana00], et Pollicott–Sharp [PS01].

Ces résultats peuvent s'obtenir grâce à un argument taubérien, en considérant les fonctions zêta tordues

$$\zeta_{\varphi,\chi}(s) = \prod_{\gamma} \left(1 - \chi([\gamma])\mathrm{e}^{-s\ell(\gamma)}\right)^{-1},$$

où le produit porte sur les géodésiques primitives et $\chi : H_1(\Sigma, \mathbb{Z}) \to \mathbb{C}^\times$ est un caractère unitaire. Ces fonctions sont étudiées *via* l'analyse spectrale d'un opérateur de Ruelle ; ce sont des analogues géométriques des séries L de Dirichlet, utilisées notamment par de La Vallée-Poussin pour montrer le théorème de la progression arithmétique.

1.1.2 Nombres d'auto-intersection

Une seconde contrainte naturelle concerne les *nombres d'auto-intersection*. Si $\gamma : \mathbb{R}/\ell(\gamma)\mathbb{Z} \to \Sigma$ est une géodésique fermée paramétrée par longueur d'arc, on définit son *nombre d'auto-intersection* par

$$\mathrm{i}(\gamma,\gamma) = \frac{1}{2}\sharp\left\{(\tau,\tau') \in (\mathbb{R}/\ell(\gamma)\mathbb{Z})^2 \ : \ \gamma(\tau) = \gamma(\tau')\right\}.$$

Une géodésique fermée sera dite *simple* si son nombre d'auto-intersection est nul. Mirzakhani [Mir08, Mir16] a étudié la croissance asymptotique des géodésiques fermées ayant un nombre d'auto-intersection prescrit.

Théorème 1.1.2 (Mirzakhani)**.** *Supposons que (Σ, g) soit hyperbolique. Alors pour tout entier naturel n, il existe $c_n > 0$ telle que, quand $L \to \infty$,*

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \mathrm{i}(\gamma,\gamma) = n\} \sim c_n L^{6(\mathrm{g}-1)}. \tag{1.1.3}$$

L'article [Mir08] de Mirzakhani porte sur les géodésiques simples, et le cas $n = 1$ du théorème précédent a d'abord été prouvé par Rivin [Riv12] ; nous mentionnons aussi les travaux de Erlandsson–Souto [ES16, ES19] qui obtiennent des résultats similaires avec une autre approche. Dans un état d'esprit un peu différent, Sapir [Sap16] et Aougab–Souto [AS18] ont étudié la croissance asymptotique du nombre de *types* de courbes sur les surfaces hyperboliques (tandis que prescrire les nombres d'auto-intersection revient à compter des géodésiques appartenant à des types fixés).

Mirzakhani montre le théorème 1.1.2 en utilisant l'ergodicité de l'action du groupe des difféotopies de la surface sur l'espace des lamination mesurées, l'exposant $6\mathrm{g}-6$ étant la dimension de cet espace. Notons que la croissance des géodésiques fermées dont les nombres d'auto-intersection sont prescrits est polynomiale et non plus exponentielle : il y en a très peu. En fait, un résultat de Lalley [Lal11] (valide aussi pour les surfaces à courbure négative variable) stipule qu'une géodésique fermée typique a un nombre d'auto-intersection proportionnel au carré de sa longueur. Plus précisément, il montre qu'il existe une constante $I > 0$ telle que pour tout $\varepsilon > 0$, on a

$$\lim_{L\to\infty} \frac{1}{N(L)}\sharp\left\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \left|\frac{\mathrm{i}(\gamma,\gamma)}{\ell(\gamma)^2} - I\right| \leqslant \varepsilon\right\} = 1. \tag{1.1.4}$$

La convergence est même exponentielle, comme cela peut être vu en utilisant un principe de grandes déviations de Kifer [Kif94] (voir Anantharaman [Ana99]).

1.1.3 Nombres d'intersection géométriques

Revenons aux surfaces à courbure négative variable. Dans le paragraphe §1.1.1, nous avons contraint la classe d'homologie des géodésiques fermées, ce qui revient à prescrire leurs nombres d'intersection *algébriques* avec des courbes simples. Que se passe-t-il si l'on considère plutôt des nombres d'intersection *géométriques* ? Pour répondre à cette question, fixons d'abord une géodésique fermée simple $\gamma_\star$. Pour toute $\gamma \in \mathcal{P}$, on note

$$\mathrm{i}(\gamma, \gamma_\star) = \inf_{\eta \sim \gamma, \eta_\star \sim \gamma_\star} |\eta \cap \eta_\star|$$

le nombre d'intersection géométrique entre γ et $\gamma_\star$, où l'infimum porte sur les courbes $\eta, \eta_\star : \mathbb{R}/\mathbb{Z} \to \Sigma$ librement homotopes à γ et $\gamma_\star$, respectivement, et

$$|\eta \cap \eta_\star| = \sharp\{(\tau, \tau_\star) \in (\mathbb{R}/\mathbb{Z})^2 \ : \ \eta(\tau) = \eta_\star(\tau_\star)\}.$$

Si n est un entier naturel, nous souhaitons étudier la croissance asymptotique de la quantité

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \mathrm{i}(\gamma, \gamma_\star) = n\}$$

quand $L \to \infty$.

Dans un premier temps, on supposera que la courbe simple $\gamma_\star$ est non séparante, dans le sens où $\Sigma \setminus \gamma_\star$ est connexe (cette condition sera relaxée plus tard).

Théorème 1.1.3. *Supposons que $\gamma_\star$ n'est pas séparante. Alors il existe des constantes $c_\star > 0$ et $h_\star \in]0, h[$ telles que pour tout entier $n > 0$, on a l'équivalent*

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \mathrm{i}(\gamma, \gamma_\star) = n\} \sim \frac{(c_\star L)^n}{n!} \frac{\mathrm{e}^{h_\star L}}{h_\star L}, \quad L \to \infty. \tag{1.1.5}$$

Le nombre $h_\star$ est l'entropie topologique du flot géodésique de la surface $\Sigma_\star$ (à bord) obtenue en découpant Σ le long de $\gamma_\star$ (voir le paragraphe 1.1.4 ci-dessous pour une définition précise). Notons que le cas $n = 0$ revient à compter les géodésiques fermées de $\Sigma_\star$ et était déjà connu grâce au travail de Dal'bo [Dal99], qui a montré que le flot géodésique des surfaces convexe co-compactes est topologiquement mélangeant, lui permettant ainsi d'utiliser le résultat de Parry–Pollicott [PP83]. En revanche, la croissance asymptotique (1.1.5) n'était pas connue pour $n > 0$, y compris en courbure constante.

Bien que plus faible que celle obtenue par Margulis, cette croissance reste exponentielle ; elle est donc strictement comprise entre celles obtenues par Mirzakhani d'un côté, et Lalley et Pollicott de l'autre. Comme nous l'avons vu, imposer un nombre d'auto-intersection est très contraignant, puisque pour une géodésique γ typique, on a $\mathrm{i}(\gamma, \gamma) \approx I\ell(\gamma)^2$. Ici c'est le nombre $\mathrm{i}(\gamma, \gamma_\star)$ que nous imposons ; en utilisant le principe de larges déviations de Kifer et la forme d'intersection de Bonahon [Bon86], nous verrons qu'on a typiquement $\mathrm{i}(\gamma, \gamma_\star) \approx I_\star \ell(\gamma)$ où $I_\star > 0$ ne dépend pas de γ (cf. la proposition 4.8.1 pour un énoncé précis dans l'esprit de (1.1.4)).

Si la courbe $\gamma_\star$ est séparante, on a le résultat suivant.

Théorème 1.1.4. *Si $\gamma_\star$ sépare Σ en deux surfaces Σ_1 et Σ_2, on désigne par $h_j \in]0, h[$ l'entropie du système ouvert $(\Sigma_j, g|_{\Sigma_j})$ pour $j = 1, 2$ (cf. le paragraphe suivant), et* *on définit* $h_\star = \max(h_1, h_2)$*. Alors il existe* $c_\star > 0$ *telle que pour tout* $n \in \mathbb{N}$ *on l'équivalent, quand* $L \to +\infty$,

$$N(2n, L) \sim \begin{cases} \dfrac{(c_\star L)^n}{n!} \dfrac{\mathrm{e}^{h_\star L}}{h_\star L} & \text{if } h_1 \neq h_2, \\ 2\dfrac{(c_\star L^2)^n}{(2n)!} \dfrac{\mathrm{e}^{h_\star L}}{h_\star L} & \text{if } h_1 = h_2. \end{cases} \tag{1.1.6}$$

La démonstration des théorèmes 1.1.3 et 1.1.4 fait notamment appel à un *opérateur de diffusion dynamique* $\mathcal{S}(s)$ agissant sur le bord du fibré unitaire tangent de $\Sigma_\star$, étudié par l'intermédiaire de la théorie des résonances pour les systèmes ouverts de Dyatlov–Guillarmou [DG16] ; la super trace bémol de $\mathcal{S}(s)^n$ est une série impliquant les géodésiques fermées γ telles que $\mathrm{i}(\gamma, \gamma_\star) = n$. Nous renvoyons à l'introduction du chapitre 4 pour une présentation plus détaillée de la stratégie adoptée. Les mêmes techniques s'emploient aussi pour obtenir des résultats asymptotiques sur des géodésiques fermées dont on a prescrit *plusieurs* nombres d'intersection géométriques avec une famille de courbes, ce que nous précisons ci-dessous.

1.1.4 Prescription des nombres d'intersection avec une famille de courbes

Soit $r \geqslant 1$ un entier et $(\gamma_{\star,1}, \ldots, \gamma_{\star,r})$ une famille de géodésiques fermées simples deux à deux disjointes. Pour tout r-uplet $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ d'entiers naturel, on souhaite comprendre le comportement asymptotique de la quantité

$$N(\mathbf{n}, L) = \sharp\left\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \mathrm{i}(\gamma, \gamma_{\star,j}) = n_j, \ j = 1, \ldots, r\right\}$$

quand $L \to +\infty$, où $\mathrm{i}(\gamma, \gamma_{\star_j})$ est le nombre d'intersection géométrique entre γ et $\gamma_{\star,j}$.

Théorème 1.1.5. *Soit* $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r \setminus \{0\}$*. Si* $N(\mathbf{n}, L) > 0$ *pour un* $L > 0$*, alors il existe des constantes* $C_\mathbf{n} > 0, d_\mathbf{n} \in \mathbb{N} \setminus \{0\}$ *et* $h_\mathbf{n} \in]0, h[$ *telles que*

$$N(\mathbf{n}, L) \sim C_\mathbf{n} L^{d_\mathbf{n} - 1} \mathrm{e}^{h_\mathbf{n} L}, \quad L \to +\infty.$$

En fait, un résultat similaire est valide si l'on impose en plus l'ordre dans lequel on veut que les intersections se produisent, comme suit. Soient $\Sigma_1, \ldots, \Sigma_q$ les composantes connexes de la surface $\Sigma_\star = \Sigma \setminus (\gamma_{\star,1} \cup \cdots \cup \gamma_{\star,r})$ obtenue en découpant Σ le long des courbes $\gamma_{\star,1}, \ldots, \gamma_{\star,r}$ (voir la figure 1.2). Si $\gamma \in \mathcal{P}$ est une géodésique fermée qui intersecte au moins une des courbes $\gamma_{\star,j}$, on désigne par $\omega(\gamma)$ la paire (u, v) de séquences

$$u = (u_1, \ldots, u_N) \quad \text{et} \quad v = (v_1, \ldots, v_N)$$

avec $N \geqslant 1$, ordonnées cycliquement, telles que γ voyage dans $\Sigma_{v_1}, \ldots, \Sigma_{v_N}$ (dans cet ordre !) et passe de Σ_{v_k} à $\Sigma_{v_{k+1}}$ en traversant $\gamma_{\star,u_k}$, où $v_{N+1} = v_1$ (cf. la figure 1.2) ; ces suites sont bien définies modulo application d'une permutation cyclique. Une telle paire ω de séquences finies est appelée *chemin admissible* si $\omega \sim \omega(\gamma)$ pour au moins une géodésique $\gamma \in \mathcal{P}$, où $\omega \sim \omega(\gamma)$ signifie que $\omega(\gamma)$ est une permutation cyclique de ω (la permutation étant la même pour les deux composantes de ω).

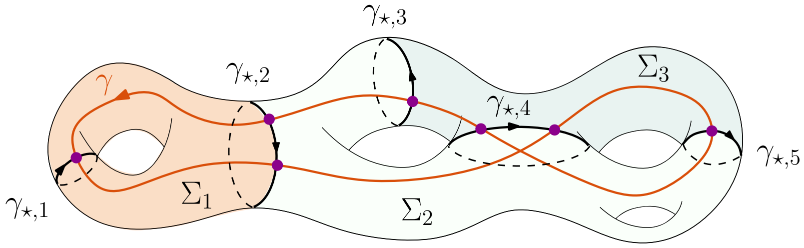

FIGURE 1.2 – Une géodésique fermée γ sur Σ. Ici, on a $r = 5$, $q = 3$, et $\omega(\gamma) \sim (u, v)$ avec $u = (1, 2, 4, 5, 4, 3, 2)$ et $v = (1, 1, 2, 3, 2, 3, 2)$ (le point de départ de γ est la flèche orangée).

Soit $S\Sigma$ le fibré unitaire tangent de (Σ, g), et $(\varphi_t)_{t\in\mathbb{R}}$ le flot géodésique associé, agissant sur $S\Sigma$. Soit $\pi : S\Sigma \to \Sigma$ la projection naturelle. On désigne par $h_j > 0$ ($j = 1, \dots, q$) l'entropie du système ouvert $(\Sigma_j, g|_{\Sigma_j})$, c'est-à-dire l'entropie topologique du flot φ restreint à l'ensemble capté

$$K_j = \overline{\{(x, w) \in S\Sigma \ : \ \pi(\varphi_t(x, w)) \in \Sigma_j, \ t \in \mathbb{R}\}},$$

où la fermeture est prise dans $S\Sigma$.

Pour tout chemin admissible $\omega = (u, v)$ de taille N, on définit

$$h_\omega = \max\{h_{v_k} \ : \ k = 1, \dots, N\} \quad \text{et} \quad d_\omega = \sharp\{k = 1, \dots, N \ : \ h_{v_k} = h_\omega\}.$$

Le nombre h_ω est le maximum des entropies des surfaces rencontrées par n'importe quelle géodésique $\gamma \in \mathcal{P}$ satisfaisant $\omega(\gamma) \sim \omega$ tandis que d_ω est le nombre de fois où une telle géodésique rencontre une surface dont l'entropie est égale à h_ω (par exemple, sur la figure 1.2, si l'entropie h_2 de Σ_2 est la plus grande, on a $h(\omega) = h_2$ et $d(\omega) = 3$, puisque γ passe trois fois dans Σ_2).

En fait, les nombres h_ω et d_ω ne dépendent que de $\mathbf{n}(\omega) = (n_1, \dots, n_r)$ où $n_j = \sharp\{k = 1, \dots, N \ : \ u_k = j\}$ (voir le paragraphe §4.9) ; ainsi, nous les désignons par $h_{\mathbf{n}(\omega)}$ and $d_{\mathbf{n}(\omega)}$ respectivement.

Théorème 1.1.6. *Soit ω un chemin admissible. Alors il existe $c(\omega) > 0$ telle que*

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \omega(\gamma) \sim \omega\} \sim c(\omega) L^{d_{\mathbf{n}(\omega)}-1} \mathrm{e}^{h_{\mathbf{n}(\omega)} L}, \quad L \to +\infty.$$

Notons que le théorème 1.1.5 peut être déduit du théorème 1.1.6 en sommant sur les chemins admissibles ω tels que $\mathbf{n}(\omega) = \mathbf{n}$, où $\mathbf{n} \in \mathbb{N}^r$ est fixé. En revanche, le théorème 1.1.3 n'est pas une conséquence directe du théorème 1.1.6 ; il découle d'un résultat plus précis — énoncé dans le paragraphe §4.9 — qui permet d'exprimer les constantes $c(\omega^k), d_{\mathbf{n}(\omega^k)}$ et $h_{\mathbf{n}(\omega^k)}$ en fonction de $c(\omega), d_{\mathbf{n}(\omega)}$ et $h_{\mathbf{n}(\omega)}$, où ω^k est le chemin obtenu en concaténant k fois le chemin ω.

1.2 Séries dynamiques et topologie

Nous relatons ici les résultats obtenus à la partie II. Celle-ci est constituée du chapitre 5, qui contient l'article *Poincaré series for surfaces with boundary* [Cha21], et du chapitre 6, qui retranscrit l'article *Dynamical torsion for contact Anosov flows* [CD19] écrit en collaboration avec Nguyen Viet Dang.

1.2.1 Séries de Poincaré pour les surfaces à bord

Soit (Σ, g) une surface riemannienne connexe, orientée, à courbure négative et dont le bord $\partial\Sigma$ est totalement géodésique. Soit $\mathcal{G}^\perp$ l'ensemble des *orthogéodésiques* de Σ, c'est-à-dire l'ensemble des arcs géodésiques $\gamma : [0, \ell] \to \Sigma$ (paramétrés par longueur d'arc) tels que $\gamma(0), \gamma(\ell) \in \partial\Sigma$, $\gamma'(0) \perp T_{\gamma(0)}\partial\Sigma$ et $\gamma'(\ell) \perp T_{\gamma(\ell)}\partial\Sigma$. Si $\mathrm{Re}(s)$ est assez grande, la série de Poincaré

$$\eta(s) = \sum_{\gamma \in \mathcal{G}^\perp} \mathrm{e}^{-s\ell(\gamma)}, \tag{1.2.1}$$

où $\ell(\gamma)$ désigne la longueur de γ, converge (voir §5.3.2). Au chapitre 5, nous montrerons le

Théorème 1.2.1. *La série de Poincaré $s \mapsto \eta(s)$ admet un prolongement méromorphe à tout le plan complexe, et s'annule à l'origine.*

Si x et y sont des points distincts de Σ, nous pouvons aussi considérer la série de Poincaré associée aux arcs géodésiques joignant x à y. Plus précisément, on définit

$$\eta_{x,y}(s) = \sum_{\gamma : x \rightsquigarrow y} \mathrm{e}^{-s\ell(\gamma)},$$

où la somme porte sur les arcs géodésiques $\gamma : [0, \ell(\gamma)] \to \Sigma$ (paramétrés par longueur d'arc) tels que $\gamma(0) = x$ et $\gamma(\ell(\gamma)) = y$. Alors nous avons le résultat suivant.

Théorème 1.2.2. *La série de Poincaré $s \mapsto \eta_{x,y}(s)$ admet un prolongement méromorphe à tout le plan complexe, et sa valeur à l'origine est donnée par*

$$\eta_{x,y}(0) = \frac{1}{\chi(\Sigma)},$$

où $\chi(\Sigma)$ est la caractéristique d'Euler de Σ.

Les nombres $\eta(0)$ et $\eta_{x,y}(0)$ peuvent être interprétés comme le nombre d'enlacement de certains noeuds lengendriens dans $S\Sigma$; pour la série η, cet enlacement est nul.

À notre connaissance, le théorème 1.2.1 est le premier résultat sur une série concernant l'*orthospectre* (l'ensemble des longueurs des orthogéodésiques) d'une surface à bord totalement géodésique, de courbure négative potentiellement *variable*. Pour les surfaces hyperboliques à bord, l'orthospectre a été largement étudié, notamment par Basmajian [Bas93], Bridgeman [Bri11], Calegari [Cal10] (voir aussi Bridgeman–Kahn [BK10]). En particulier, il est connu que si (Σ, g) est une surface hyperbolique compacte à bord totalement géodésique, on a

$$\ell(\partial\Sigma) = \sum_{\gamma \in \mathcal{G}^\perp} 2 \log \coth(\ell(\gamma)/2), \quad \mathrm{vol}(\Sigma) = \frac{2}{\pi} \sum_{\gamma \in \mathcal{G}^\perp} \mathcal{R}\left(\mathrm{sech}^2(\ell(\gamma)/2)\right),$$

où $\ell(\partial\Sigma)$ est la longueur Σ, $\mathrm{vol}(\Sigma)$ est son volume, et $\mathcal{R}$ est la fonction dilogarithmique. Nous renvoyons à [BT16] pour une exposition détaillée de ces résultats.

Afin d'étudier les séries de Poincaré $\eta(s)$ et $\eta_{x,y}(s)$, nous adopterons la stratégie élégante de Dang et Rivière [DR20a], qui consiste à réécrire les deux séries comme des appariements distributionnels impliquant la résolvante du flot géodésique. Sur une surface *fermée* à courbure négative, Dang et Rivière ont prouvé que les séries de Poincaré associées aux arcs orthogéodésiques joignant deux géodésiques fermées triviales en homologie, mais aussi celles comptant les arcs géodésiques joignant deux points, admettent un prolongement méromorphe à tout le plan complexe ; ils ont montré que leurs valeurs à l'origine coïncident avec l'enlacement de certains noeuds legendriens dans le fibré unitaire de la surface — pour la série comptant les arcs géodésiques reliant deux points, ils obtiennent (comme ici) que cette valeur coïncide avec l'inverse de la caractéristique d'Euler de la surface. Le travail de Dang–Rivière généralise un résultat antérieur de Paternain [Pat00] qui stipule que si (Σ, g) est fermée et hyperbolique, alors

$$\int_\Sigma \eta_{x,y}(s)\mathrm{dvol}_g(x)\mathrm{dvol}_g(y) = \frac{4\pi\chi(\Sigma)}{1-s^2},$$

où $\eta_{x,y}$ est la série associée aux arcs reliant x et y, et où vol_g est la mesure de volume riemannienne. La principale nouveauté de nos résultats est que nous travaillons avec des surfaces à bords. Ceci nous conduira (encore !) à utiliser la théorie des résonances de Pollicott–Ruelle pour les systèmes ouverts développée par Dyatlov–Guillarmou [DG16] ainsi qu'un résultat de Hadfield [Had18] sur la topologie des états résonants.

1.2.2 Torsion dynamique pour les flots d'Anosov de contact

Soit M une variété fermée de dimension impaire et (E, ∇) un fibré plat de rang d sur M. Le transport parallèle de la connexion ∇ induit une représentation $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{GL}(\mathbb{C}^d))$ du groupe fondamental. De plus, ∇ induit une différentielle tordue sur le complexe $\Omega^\bullet(M, E)$ des formes différentielles sur M à valeurs dans E, donnant lieu à des groupes de cohomologie $H^\bullet(M, \nabla) = H^\bullet(M, \rho)$. On dira que ∇ (ou ρ) est *acyclique* si ces groupes de cohomologie sont triviaux. Si ρ est unitaire (c'est-à-dire s'il existe une structure hermitienne sur E qui est préservée par ∇) et acyclique, Reidemeister [Rei35] a introduit un invariant combinatoire $\tau_{\mathrm{R}}(\rho)$ de la paire (M, ρ), appelé *torsion de Franz-Reidemeister* (ou R-torsion), qui est un nombre strictement positif. Cela lui a permis de classifier (à homéomorphisme près) les espaces lenticulaires en dimension 3 ; ce résultat a été étendu aux dimensions supérieures par Franz [Fra35] et de Rham [dR36].

Côté analytique, Ray-Singer [RS71] ont introduit un autre invariant $\tau_{\mathrm{RS}}(\rho)$ — la *torsion analytique* — défini grâce à la fonction zêta spectrale du laplacien induit par la structure hermitienne sur E et une métrique riemannienne sur M. Ils ont conjecturé l'égalité entre la torsion analytique et celle de Reidemeister. Cette conjecture a été démontrée indépendamment par Cheeger [Che79] et Müller [Mül78], dans le cas où on suppose seulement ρ unitaire (la R-torsion et la torsion analytique ont une extension naturelle dans le cas où ρ n'est pas acyclique). Le théorème de Cheeger-Müller a été étendu aux fibrés plats unimodulaires par Müller [Mul93] et à tous les fibrés plats par Bismut-Zhang [BZ92].

Fried [Fri87] s'est intéressé au lien entre la R-torsion la fonction zêta de Ruelle d'un flot d'Anosov φ généré par un champ X et *tordue* par une représentation ρ.

Plus précisément, on pose

$$\zeta_{X,\rho}(s) = \prod_{\gamma \in \mathcal{G}_\varphi^\sharp} \det\Big(1 - \varepsilon_\gamma \rho([\gamma]) \mathrm{e}^{-s\tau(\gamma)}\Big), \quad \mathrm{Re}(s) \gg 1,$$

où $\mathcal{G}_\varphi^\sharp$ est l'ensemble des orbites périodiques primitives de φ, $\tau(\gamma)$ est la période de γ et $\varepsilon_\gamma = 1$ si le fibré stable de γ est orientable et $\varepsilon_\gamma = -1$ sinon. Le théorème 1.0.3 s'étend naturellement à ce contexte, et $\zeta_{X,\rho}$ admet un prolongement méromorphe à tout le plan complexe. En utilisant la formule des traces de Selberg, Fried [Fri86a] a pu relier le comportement de $\zeta_{X,\rho}(s)$ près de $s = 0$ avec τ_{R}, dans l'esprit du théorème 1.0.4, comme suit.

Théorème 1.2.3 (Fried)**.** *Soit $M = SZ$ le fibré unitaire tangent d'une variété hyperbolique fermée Z, et X le champ de vecteur géodésique associé. Supposons que $\rho : \pi_1(M) \to O(d)$ est une représentation unitaire et acyclique. Alors $\zeta_{X,\rho}(s)$ est analytique près de $s = 0$ et*

$$|\zeta_{X,\rho}(0)|^{(-1)^r} = \tau_R(\rho), \tag{1.2.2}$$

où $2r + 1 = \dim M$.

Dans son article [Fri95], Fried a proposé la

Conjecture 1.2.1 (Fried)**.** *L'égalité* (1.2.2) *est valable pour les flots géodésiques des variétés à courbure négative.*

Fried avait déjà conjecturé la validité de l'égalité (1.2.2) pour les variétés localement symétriques à courbure négative dans [Fri87] ; cela a été prouvé par Shen [She17] après des contributions de Moscovici-Stanton [MS91].

Plus généralement, on peut se poser la question de la validité de (1.2.2) pour des flots hyperboliques généraux. Pour les flots d'Anosov analytiques, Sanchez-Morgado [SM93, SM96] a montré, en dimension 3, que si ρ est acyclique, unitaire, et vérifie que $\rho([\gamma]) - \varepsilon_\gamma^j$ est inversible pour $j \in \{0, 1\}$ pour une certaine orbite γ, alors l'égalité (1.2.2) est satisfaite. La preuve de Sanchez-Morgado repose cependant sur l'existence de partitions de Markov analytiques et ne s'étend donc pas, *a priori*, aux flots C^∞.

Dang–Guillarmou–Rivière–Shen [DGRS20] ont contourné le problème en s'appuyant sur la théorie spectrale moderne des flots hyperboliques évoquée plus haut (voir aussi Dang–Rivière [DR19b] pour les flots Morse–Smale). En effet, le théorème 1.0.2 est valide ici et permet de définir un spectre de résonances de Ruelle pour la dérivée de Lie tordue

$$\mathcal{L}_X^\nabla = \nabla \iota_X + \iota_X \nabla,$$

où ι_X est le produit intérieur avec X agissant sur $\Omega^\bullet(M, E)$; ce spectre est noté $\mathrm{Res}(\mathcal{L}_X^\nabla)$.

Théorème 1.2.4 (Dang–Rivière–Guillarmou–Shen)**.** *Soit ρ une représentation acyclique de $\pi_1(M)$. Alors l'application*

$$X \mapsto \zeta_{X,\rho}(0)$$

est localement constante sur l'espace des champs de vecteurs lisses X d'Anosov pour lesquels $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$. Si le flot φ_t préserve une forme volume lisse et $\dim(M) = 3$, alors l'égalité (1.2.2) est satisfaite si $b_1(M) \neq 0$ ou sous la même hypothèse demandée par Sanchez-Morgado [SM96].

Ce résultat a permis à Dang–Guillarmou–Rivière–Shen, par le biais d'un argument d'approximation, d'utiliser le résultat de Sanchez-Morgado pour montrer que l'égalité (1.2.2) est satisfaite pour les flots d'Anosov qui préservent une forme volume en dimension 3. Il y a cependant deux restrictions au théorème 1.2.4. La première est que l'égalité $|\zeta_{X,\rho}(0)|^{(-1)^r} = \tau_\mathrm{R}(\rho)$ concerne deux nombres strictement positifs, du fait que la representation ρ est unitaire ; or il se pourrait que, si ρ est non unitaire, la phase du nombre complexe $\zeta_{X,\rho}(0)$ contienne des informations topologiques. La seconde concerne l'hypothèse que 0 n'est pas une résonance de Ruelle. À l'exception des petites dimensions étudiées dans [DGRS20], cette hypothèse est difficile à contrôler, même sur des exemples explicites. Par ailleurs, dans le cas non-acyclique, les travaux récents de Cekic–Paternain [CP21] et Cekic–Dyatlov–Küster–Paternain [CDDP22] montrent que les dimensions des espaces propres de $\mathcal{L}_X^\nabla$ pour la résonance $s = 0$ — qui sont étroitement liées à l'ordre de la singularité de $\zeta_{X,\rho}$ à l'origine — ne sont pas nécessairement stables par perturbations du champ X. Ainsi, rien ne garantit *a priori* que le nombre $\zeta_{X,\rho}(0)$ soit bien défini, même si la représentation ρ est supposée acyclique.

Pour surmonter ces restrictions (au moins dans le cas où X engendre un flot d'Anosov de *contact*), nous avons, dans un travail en collaboration avec Nguyen Viet Dang [CD19], introduit un nouvel invariant — la *torsion dynamique* — bien défini pour n'importe quelle représentation ρ et qui coïncide avec $\zeta_{X,\rho}(0)^{\pm 1}$ si $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$. Avant d'introduire cet invariant, nous discutons de quelques versions raffinées des torsions combinatoire et analytique présentes dans la littérature, dont certaines seront reliées à la torsion dynamique.

1.2.2.1 Des versions raffinées de la torsion

La torsion de Franz–Reidemeister τ_R est donnée par le module d'un certain produit alterné de déterminants ; le module est important, car des choix doivent être faits pour définir la torsion combinatoire, et ces ambiguïtés ont des répercussions sur les valeurs des déterminants. Pour résoudre ce problème, Turaev [Tur86, Tur90, Tur97] a introduit une version raffinée de la R-torsion combinatoire, appelée *torsion combinatoire raffinée*. C'est un nombre $\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ qui dépend d'autres données combinatoires, à savoir une *structure d'Euler* $\mathfrak{e}$ et un choix d'*orientation cohomologique* $\mathfrak{o}$ de M ; si ρ est acyclique et unitaire, on a $|\tau_{\mathfrak{e},\mathfrak{o}}(\rho)| = \tau_\mathrm{R}(\rho)$. Nous renvoyons au paragraphe §6.7.2 pour des définitions précises. Plus tard, Farber-Turaev [FT00] ont généralisé la définition pour les représentations non-acycliques. Dans ce cas, $\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ est un élément du fibré déterminant $\det H^\bullet(M,\rho)$.

Motivés par le travail de Turaev, Braverman-Kappeler [BK07c, BK+08, BK07b] ont introduit une version raffinée de la torsion analytique de Ray–Singer, la *torsion analytique raffinée* $\tau_\mathrm{an}(\rho)$, qui est à valeurs complexe si ρ est acyclique. Leur construction repose sur l'existence d'un opérateur de *chiralité*

$$\Gamma_g : \Omega^\bullet(M,E) \to \Omega^{n-\bullet}(M,E), \quad \Gamma_g^2 = \mathrm{Id},$$

une version renormalisée de l'étoile de Hodge associée à une métrique g. Ils ont montré que le ratio

$$\rho \mapsto \frac{\tau_{\mathrm{an}}(\rho)}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho)}$$

est une fonction holomorphe sur la variété des representations, donnée par une expression locale explicite, à multiplication par une constante près. Ce résultat généralise le théorème de Cheeger-Müller. Simultanément, Burghelea-Haller [BH07] ont introduit une torsion analytique complexe, étroitement liée à celle de [BK07a] quand elle est définie ; nous renvoyons au travail de Huang [H+07] pour plus de détails sur ce sujet.

1.2.2.2 La torsion dynamique

Supposons maintenant que $X = X_\vartheta$ est le champ de Reeb associé à une forme de contact ϑ sur M. La forme de contact ϑ induit un opérateur de chiralité

$$\Gamma_\vartheta : \Omega^\bullet(M,E) \to \Omega^{n-\bullet}(M,E), \quad \Gamma_\vartheta^2 = \mathrm{Id},$$

cf. §6.4, une version contact de l'étoile de Hodge. Soit $C^\bullet \subset \mathcal{D}'^\bullet(M,E)$ l'espace (de dimension finie) des états résonants de Ruelle généralisés de $\mathcal{L}_X^\nabla$ pour la résonance 0. Plus précisément, on pose

$$C^\bullet = \left\{ u \in \mathcal{D}'^\bullet(M,E) \ : \ \mathrm{WF}(u) \subset E_u^*, \ \exists N \in \mathbb{N}, \ \left(\mathcal{L}_X^\nabla\right)^N u = 0 \right\},$$

où WF est le front d'onde de Hörmander, $E_u^* \subset T^*M$ est le fibré co-instable de X [1], et $\mathcal{D}'(M,E)$ est l'espace des courants à valeurs dans E. Alors ∇ induit une différentielle sur $C^\bullet$, ce qui nous donne un complexe de co-chaines de dimension finie. Un résultat de Dang–Rivière [DR19b] implique que le complexe $(C^\bullet, \nabla)$ est acyclique dès que ∇ l'est. Parce que la chiralité Γ_ϑ commute avec $\mathcal{L}_X^\nabla$, elle induit une chiralité sur $C^\bullet$; en particulier, on peut calculer la torsion $\tau(C^\bullet, \Gamma_\vartheta)$ du complexe de dimension finie $(C^\bullet, \nabla)$, respectivement à la chiralité Γ_ϑ, comme défini dans [BK07c] (voir le paragraphe §6.2). La *torsion dynamique* τ_ϑ est alors définie par

$$\tau_\vartheta(\rho)^{(-1)^q} = \pm \underbrace{\tau(C^\bullet, \Gamma_\vartheta)^{(-1)^q}}_{\text{torsion en dimension finie}} \times \underbrace{\lim_{s\to 0} s^{-m(X,\rho)} \zeta_{X,\rho}(s)}_{\text{fonction zeta renormalisée}} \tag{1.2.3}$$

où le signe $\pm$ sera donné plus tard, $m(X,\rho) \in \mathbb{Z}$ est l'ordre de la singularité de $\zeta_{X,\rho}(s)$ au point $s = 0$, et $q = (\dim(M) - 1)/2$ est la dimension du fibré instable de X. Une remarque cruciale est que ni $m(X,\rho)$, ni chacun des deux termes dans le produit (1.2.3), ne sont *a priori* stables par perturbations de (X,ρ) ; la torsion dynamique τ_ϑ a en revanche d'intéressantes propriétés d'invariance, comme nous le verrons dans le paragraphe suivant.

1.2.2.3 Résultats obtenus

Soit $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ l'ensemble des représentations acycliques $\pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$ du groupe fondamental et $\mathcal{A} \subset \mathcal{C}^\infty(M, T^*M)$ l'ensemble des formes de contact sur M

1. Ici E_u^* est l'anihilateur de $E_u \oplus \mathbb{R}X$, où $E_u \subset TM$ est le fibré instable du flot, cf. §6.3

dont le champ de Reeb induit un flot d'Anosov. Dans l'esprit du travail de Ray–Singer [RS71] sur l'indépendance de la torsion analytique relativement à un choix d'une métrique riemannienne, le premier résultat de notre article [CD19] montre que $\tau_\vartheta(\rho)$ est invariant par des petites perturbations de la forme de contact $\vartheta \in \mathcal{A}$.

Théorème 1.2.5 (C.–Dang)**.** *Soit* $(\vartheta_\tau)_{\tau\in(-\varepsilon,\varepsilon)}$ *une famille lisse de formes de contact de* $\mathcal{A}$*. Alors*

$$\partial_\tau \log \tau_{\vartheta_\tau}(\rho) = 0$$

pour toute $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d)$*.*

Dans le cas où la représentation ρ n'est pas acyclique, il est toujours possible de définir $\tau_\vartheta(\rho)$ comme un élément du fibré déterminant $\det H^\bullet(M,\rho)$ et cet élément est encore invariant par perturbations de $\vartheta \in \mathcal{A}$, comme ce sera expliqué dans les remarques 6.4.5 et 6.5.2.

Nous comparons ensuite τ_ϑ avec la torsion de Turaev $\tau_{\mathfrak{e},\mathfrak{o}}$, qui dépend des choix d'une structure d'Euler $\mathfrak{e}$ et d'une orientation cohomologique $\mathfrak{o}$.

Théorème 1.2.6 (C.–Dang)**.** *Soit* (M,ϑ) *une variété de contact telle que le champ de Reeb de* ϑ *induit un flot d'Anosov. Alors* $\rho \mapsto \tau_\vartheta(\rho)$ *est holomorphe*[2] *et il existe une structure d'Euler* $\mathfrak{e}$ *telle que pour toute orientation* $\mathfrak{o}$ *et toute famille lisse* $(\rho_u)_{u\in(-\varepsilon,\varepsilon)}$ *de* $\mathrm{Rep}_{\mathrm{ac}}(M,d)$*,*

$$\partial_u \log \tau_\vartheta(\rho_u) = \partial_u \log \tau_{\mathfrak{e},\mathfrak{o}}(\rho_u)$$

De plus, si $\dim M = 3$ *et* $b_1(M) \neq 0$*, l'application* $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ *est de module* 1 *sur les composantes connexes de* $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ *qui contiennent une représentation acyclique et unitaire.*

Ce résultat nous permet de comparer directement les comportements de $\tau_\vartheta(\rho)$ et $\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ — en tant que fonctions de la représentation ρ — tandis que dans [DGRS20], les auteurs se basent sur l'existence d'un lien *a priori* entre $\zeta_{X,\rho}(0)$ et $\tau_{\mathrm{R}}(\rho)$ (donné par Sanchez-Morgado [SM96]).

Finalement, énonçons un dernier résultat qui s'intéresse à la façon dont $\partial_u \log \tau_\vartheta(\rho_u)$ dépend du choix du champ de vecteurs X_ϑ.

Théorème 1.2.7 (C.–Dang)**.** *Soit* (M,ϑ) *une variété de contact telle que le champ de Reeb induit un flot d'Anosov. Soit* $(\rho_u)_{|u|\leqslant\varepsilon}$ *une famille lisse de* $\mathrm{Rep}_{\mathrm{ac}}(M,d)$*. Alors, pour tout* $\eta \in \mathcal{A}$*, on a la formule variationnelle*

$$\partial_u \log \tau_\eta(\rho_u) = \partial_u \log \tau_\vartheta(\rho_u) + \partial_u \log \underbrace{\det \langle \rho_u, \mathrm{cs}(X_\vartheta, X_\eta)\rangle}_{\text{topologique}}$$

où $\mathrm{cs}(X_\vartheta, X_\eta) \in H_1(M,\mathbb{Z})$ *est la classe de Chern-Simons de la paire* (X_ϑ, X_η) *(voir le paragraphe 6.7.1).*

Le terme $\det\langle\rho, \mathrm{cs}(X_\vartheta, X_\eta)\rangle$ est topologique car c'est le déterminant de la représentation ρ calculée sur la classe Chern–Simons $\mathrm{cs}(X_\vartheta, X_\eta) \in H_1(M,\mathbb{Z})$[3] ; cette

2. $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ est une variété algébrique sur $\mathbb{C}$, cf. §6.9.2.

3. Il est à noter que le déterminant $\det\langle\rho, \mathrm{cs}(X_\vartheta, X_\eta)\rangle$ ne dépend pas du choix du représentant $\mathrm{cs}(X_\vartheta, X_\eta)$ dans $\pi_1(M)$.

dernière mesure l'obstruction à trouver une homotopie connectant X_ϑ et X_η parmi des champs de vecteurs de s'annulant pas — par exemple, s'il existe une famille continue $(X_t)_{t\in[0,1]}$ de champs de vecteurs sans zéros telle que $X_0 = X_\vartheta$ et $X_1 = X_\eta$, alors $\mathrm{cs}(X_\vartheta, X_\eta) = 0$.

Puisque la torsion dynamique est définie à l'aide de la fonction zêta de Ruelle, les résultats énoncés ci-dessus permettent de récupérer des informations sur le comportement de $\zeta_{X,\rho}$ près de l'origine (voir en particulier le corollaire 6.1.5).

1.2.2.4 Travaux liés

Certains analogues de notre torsion dynamique ont été introduits par Burghelea–Haller [BH08b] pour les champs de vecteurs admettant une 1-forme de Lyapunov fermée, généralisant des résultats de Hutchings [Hut02] et Hutchings–Lee [HL99b, HL99a] sur les flots de Morse–Novikov. Dans ce cas, la torsion dynamique dépend du choix d'une structure d'Euler et est une fonction définie sur un sous-ensemble de $\mathrm{Rep}_{\mathrm{ac}}(M, d)$; si $d = 1$, il est montré dans [BH08a] qu'elle s'étend en une fonction rationnelle sur la fermeture de Zariski de $\mathrm{Rep}_{\mathrm{ac}}(M, 1)$ qui coïncide, au signe près, avec la torsion de Turaev. Dans ces travaux, la torsion considérée est de la forme

$$\text{fonction zêta dynamique en zéro} \times \text{terme correctif}$$

où le terme correctif est la torsion d'un complexe de dimension finie dont les chaînes sont générées par les zéros du champ vecteurs. Le choix de la structure d'Euler donne une base distinguée du complexe et donc une valeur bien définie pour la torsion. Pour les flots d'Anosov, il n'y a pas de choix canonique de courants dans $C^\bullet$; c'est précisément là où notre chiralité Γ_ϑ intervient, puisqu'elle permet de définir une classe de bases de $C^\bullet$ invariantes par Γ_ϑ.

Nous mentionnons aussi les résultats de Rumin–Seshadri [RS12] sur les 3-variétés CR de Seifert, qui relient une fonction zêta dynamique à une certaine torsion de contact analytique. Plus récemment, Spilioti [Spi20] et Müller [Mue20] ont été en mesure de comparer la fonction zêta de Ruelle associée à une variété hyperbolique compacte de dimension impaire avec certaines torsions analytiques raffinées. Enfin, pour les flots géodésiques des orbisurfaces hyperboliques compactes, Bénard–Frahm–Spilioti [BFS21] ont montré que $\zeta_{X,\rho}(0)$ coïncide avec la torsion de Turaev (au signe près, pour un certain choix de structure d'Euler) en utilisant la formule des traces de Selberg ; ceci constitue, pour les orbisurfaces, une généralisation de notre théorème 1.2.6.

1.3 Résultats sur les flots de billards

Nous présentons ici les résultats des chapitres 7 et 8 qui forment la partie III ; ils contiennent respectivement les articles *Closed billiard trajectories with prescribed bounces* [Chaa] et *Dynamical torsion for contact Anosov flows* [CP22] — ce dernier est écrit en collaboration avec Vesselin Petkov. Soit $r \geqslant 3$ un entier, et $D_1, \dots, D_r \subset \mathbb{R}^d$ une famille d'obstacles lisses et strictement convexes, vérifiant la condition de non-éclipse

$$\mathrm{conv}(D_i \cup D_j) \cap D_k = \emptyset, \quad i \neq k, \quad j \neq k,$$

où conv désigne l'enveloppe convexe. Ces obstacles donnent lieu à un *flot de billard*, qui généralise le flot géodésique, pour lequel les trajectoires se réfléchissent sur le bord des obstacles selon la loi de Fresnel–Descartes. On désignera par $\mathcal{P}$ l'ensemble des trajectoires périodiques primitives du flot du billard. Dans ce cadre, on a encore le théorème des orbites primitives

$$\sharp\{\gamma \in \mathcal{P} \ : \ \tau(\gamma) \leqslant t\} \sim \frac{e^{h_{\mathbf{B}} t}}{h_{\mathbf{B}} t},$$

où $\tau(\gamma)$ est la période de γ et $h_{\mathbf{B}} > 0$ est l'entropie topologique du flot du billard $\mathbf{B} = \{D_1, \dots D_r\}$.

1.3.1 Comptage sous contrainte

Dans le chapitre 7, nous généralisons le théorème 1.1.3 au cadre des flots des billards. Plus précisément, on suppose que $d = 2$ et on se donne un autre obstacle $D_0 \subset \mathbb{R}^2$ de sorte que la famille $D_0, \dots, D_r$ satisfasse toujours la condition de non-éclipse. Pour toute trajectoire périodique $\gamma \in \mathcal{P}$, on note $m_0(\gamma)$ le nombre de réflexions de γ sur D_0.

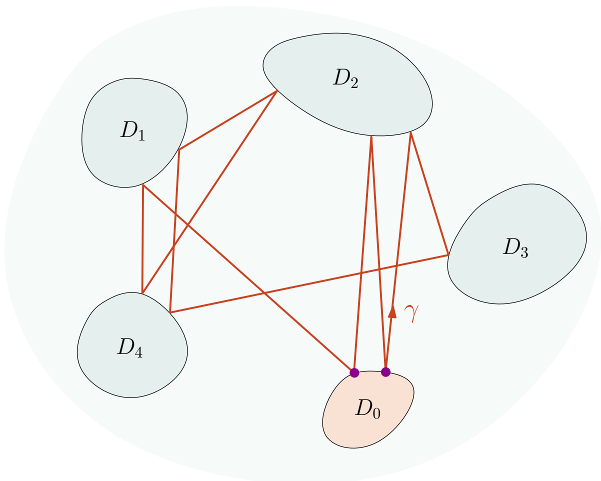

FIGURE 1.3 – Une trajectoire fermée γ du flot de billard avec $m_0(\gamma) = 2$.

Théorème 1.3.1. *Il existe une constante $c > 0$ telle que pour tout entier naturel n on a, quand $t \to \infty$,*

$$\sharp\{\gamma \in \mathcal{P} \ : \ \tau(\gamma) \leqslant t, \ m_0(\gamma) = n\} \ \sim \frac{(ct)^n}{n!} \frac{e^{h_{\mathbf{B}} t}}{h_{\mathbf{B}} t}.$$

Ce résultat sera obtenu grâce à des méthodes similaires à celles utilisées au chapitre 4, notamment en faisant appel à un travail récent de Küster–Schütte–Weich [KSW21] qui permet de voir le flot de billard comme un flot régulier sur une variété lisse, de sorte que la théorie de Dyatlov–Guillarmou [DG16] peut être utilisée pour comprendre la résolvante du flot.

1.3.2 Séries de Dirichlet et résonances du laplacien

Dans le chapitre 8, nous obtenons un prolongement méromorphe pour certaines séries de Dirichlet liées aux résonances du laplacien sur $\mathbb{R}^d \setminus \cup_{j=1}^r D_j$. Plus précisément, pour tout entier naturel strictement positif q, posons

$$\eta_q(s) = \sum_{m(\gamma)\in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|1-P_\gamma|^{1/2}}, \quad \mathrm{Re}(s) \gg 1,$$

où la somme porte sur toutes les orbites périodiques (pas nécessairement primitives), $m(\gamma)$ est le nombre de réflexions de γ sur les obstacles $D_1, \dots, D_r$, P_γ est l'application de Poincaré linéarisée de γ et $|1-P_\gamma| = |\det(1-P_\gamma)|$.

Théorème 1.3.2 (C.–Petkov). *La série η_q admet un prolongement méromorphe à tout le plan complexe, avec pôles simples et résidus dans $\mathbb{Z}/q$.*

Ce théorème est démontré en faisant usage du modèle lisse de [KSW21], en relevant le flot du billard sur un fibré en grassmanniennes, suivant la méthode de Faure–Tsujii [FT17] utilisée pour étudier des flots géodésiques, et en introduisant un fibré de q-réflexion, qui permet de faire abstraction des orbites γ telles que $m(\gamma) \notin q\mathbb{Z}$.

En particulier, le théorème 1.3.2 implique le prolongement méromorphe de la série

$$\eta_\mathrm{D}(s) = \sum_\gamma (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|1-P_\gamma|^{1/2}}, \quad \mathrm{Re}(s) \gg 1,$$

en écrivant $\eta_\mathrm{D}(s) = 2\eta_2(s) - \eta_1(s)$. Cette dernière série est intimement reliée aux résonances $\{\mu_j\} \subset \mathbb{C}$ du laplacien de Dirichlet Δ sur $\mathbb{R}^d \setminus \cup_{j=1}^r D_j$, *via* la formule des traces de Bardos–Guillot–Ralston [BGR82]. Plus précisément, pour $\mu \in \mathbb{C}$ avec $\mathrm{Im}(\mu) < 0$, la résolvante

$$\mathrm{R}_\Delta(\mu) = (-\Delta - \mu^2)^{-1} : L^2(\Omega) \to L^2(\Omega),$$

où $\Omega = \mathbb{R}^d \setminus D$ et $D = \cup_{j=1}^r D_j$, est bien définie. On sait depuis le travail de Lax–Phillips [LP67, LP89] que $\mu \mapsto \mathrm{R}_\Delta(\mu)$ admet un prolongement méromorphe, en tant que famille d'opérateurs

$$L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega),$$

pour $\mu \in \mathbb{C}$ si la dimension d est impaire et pour μ dans un revêtement logarithmique $\{z \in \mathbb{C} \;:\; -\infty < \arg(z) < \infty\}$ sinon ; les résonances quantiques $\{\mu_j\}$ du système sont par définition les pôles de $\mathrm{R}_\Delta(\mu)$.

La distribution de ces résonances — et notamment l'existence d'un gap spectral — est liée à la décroissance de l'énergie locale des solutions de l'équation des ondes. Sous des conditions de pression topologique, un gap a été obtenu par Ikawa [Ika88a], puis par Nonnenmacher–Zworski [NZ09] dans un cadre très général. Plus récemment, en dimension 2, Vacossin [Vac22] a montré que la condition de pression pouvait être omise pour les systèmes d'obstacles.

Lax–Phillips [LP67] ont conjecturé que si $D \subset \mathbb{R}^d$ est un ensemble compact piégeant (dans le sens où il existe une orbite périodique pour le flot de billard dans $\mathbb{R}^d \setminus D$), alors on peut trouver une suite (μ_{j_k}) de résonances avec $\mathrm{Im}(\mu_{j_k}) \to 0$. Ikawa [Ika82] et Gérard [Gér88] ont démontré que cette conjecture était fausse si D est formé de deux obstacles strictement convexes. Cela a conduit Ikawa [Ika88b] à formuler la *conjecture de Lax–Phillips modifiée* (CLPM), comme suit.

Conjecture 1.3.1 (Ikawa)**.** *Si D est piégeant, alors il existe $\delta > 0$ tel que* $\mathrm{Im}(\mu_j) < \delta$ *pour une infinité d'indices j.*

Si la dimension d est paire, il est implicite qu'on ne considère que les résonances μ_j telles que $0 < \mathrm{Im}(\mu_j) \leqslant \delta$ avec $0 < \arg(\mu_j) < \pi$. Ikawa [Ika88b] a montré que cette conjecture est valide dès que la série η_{D} a un pôle — pour les résonances du laplacien avec conditions aux bords de Neumann, la même implication est valide si l'on remplace η_{D} par η_1 ; l'existence d'un pôle est alors automatique, puisque les coefficients de la série η_1 sont strictement positifs. Dans le cas où D est une union finie de boules $D_j = B(x_j, \varepsilon)$ centrées en $x_j \in \mathbb{R}^d$, Ikawa [Ika88b] a montré que η_{D} a un pôle, et donc la CLPM est vérifiée, dès que $\varepsilon > 0$ est assez petit. Plus tard, Stoyanov [Sto09] a étendu ce résultat à des obstacles généraux, mais toujours sous une condition de petitesse.

En utilisant les travaux d'Ikawa [Ika88b, Ika90a] et de Fried [Fri95], nous montrerons la CLPM pour des obstacles analytiques.

Théorème 1.3.3 (C.–Petkov)**.** *La conjecture de Lax–Phillips modifiée est valide pour une union d'obstacles strictement convexes, analytiques réels et satisfaisant la condition de non éclipse.*

Chapitre 2

Introduction (anglais)

In this thesis we study certain dynamical series associated to *hyperbolic* systems. The latter participate in the so-called *chaotic* systems — strongly recurrent and sensitive to initial conditions —, whose most famous ambassador is perhaps the three-body celestial system, studied by Poincaré at the end of the XIXth century [Poi90]. Even though the dynamics are governed by deterministic laws, the trajectories of evolution seem completely unpredictable, even random. Nevertheless, some of them are found to be *periodic* (they reproduce themselves indefinitely) and, in this thesis, it is mainly on these that we will focus our interest. The existence of periodic orbits in a context of chaos may seem counter-intuitive ; however, they are abundant and the knowledge of their periods is often useful to recover essential information on the system, in particular through the use of dynamical series and other zeta functions. Before exposing in details the problems which will be discussed in this manuscript — and to motivate them a little — we first present some results about the spectral theory of hyperbolic flows.

Anosov flows and periodic orbits

In 1898, Hadamard showed that chaos could arise in a very simple geometric context, exhibiting the instability of geodesic lines on surfaces with negative curvature. He further showed that each free homotopy class of curves contains a single closed geodesic ; the distribution of the lengths of these particular curves has since been the subject of many works. For compact hyperbolic surfaces — that are, surfaces of *constant* curvature -1 —, Selberg [Sel56] introduced a zeta function that counts closed geodesics and related their lengths to the eigenvalues of the hyperbolic Laplacian *via* a trace formula. Huber [Hub61] showed later a geometric analogue of the prime number Theorem : the number of closed geodesics whose length is not greater than L is asymptotic to $\exp(L)/L$ when L goes to infinity. Margulis [Mar69] then obtained a similar result for surfaces with negative curvature *variable*.

Geodesic flows with negative curvature are special cases of *hyperbolic* dynamical systems, in the sense of the definition given by Anosov in 1967 in a seminal paper [Ano67].

Definition 2.0.1 (Anosov)**.** Let $\varphi = (\varphi_t)_{t\in\mathbb{R}}$ be a smooth flow acting on a closed manifold M, and $X = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \varphi_t$ be its generator. The flow φ will be said to be hyperbolic, or Anosov, if for any $z \in M$ there exists a decomposition

$$T_z M = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z)$$

depending continuously on z, such that $\mathrm{d}\varphi_t(z)E_\mathrm{b}(z) = E_\mathrm{b}(\varphi_t(z))$ for $b = u, s$, and such that

$$|\mathrm{d}\varphi_t(z)v| \leqslant C\mathrm{e}^{-\nu t}|v|, \qquad t \geqslant 0, \quad v \in E_s(z),$$
$$|\mathrm{d}\varphi_t(z)v| \leqslant C\mathrm{e}^{-\nu |t|}|v|, \qquad t \leqslant 0, \quad v \in E_u(z),$$

for some constants $C, \nu > 0$, where $|\cdot|$ is some norm on TM.

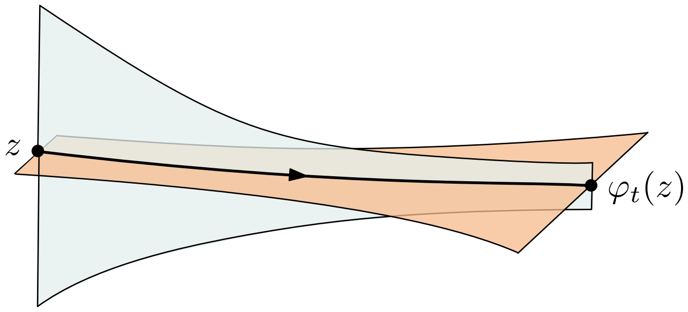

FIGURE 2.1 – An Anosov flow.

The hyperbolicity property means that some directions, called *stable* (the directions of E_s), are contracted by the dynamics, while others, called *unstable* (the directions of E_u), are dilated. In this context, the result of Margulis mentioned above is still valid and reads as follows.

Theorem 2.0.2 (Margulis)**.** *For any topologically mixing Anosov flow $\varphi = (\varphi_t)_{t\in\mathbb{R}}$ there is $h > 0$ such that it holds*

$$N(\varphi, t) \sim \frac{\mathrm{e}^{ht}}{ht} \tag{2.0.1}$$

as t goes to infinity, where $N(\varphi, t)$ is the number of primitive periodic orbits of the flow φ, whose period not greater than t.

Here, the number h denotes the *topological entropy* of the flow, it is a measure of chaos — for geodesic flows of compact hyperbolic surfaces, this entropy is equal to 1 according to Huber's result. Parry and Pollicott [PP83] proved that (2.0.1) also holds for *Axiom A* flows (a class of flows which generalizes the Anosov flows introduced by Smale [Sma67]) after important contributions of Bowen [Bow72].

Zeta functions and Ruelle resonances

Unlike Margulis who uses ergodic theory, Parry and Pollicott prove the primitive orbit Theorem by using a zeta function introduced by Ruelle [Rue76] — a slightly modified version of Selberg's zeta function — which counts periodic orbits. The Ruelle zeta function is a dynamical counterpart of the Riemann zeta function ; it is defined by the formula

$$\zeta_\varphi(s) = \prod_\gamma \left(1 - \mathrm{e}^{-s\tau(\gamma)}\right)^{-1}, \quad \mathrm{Re}(s) > h,$$

where the product runs over *primitive* periodic orbits γ of the flow φ, and $\tau(\gamma)$ is the period of γ. Relying in particular on the symbolic coding of hyperbolic flows developed by Bowen [Bow73], Parry and Pollicott proved in [PP83] that ζ_φ extends analytically to an open neighborhood of the half-plane $\{\mathrm{Re}(s) \geqslant h\}$, except at $s = h$, where there is a simple pole. Then they are able to obtain (2.0.1) by reproducing the proof of Wiener–Ikehara of the prime number Theorem [Wie88] which relies on a Tauberian argument : the distribution of periods $\tau(\gamma)$ can be understood through the analytic singularities of the function ζ_φ.

Smale [Sma67] wondered if it was possible, for Axiom A flows, to obtain a meromorphic extension to the whole complex plane for the function ζ_φ, saying "I must admit that a positive answer would be a little shocking !". This question was much discussed and took almost fifty years to be solved. Ruelle [Rue76] obtained such an extension, under the condition that the flow is analytic, as well as its stable and unstable distributions. Later, Rugh [Rug96] showed that, for three-dimensional Anosov flows, the analyticity assumption on the stable and unstable distributions (but not on the flow !) could be omitted, which was generalized in any dimension by Fried [Fri95]. For smooth Anosov flows (of class C^∞), Pollicott obtained an extension of ζ_φ in a half-plane $\{\mathrm{Re}(s) > h - \varepsilon\}$ for some $\varepsilon > 0$ depending on φ and this was extended to Axiom A flows by Parry–Pollicott [PP90].

These results are typically obtained by encoding the dynamics with the help of Markov partitions and expressing the function ζ_φ as an alternating product of Fredholm determinants of some operators acting on the space of Hölder functions on a sub-shift of finite type, relating the zeros and poles of ζ_φ to the spectrum of the aforementioned operators. However, this method does not take into account the regularity of the flow and the work of [Kit99] suggests that the regularity of the dynamics is closely related to the depth of the half-plane on which an analytic extension can be obtained.

Blank, Keller and Liverani [BKL02] introduced in the early 2000s some functional spaces tailored for a hyperbolic *diffeomorphism* f (the discrete version of Anosov flows), on which the Koopman operator $u \mapsto u \circ f$ is quasi-compact. The key is to consider certain distributions with *anisotropic* regularity, requiring a high level of regularity in stable directions and a low level in unstable directions. These results were then refined by Baladi [Bal05], Gouëzel–Liverani [GL06] and Baladi–Tsujii [BT07]. Later, Faure–Roy–Sjöstrand [FRS08] proposed a semi-classical approach. Liverani [Liv04] (for contact flows) and Butterley–Liverani [BL07] adapted this work to the continuous setting, constructing functional spaces on which the generator $X : u \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} u \circ \varphi_t$ of an Anosov flow (φ_t) has a quasi-compact resolvent. As in the discrete case, Faure–Sjöstrand [FS11] then proposed a micro-local version of these spaces.

Let us briefly specify these results. Let φ be an Anosov flow on a manifold M, and X be its generator. If s is a complex number, the resolvent $\mathrm{R}_\varphi(s)$ of φ is defined by the integral

$$\mathrm{R}_\varphi(s) = \int_0^\infty \mathrm{e}^{-ts} \varphi_{-t}^* \mathrm{d}t,$$

where φ_{-t}^* is the pull-back by φ_{-t}, acting on the space $\Omega^\bullet(M)$ of differential forms. As soon as the real part of s is large enough, this integral is convergent and gives rise to an operator $\mathrm{R}_\varphi(s) : \Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$, where $\mathcal{D}'^\bullet(M)$ denotes the space of *currents* — the topological dual of $\Omega^\bullet(M)$. The "resolvent" terminology is justified by the identities

$$(\mathcal{L}_X + s)\mathrm{R}_\varphi(s) = \mathrm{R}_\varphi(s)(\mathcal{L}_X + s) = \mathrm{Id}_{\Omega^\bullet(M)},$$

where $\mathcal{L}_X$ is the Lie derivative in the X direction.

Theorem 2.0.3 (Butterley–Liverani, Faure–Sjöstrand)**.** *The resolvent* $\mathrm{R}_\varphi(s)$*, which is well defined on a half-plane* $\{\mathrm{Re}(s) > C\}$*, admits a meromorphic extension in the variable* s*, to the whole complex plane, as a family of operators* $\Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$*, whose residues are finite-rank projectors. Its poles are the* Ruelle resonances *of* φ*.*

A spectrum of Ruelle resonances was later obtained by Dyatlov–Guillarmou [DG16] for *open* hyperbolic systems (i.e. systems with trajectories that can escape to infinity) and more recently by Meddane for Axiom A flows [Med21], after contributions of Dang–Rivière on Morse-Smale flows [DR20b, DR20c].

With these modern techniques, first Giulietti–Liverani–Pollicott [GLP13], then Dyatlov–Zworski [DZ16] with a semiclassical approach, were able to obtain the analytic extension of ζ_φ to the whole complex plane, thus obtaining the

Theorem 2.0.4 (Giulietti–Liverani–Pollicott, Dyatlov–Zworski)**.** *If* φ *is an Anosov flow, the function* ζ_φ *admits a meromorphic extension to the whole complex plane ; its poles and zeros are included in the Ruelle resonances.*

Dyatlov–Guillarmou extended later this Theorem for any Axiom A flow [DG18] thanks to their work on open systems, thus answering positively Smale's question. The strategy consists in linking $\zeta_\varphi(s)$ and the resolvent $\mathrm{R}_\varphi(s)$: we can show thanks to the Guillemin trace formula [Gui77] that if $\mathrm{Re}(s)$ is large enough then

$$\frac{\zeta_\varphi'(s)}{\zeta_\varphi(s)} = \mathrm{e}^{\varepsilon s}\mathrm{tr}^\flat_{\mathrm{gr}}\left(\varphi_{-\varepsilon}^*\mathrm{R}_\varphi(s)\right), \tag{2.0.2}$$

where $\varepsilon > 0$ is a small number and $\mathrm{tr}^\flat_{\mathrm{gr}}$ denotes the *graduated flat trace* — an extension of the L^2 graduated trace which is well defined for operators satisfying certain *wavefront set* conditions ; we refer to Appendix B.3 for a precise definition. Using semi-classical methods (singularity propagation and radial estimates), Dyatlov and Zworski gave a precise description of the wavefront of the Schwartz kernel of the resolvent and deduced that the flat trace of $\varphi_{-\varepsilon}^*\mathrm{R}_\varphi(s)$ is well defined ; Theorem 2.0.4 is then a consequence of equality (2.0.2) and Theorem 2.0.3.

We finally mention a result obtained by Dyatlov and Zworski [DZ17] on the order of the singularity of $\zeta_\varphi(s)$ at $s = 0$ for geodesic flows of surfaces.

Theorem 2.0.5 (Dyatlov–Zworski)**.** *If* φ *is the geodesic flow of a surface with negative curvature* Σ*, then* $\zeta_\varphi(s)$ *has a pole of order* $|\chi(\Sigma)|$ *at* $s = 0$*, where* $\chi(\Sigma)$ *is the Euler characteristic of* Σ*.*

This Theorem, known for hyperbolic surfaces since Fried's work [Fri86b], tells us that the behavior of the function ζ_φ near the origin is related to the underlying topology. We will see in the following that this phenomenon does not concern only geodesic flows and that some topological invariants can be recovered with the help of dynamical zeta functions.

Plan of this thesis

In this dissertation, we propose some contributions on certain questions related to those mentioned above. Theorems 2.0.2, 2.0.4 and 2.0.5 are prototypical models of the various results we will present : counting periodic orbits, analytic extension of zeta functions or dynamical series and weaving a link with the underlying topology. Our results will be obtained by using systematically the spectral theory of hyperbolic flows and in particular Theorem 2.0.3, as well as its counterpart for open systems. We have chosen to divide the thesis in three parts (independently of the previous trichotomy), as follows.

In the first part, consisting of Chapters 3 and 4, we address a counting problem with constraints. After illustrating the problem on a toy model in chapter 3, we show in chapter 4 an asymptotic result in the spirit of (1.0.1) for closed geodesics of a negatively curved surface whose intersection numbers with a family of simple curves are prescribed.

The second part, consisting of Chapters 5 and 6, is more focused on topology. In Chapter 5, we compute the value at the origin of some Poincaré series counting geodesic arcs of a surface with boundary. Then, in a rather different framework, we construct in chapter 6 a topological invariant — called *dynamical torsion* — defined with the help of a Ruelle zeta function *twisted* by a representation of the fundamental group ; we finally connect the dynamical torsion to another topological invariant, the *Turaev torsion*.

The last part is devoted to billiard flows associated with a finite family of convex obstacles in the Euclidean space and contains Chapters 7 and 8. First, in Chapter 7, we extend a counting result obtained in the first part to the setting of billiard flows. Then, in Chapter 8, we show that some dynamical Dirichlet series related to the quantum resonances of the system admit a meromorphic extension to the whole complex plane.

These results are detailed in the following paragraphs.

2.1 Counting closed geodesics under intersection constraints

In this paragraph, we detail the results obtained in Chapter 4, which contains the article *Closed geodesics with prescribed intersection numbers* [Chab].

Let (Σ, g) be a closed, oriented, Riemannian surface with negative curvature. Let $\mathcal{P}$ be the set of its primitive closed geodesics, i.e. the set of closed geodesics which are not multiple of a shorter geodesic. For all $L > 0$, we denote by

$$N(L) = \sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L\} \tag{2.1.1}$$

the number of these geodesics that are of length less not greater than L. Recall Margulis' result : when L tends to infinity, we have the asymptotics

$$N(L) \sim \frac{e^{hL}}{hL}$$

where $h > 0$ is the topological entropy of the geodesic flow. Other similar counting results exist for non-compact Riemann surfaces, cf. Sarnak [Sar80], Guillopé [Gui86], or Lalley [Lal89] ; we refer to Paulin–Pollicott–Schapira [PPS12] for precise references on counting results in more general contexts.

In what follows, we will ask the

Question 2.1.1. Can we count primitive closed geodesics subject to certain constraints (topological or geometric) ?

2.1.1 Homological constraints

A first constraint that one may want to impose to the geodesics is of homological nature. Lalley [Lal88] and Pollicott [Pol91] independently obtained the following result.

Theorem 2.1.2 (Lalley, Pollicott)**.** *There exists a constant $c > 0$ such that for any homology class $\xi \in H_1(\Sigma, \mathbb{Z})$, we have*

$$\sharp(\gamma) \in \mathcal{P} \;:\; \ell(\gamma) \leqslant L,\; [\gamma] = \xi\} \sim c\frac{\mathrm{e}^{hL}}{L^{\mathrm{g}+1}}, \tag{2.1.2}$$

when $L \to \infty$, where g is the genus of the surface.

Similar results had already been obtained for hyperbolic surfaces (surfaces with curvature *constant*, equal to -1) by Phillips–Sarnak [PS87] and Katsuda–Sunada [KS88]. Without stating them, we mention that much more precise results — for example valid for a more general class of hyperbolic flows, with asymptotic developments including more terms or authorizing the homology class ξ to depend on L — were obtained later by Sharp [Sha93], Babillot–Ledrappier [BL98], Anantharaman [Ana00], and Pollicott–Sharp [PS01].

These results are typically obtained by using a Tauberian argument, with the twisted zeta functions

$$L_\varphi(\chi, s) = \prod_\gamma \left(1 - \chi([\gamma])\mathrm{e}^{-s\ell(\gamma)}\right)^{-1},$$

where the product is on primitive closed geodesics, $\chi : H_1(\Sigma, \mathbb{Z}) \to \mathbb{C}^\times$ is a unitary character and $[\gamma]$ is the homology class generated by γ ; those functions are then studied with the help of the spectral theory of certain Ruelle operators.

2.1.2 Self-intersection numbers

A second natural constraint concerns the *self-intersection numbers*. If $\gamma : \mathbb{R}/\ell(\gamma)\mathbb{Z} \to \Sigma$ is a closed geodesic parameterized by arc length, we define its *self-intersection number* by

$$\mathrm{i}(\gamma, \gamma) = \frac{1}{2}\sharp\left\{(\tau, \tau') \in (\mathbb{R}/\ell(\gamma)\mathbb{Z})^2 \;:\; \gamma(\tau) = \gamma(\tau')\right\}.$$

A closed geodesic will be said to be *simple* if its self-intersection number is zero. Mirzakhani [Mir08, Mir16] studied the asymptotic growth of closed geodesics with a prescribed number of self-intersections.

Theorem 2.1.3 (Mirzakhani)**.** *Suppose that (Σ, g) is hyperbolic. Then for any natural number k, there exists $c_k > 0$ such that, when $L \to \infty$,*

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L,\ \mathrm{i}(\gamma, \gamma) = k\} \sim c_k L^{6(\mathrm{g}-1)}. \tag{2.1.3}$$

Mirzakhani's paper [Mir08] deals with simple geodesics, and for $k = 1$ the previous theorem was first proved by Rivin [Riv12] ; we also mention the work of Erlandsson–Souto [ES16, ES19] who obtain similar results with another approach. In a slightly spirit, Sapir [Sap16] and Aougab–Souto [AS18] studied the asymptotic growth of the number of *types* of curves on hyperbolic surfaces (while prescribing the self-intersection numbers amounts to counting geodesics belonging in fixed types).

To obtain the Theorem 2.1.3, Mirzakhani uses the ergodicity of the action of the mapping class group of the surface on the space of measured lamination, the exponent $6\mathrm{g} - 6$ being the dimension of this space. Note that the growth of closed geodesics with prescribed self-intersection numbers is polynomial and not exponential : there are very few of them. In fact, a result of Lalley [Lal11] (valid also for surfaces with variable negative curvature) states that a typical closed geodesic has a number of self-intersections proportional to the square of its length. More precisely, he shows that there exists a constant $I > 0$ such that for any $\varepsilon > 0$, we have

$$\lim_{L\to\infty} \frac{1}{N(L)} \sharp \left\{ \gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L,\ \left| \frac{\mathrm{i}(\gamma,\gamma)}{\ell(\gamma)^2} - I \right| \leqslant \varepsilon \right\} = 1. \tag{2.1.4}$$

The convergence is in fact exponential, as can be seen by using a principle of large deviations of Kifer [Kif94] (see Anantharaman [Ana99]).

2.1.3 Geometric intersection constraints

Let us return to the case of surfaces with variable negative curvature. In §2.1.1, we constrained the homology class of closed geodesics, which amounts to prescribe certain *algebraic* intersection numbers with a family of simple curves. What happens if we constrain *geometric* intersection numbers instead ? Let us fix $\gamma_\star$ a simple closed geodesic. For any $\gamma \in \mathcal{P}$, we denote by

$$\mathrm{i}(\gamma, \gamma_\star) = \inf_{\eta\sim\gamma, \eta_\star\sim\gamma_\star} |\eta \cap \eta_\star|$$

the geometric intersection number between γ and $\gamma_\star$, where the infimum runs over curves $\eta, \eta_\star : \mathbb{R}/\mathbb{Z} \to \Sigma$ freely homotopic to γ and $\gamma_\star$, respectively, and

$$|\eta \cap \eta_\star| = \sharp\{(\tau, \tau_\star) \in (\mathbb{R}/\mathbb{Z})^2 \ : \ \eta(\tau) = \eta_\star(\tau_\star)\}.$$

If n is a natural number, we wish to study the asymptotic growth of the quantity

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L,\ \mathrm{i}(\gamma, \gamma_\star) = n\}$$

when $L \to \infty$.

We will first assume that the simple curve $\gamma_\star$ is non-separating, in the sense that $\Sigma \backslash \gamma_\star$ is connected (this condition will be relaxed later). Then we will show in Chapter 4 the following

Theorem 2.1.4. *Suppose that $\gamma_\star$ is not separating. Then there exist constants $c_\star > 0$ and $h_\star \in]0, h[$ such that for any positive integer n, we have the asymptotics*

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \mathrm{i}(\gamma, \gamma_\star) = n\} \sim \frac{(c_\star L)^n}{n!} \frac{\mathrm{e}^{h_\star L}}{h_\star L}, \quad L \to \infty. \tag{2.1.5}$$

The number $h_\star$ is the topological entropy of the geodesic flow of the surface with boundary $\Sigma_\star$ obtained by cutting Σ along $\gamma_\star$ (see the paragraph 2.1.4 below for a precise definition). The case $n = 0$ amounts to counting the closed geodesics of $\Sigma_\star$ and was already known thanks to the work of Dal'bo [Dal99], who showed that the geodesic flow of co-compact convex surfaces was topologically mixing, thus allowing the use of Parry–Pollicott's result [PP83]. However, our result was not known for $n > 0$, even for hyperbolic surfaces.

Note that the asymptotic growth (2.1.5) remains exponential, although weaker than that of Margulis' formula (2.1.1). In particular, this growth is somehow between those obtained by Mirzakhani on the one hand, and Lalley and Pollicott on the other. As said above, prescribing the number of self-intersections is very restrictive, since for a typical closed geodesic γ, we have $\mathrm{i}(\gamma, \gamma) \sim I\ell(\gamma)^2$. Here it is rather the number $\mathrm{i}(\gamma, \gamma_\star)$ that we constrain ; using Kifer's principle of large deviations, and Bonahon's intersection form [Bon86], we will in fact show that typically, the number $\mathrm{i}(\gamma, \gamma_\star)$ is proportional to $\ell(\gamma)$ (see Proposition 4.8.1 for a precise statement in the spirit of (2.1.4)).

If the curve $\gamma_\star$ is separating, we have the following result.

Theorem 2.1.5. *If $\gamma_\star$ separates Σ into two surfaces Σ_1 and Σ_2, we denote by $h_j \in]0, h[$ the entropy of the open system $(\Sigma_j, g|_{\Sigma_j})$ for $j = 1, 2$ (cf. the next paragraph), and we define $h_\star = \max(h_1, h_2)$. Then there exists $c_\star > 0$ such that for all $n \in \mathbb{N}$ we have the asymptotics, when $L \to +\infty$,*

$$N(2n, L) \sim \begin{cases} \dfrac{(c_\star L)^n}{n!} \dfrac{\mathrm{e}^{h_\star L}}{h_\star L} & \text{if } h_1 \neq h_2, \\ 2\dfrac{(c_\star L^2)^n}{(2n)!} \dfrac{\mathrm{e}^{h_\star L}}{h_\star L} & \text{if } \ h_1 = h_2. \end{cases} \tag{2.1.6}$$

The proofs of Theorems 2.1.4 and 2.1.5 make use of a *dynamical scattering operator* $\mathcal{S}(s)$, acting on the boundary of the unit tangent bundle of $\Sigma_\star$, and studied with the help of the spectral theory of open systems by Dyatlov–Guillarmou [DG16] — we refer to the introduction of Chapter 4 for a detailed presentation of the strategy of proof. In fact, our techniques allow to obtain asymptotic results concerning closed geodesic of which *several* intersection numbers (with a family of simple curves) are prescribed, as we will see below.

2.1.4 Intersection numbers with several curves

Let $r \geqslant 1$ be an integer, and take a family $\gamma_{\star,1}, \ldots, \gamma_{\star,r}$ of pairwise disjoint simple closed geodesics. For any r-uplet $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ of natural numbers, we wish to understand the asymptotic behavior of the quantity

$$N(\mathbf{n}, L) = \sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \mathrm{i}(\gamma, \gamma_{\star,j}) = n_j, \ j = 1, \ldots, r\}$$

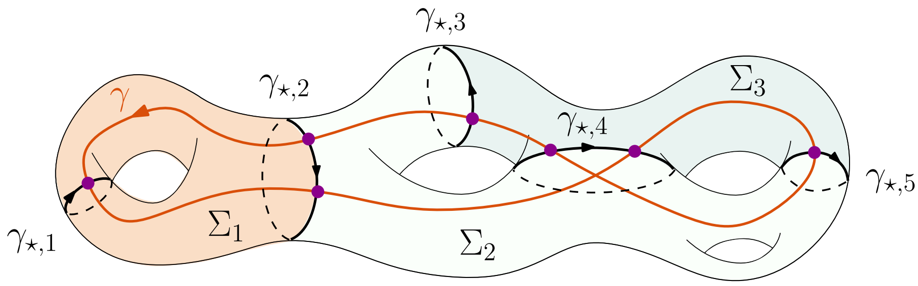

FIGURE 2.2 – A closed geodesic γ on Σ. Here we have $r = 5$, $q = 3$, and $\omega(\gamma) \sim (u, v)$ where $u = (1, 2, 4, 5, 4, 3, 2)$ and $v = (1, 1, 2, 3, 2, 3, 2)$ (the starting point of γ is the red arrow).

when $L \to +\infty$, where $\mathrm{i}(\gamma, \gamma_{\star_j})$ is the number of geometric intersection between γ and $\gamma_{\star,j}$.

Theorem 2.1.6. *Let $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. If $N(\mathbf{n}, L) > 0$ for an $L > 0$, then there are constants $C_{\mathbf{n}} > 0, d_{\mathbf{n}} \in \mathbb{N} \setminus \{0\}$ and $h_{\mathbf{n}} \in]0, h[$ such that*

$$N(\mathbf{n}, L) \sim C_{\mathbf{n}} L^{d_{\mathbf{n}}-1} \mathrm{e}^{h_{\mathbf{n}} L}, \quad L \to +\infty.$$

In fact, a similar result is valid if we additionally impose the order in which we want the intersections to occur, as follows. Let $\Sigma_1, \ldots, \Sigma_q$ be the connected components of the surface $\Sigma_\star = \Sigma \setminus (\gamma_{\star,1} \cup \cdots \cup \gamma_{\star,r})$ obtained by cutting Σ along the curves $\gamma_{\star,1}, \ldots, \gamma_{\star,r}$ (see Figure 2.2). For any $\gamma \in \mathcal{P}$ which intersects at least one of the curves $\gamma_{\star,j}$ we denote by $\omega(\gamma)$ the pair (u, v) of sequences

$$u = (u_1, \ldots, u_N) \quad \text{and} \quad v = (v_1, \ldots, v_N)$$

with $N \geqslant 1$, cyclically ordered, such that γ travels in $\Sigma_{v_1}, \ldots, \Sigma_{v_N}$ (in this order !) and passes from Σ_{v_k} to $\Sigma_{v_{k+1}}$ by crossing $\gamma_{\star,u_k}$, where $v_{N+1} = v_1$ (see Figure 2.2) ; these sequences are well defined modulo application of a cyclic permutation. Such a pair ω of finite sequences will be called an *admissible path* if $\omega \sim \omega(\gamma)$ for at least one closed geodesic $\gamma \in \mathcal{P}$, where $\omega \sim \omega(\gamma)$ means that $\omega(\gamma)$ is a cyclic permutation of ω (the permutation being the same for both components of ω).

Let $S\Sigma$ be the unit tangent bundle of (Σ, g), and $(\varphi_t)_{t\in\mathbb{R}}$ the associated geodesic flow, acting on $S\Sigma$. Let $\pi : S\Sigma \to \Sigma$ be the natural projection. We denote by $h_j > 0$ ($j = 1, \ldots, q$) the entropy of the open system $(\Sigma_j, g|_{\Sigma_j})$, i.e. the topological entropy of the flow φ restricted to the trapped set

$$K_j = \overline{(x, v) \in S\Sigma \ : \ \pi(\varphi_t(x, w)) \in \Sigma_j, \ t \in \mathbb{R}\}},$$

where the closure is taken in Σ.

For any admissible path $\omega = (u, v)$ of size N, we define

$$h_\omega = \max\{h_{v_k} \ : \ k = 1, \ldots, N\}, \quad d_\omega = \sharp\{k = 1, \ldots, N \ : \ h_{v_k} = h_\omega\}.$$

The number h_ω is the maximum of the entropies of the surfaces encountered by any $\gamma \in \mathcal{P}$ geodesic satisfying $\omega(\gamma) \sim \omega$ while d_ω is the number of times such a geodesic encounters a surface whose entropy equals h_ω (for example, in Figure 2.2, if the entropy h_2 of Σ_2 is greatest, we have $h(\omega) = h_2$ and $d(\omega) = 3$, since γ passes three times through Σ_2).

In fact, the numbers h_ω and d_ω only depend on $\mathbf{n}(\omega) = (n_1, \ldots, n_r)$ where $n_j = \sharp\{k = 1, \ldots, N \ : \ u_k = j\}$ (see §4.9) ; thus we will denote then by $h_{\mathbf{n}(\omega)}$ and $d_{\mathbf{n}(\omega)}$ respectively.

Theorem 2.1.7. *Let ω be an admissible path. Then, there is $c(\omega) > 0$ such that*

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \omega(\gamma) \sim \omega\} \sim c(\omega) L^{d_{\mathbf{n}(\omega)}-1} \mathrm{e}^{h_{\mathbf{n}(\omega)} L}, \quad L \to +\infty.$$

Note that Theorem 2.1.6 may be deduced from Theorem 2.1.7 by summing over admissible paths ω such that $\mathbf{n}(\omega) = \mathbf{n}$, where $\mathbf{n} \in \mathbb{N}^r$ is fixed. However, Theorem 2.1.4 is not an immediate consequence of Theorem 2.1.7 ; it will be a consequence of a more precise result proved in §4.9, which allows to compute the numbers $c(\omega^k), d_{\mathbf{n}(\omega^k)}$ et $h_{\mathbf{n}(\omega^k)}$ in terms of $c(\omega), d_{\mathbf{n}(\omega)}$ et $h_{\mathbf{n}(\omega)}$, where ω^k is the path obtained by concatenating k times ω.

2.2 Dynamical series and topology

We relate here the results obtained in Part II. The latter consists of Chapter 5, which contains the article *Poincaré series for surfaces with boundary* [Cha21], and of Chapter 6, which transcribes the article *Dynamical torsion for contact Anosov flows* [CD19] written in collaboration with Nguyen Viet Dang.

2.2.1 Poincaré series for surfaces with boundary

Let (Σ, g) be a connected, oriented, negatively curved Riemannian surface, with totally geodesic boundary $\partial\Sigma$. Let $\mathcal{G}^\perp$ be the set of *orthogeodesics* of Σ, that is the set of geodesic arcs $\gamma : [0, \ell] \to \Sigma$ (parameterized by arc length) such that $\gamma(0), \gamma(\ell) \in \partial\Sigma$, $\gamma'(0) \perp T_{\gamma(0)}\partial\Sigma$ and $\gamma'(\ell) \perp T_{\gamma(\ell)}\partial\Sigma$. For $\mathrm{Re}(s)$ large, the Poincaré series

$$\eta(s) = \sum_{\gamma \in \mathcal{G}^\perp} \mathrm{e}^{-s\ell(\gamma)}, \tag{2.2.1}$$

where $\ell(\gamma)$ is the length of γ, converges (see §5.3.2). We will prove the following

Theorem 2.2.1. *The Poincaré series $s \mapsto \eta(s)$ admits a meromorphic continuation to the whole complex plane, and vanishes at the origin.*

If $x \neq y \in \Sigma$, we may also consider the Poincaré series associated to the geodesic arcs joining x to y. Namely, we set for $\mathrm{Re}(s)$ large enough

$$\eta_{x,y}(s) = \sum_{\gamma : x \rightsquigarrow y} \mathrm{e}^{-s\ell(\gamma)},$$

where the sum runs over all geodesic arcs $\gamma : [0, \ell] \to \Sigma$ (parameterized by arc length) such that $\gamma(0) = x$ and $\gamma(\ell) = y$ and $\ell(\gamma) = \ell$ is the length of γ. Then we have the following result.

Theorem 2.2.2. *The Poincaré series $s \mapsto \eta_{x,y}(s)$ extends meromorphically to the whole complex plane and*

$$\eta_{x,y}(0) = \frac{1}{\chi(\Sigma)},$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

The numbers $\eta(0)$ and $\eta_{x,y}(0)$ may be interpreted as some linking numbers of certain Legendrian knots in $S\Sigma$; for the series η, this linking number vanishes.

To the best of our knowledge, Theorem 2.2.1 is the first result on a series involving the orthospectrum (that is, the set of lengths of orthogeodesics) of a surface with totally geodesic boundary which has *variable* negative curvature. For hyperbolic surfaces (i.e. surfaces with *constant* curvature -1) with totally geodesic boundary, the orthospectrum has been studied by many authors, among others Basmajian [Bas93], Bridgeman [Bri11], Calegari [Cal10] (see also Bridgeman–Kahn [BK10]). In particular they show that if (Σ, g) is a compact hyperbolic surface with totally geodesic boundary, one has

$$\ell(\partial\Sigma) = \sum_{\gamma \in \mathcal{G}^\perp} 2 \log \coth(\ell(\gamma)/2), \quad \mathrm{vol}(\Sigma) = \frac{2}{\pi} \sum_{\gamma \in \mathcal{G}^\perp} \mathcal{R}\left(\mathrm{sech}^2(\ell(\gamma)/2)\right),$$

where $\ell(\partial\Sigma)$ is the length of the boundary of Σ, $\mathrm{vol}(\Sigma)$ is the area of Σ and $\mathcal{R}$ is the Rogers dilogarithm function. We refer to [BT16] for a detailed exposition of those results.

In order to study the Poincaré series $\eta(s)$ and $\eta_{x,y}(s)$, we will adopt the elegant approach of Dang and Rivière [DR20a], which consists in interpreting both series as distributional pairings involving the resolvent of the geodesic flow. On a *closed* surface with negative curvature, Dang and Rivière proved that Poincaré series associated to orthogeodesic arcs joining any two homologically trivial closed geodesics, as well as Poincaré series associated to geodesic arcs linking two points, admit a meromorphic extension to the whole complex plane ; moreover they computed their values at zero — for the series associated to geodesic arcs linking two points, they found (as here) that this value coincides with the inverse of the Euler characteristic of the surface. The work of Dang–Rivière extends a previous result of Paternain [Pat00] which says that if (Σ, g) is a closed hyperbolic surface, then

$$\int_\Sigma \eta_{x,y}(s) \mathrm{dvol}_g(x) \mathrm{dvol}_g(y) = \frac{4\pi\chi(\Sigma)}{1 - s^2},$$

where vol_g is the Riemannian measure on Σ ; we refer to [DR20a] for precise references about Poincaré series counting geodesic arcs.

The main novelty of our work is that we deal with the open case, which leads us to use the theory of Pollicott–Ruelle resonances for open systems developed by Dyatlov and Guillarmou [DG16], as well as a result of Hadfield [Had18] about the topology of resonant states for surfaces with boundary.

2.2.2 Dynamical torsion for contact Anosov flows

Let M be a closed odd dimensional manifold and (E, ∇) be a flat vector bundle over M. The parallel transport of the connection ∇ induces a conjugacy class of representation $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{GL}(\mathbb{C}^d))$. Moreover, ∇ defines a differential on the complex $\Omega^\bullet(M, E)$ of E-valued differential forms on M and thus cohomology groups $H^\bullet(M, \nabla) = H^\bullet(M, \rho)$ (note that we use the notation ∇ also for the twisted differential induced by ∇ whereas it can be denoted by d^∇ in other references). We will say that ∇ (or ρ) is acyclic if those cohomology groups are trivial. If ρ is unitary (or equivalently, if there exists a hermitian structure on E preserved by ∇) and acyclic, Reidemeister [Rei35] introduced a combinatorial invariant $\tau_{\mathrm{R}}(\rho)$ of the pair (M, ρ), the so-called *Franz-Reidemeister torsion* (or R-torsion), which is a positive number. This allowed him to classify lens spaces in dimension 3 ; this result was then extended in higher dimension by Franz [Fra35] and De Rham [dR36].

On the analytic side, Ray-Singer [RS71] introduced another invariant $\tau_{\mathrm{RS}}(\rho)$, the *analytic torsion*, defined via the derivative at 0 of the spectral zeta function of the Laplacian given by the Hermitian metric on E and some Riemannian metric on M. They conjectured the equality of the analytic and Reidemeister torsions. This conjecture was proved independently by Cheeger [Che79] and Müller [Mül78], assuming only that ρ is unitary (both R-torsion and analytic torsion have a natural extension if ρ is unitary and not acyclic). The Cheeger-Müller theorem was extended to unimodular flat vector bundles by Müller [Mul93] and to arbitrary flat vector bundles by Bismut-Zhang [BZ92].

In the context of hyperbolic dynamical systems, Fried [Fri87] was interested in the link between the R-torsion and the Ruelle zeta function of an Anosov flow X twisted by ρ, which is defined by

$$\zeta_{X,\rho}(s) = \prod_{\gamma \in \mathcal{G}_X^\#} \det\Big(1 - \varepsilon_\gamma \rho([\gamma]) \mathrm{e}^{-s\ell(\gamma)}\Big), \quad \mathrm{Re}(s) \gg 0,$$

where $\mathcal{G}_X^\#$ is the set of primitive closed orbits of X, $\ell(\gamma)$ is the period of γ and $\varepsilon_\gamma = 1$ if the stable bundle of γ is orientable and $\varepsilon_\gamma = -1$ otherwise. Theorem 2.0.4 naturally extends in this framework, and $\zeta_{X,\rho}$ admits a meromorphic continuation to the whole complex plane. Using Selberg's trace formula Fried [Fri86a] could relate, in the spirit of Theorem 2.0.5, the behavior of $\zeta_{X,\rho}(s)$ near $s = 0$ with τ_{R}, as follows.

Theorem 2.2.3 (Fried). *Let $M = SZ$ be the unit tangent bundle of some closed oriented hyperbolic manifold Z, and denote by X its geodesic vector field on M. Assume that $\rho : \pi_1(M) \to O(d)$ is an acyclic and unitary representation. Then $\zeta_{X,\rho}$ extends meromorphically to $\mathbb{C}$. Moreover, it is holomorphic near $s = 0$ and*

$$|\zeta_{X,\rho}(0)|^{(-1)^r} = \tau_R(\rho), \tag{2.2.2}$$

where $2r + 1 = \dim M$, and $\tau_R(\rho)$ is the Reidemeister torsion of (M, ρ).

In his article [Fri95], Fried proposed the

Conjecture 2.2.1 (Fried). *Equality (1.2.2) is true for any geodesic flow of a negatively curved compact manifold.*

Fried had already conjectured that the same holds true for geodesic flows of negatively curved locally symmetric spaces in [Fri87] ; this was proved by Moscovici-Stanton [MS91] and Shen [She17]. For *analytic* Anosov flows, Sanchez-Morgado [SM93, SM96] proved in dimension 3 that (2.2.2) holds true if ρ is acyclic, unitary, and satisfies that $\rho([\gamma]) - \varepsilon_\gamma^j$ is invertible for $j \in \{0,1\}$ for some closed orbit γ. However the proof of Sanchez-Morgado relies on the existence of an analytic Markov partition and does not extend, *a priori*, to C^∞ flows.

Dang–Guillarmou–Rivière–Shen [DGRS20] overcame this problem thanks to the help of the modern spectral theory for hyperbolic systems mentioned above (see also [DR19b] for Morse–Smale flows). Indeed, Theorem 2.0.3 is still valid in this context, and allows to define a spectrum of Ruelle resonances for the twisted Lie derivative

$$\mathcal{L}_X^\nabla = \nabla \iota_X + \iota_X \nabla,$$

where ι_X is the interior product with X acting on $\Omega^\bullet(M, E)$; this spectrum is denoted by $\mathrm{Res}(\mathcal{L}_X^\nabla)$.

Theorem 2.2.4 (Dang–Rivière–Guillarmou–Shen)**.** *Let ρ be an acyclic representation of $\pi_1(M)$. Then the map*

$$X \mapsto \zeta_{X,\rho}(0)$$

is locally constant on the open set of smooth vector fields which are Anosov and for which 0 *is not a Ruelle resonance, that is,* $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$*. If X preserves a smooth volume form and* $\dim(M) = 3$*, equation (2.2.2) holds true if* $b_1(M) \neq 0$ *or under the same assumption used in [SM96].*

Though the above theorem is the first result dealing with Fried's conjecture for general Anosov flows, there are two restrictions. The first one is that $|\zeta_{X,\rho}(0)|^{(-1)^r} = \tau_{\mathrm{R}}(\rho)$ is an equality of positive real numbers and the representation ρ is unitary. For arbitrary acyclic representations $\rho : \pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$, one could wonder if the phase of the complex number $\zeta_{X,\rho}(0)$ contains topological information. For instance, if it can be compared with some complex valued torsion defined for general acyclic representations $\rho : \pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$. The second restriction concerns the assumption that 0 is not a Ruelle resonance. Apart from the low dimension cases studied in [DGRS20], this assumption is particularly hard to control and is difficult to check for explicit examples. Moreover, in the non-acyclic case, the recent works of Cekic–Paternain [CP21] and Cekic–Dyatlov–Küster–Paternain [CDDP22] show that the dimension of the spaces of resonant states for $\mathcal{L}_X^\nabla$ for the resonance $s = 0$ — which are intimately linked with the singularity of $\zeta_{X,\rho}$ at the origin — may be unstable under perturbations of X. In particular, nothing guarantees *a priori* that the number $\zeta_{X,\rho}(0)$ is well defined.

In order to partially overcome these two obstacles in the framework of *contact* Anosov flows, we introduced, in a work in collaboration with Nguyen Viet Dang, a new object — the *dynamical torsion* — which is defined for any acyclic ρ and which coincides with $\zeta_{X,\rho}(0)^{\pm 1}$ whenever $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$. Before stating our main results, let us introduce the two main characters of our discussion in the following paragraphs.

2.2.2.1 Refined versions of torsion

The Franz–Reidemeister torsion τ_{R} is given by the modulus of some alternate product of determinants and is therefore real valued. One cannot get a canonical object by removing the modulus since one has to make some choices to define the combinatorial torsion, and the ambiguities in these choices affect the determinants. To remove indeterminacies arising in the definition of the combinatorial torsion, Turaev [Tur86, Tur90, Tur97] introduced in the acyclic case a refined version of the combinatorial R-torsion, the *refined combinatorial torsion*. It is a complex number $\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ which depends on additional combinatorial data, namely an Euler structure $\mathfrak{e}$ and a cohomological orientation $\mathfrak{o}$ of M, and which satisfies $|\tau_{\mathfrak{e},\mathfrak{o}}(\rho)| = \tau_{\mathrm{R}}(\rho)$ if ρ is acyclic and unitary. We refer the reader to subsection 6.7.2 for precise definitions. Later, Farber-Turaev [FT00] extended this object to non-acyclic representations. In this case, $\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is an element of the determinant line of cohomology $\det H^\bullet(M,\rho)$.

Motivated by the work of Turaev, but from the analytic side, Braverman-Kappeler [BK07c, BK+08, BK07b] introduced a refined version of the Ray-Singer analytic torsion called *refined analytic torsion* $\tau_{\mathrm{an}}(\rho)$. It is complex valued in the acyclic case. Their construction heavily relies on the existence of a chirality operator Γ_g, that is,

$$\Gamma_g : \Omega^\bullet(M,E) \to \Omega^{n-\bullet}(M,E), \quad \Gamma_g^2 = \mathrm{Id},$$

which is a renormalized version of the Hodge star operator associated to some metric g. They showed that the ratio

$$\rho \mapsto \frac{\tau_{\mathrm{an}}(\rho)}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho)}$$

is a holomorphic function on the representation variety given by an explicit local expression, up to a local constant of modulus one. This result is an extension of the Cheeger-Müller theorem. Simultaneously, Burghelea-Haller [BH07] introduced a complex valued analytic torsion, which is closely related to the refined analytic torsion [BK07a] when it is defined ; see [H+07] for comparison theorems.

2.2.2.2 Dynamical torsion

We now assume that $X = X_\vartheta$ is the Reeb vector field of some contact form ϑ on M. Let us briefly describe the construction of the dynamical torsion. In the spirit of [BK07c], we use a chirality operator associated to the contact form ϑ,

$$\Gamma_\vartheta : \Omega^\bullet(M,E) \to \Omega^{n-\bullet}(M,E), \quad \Gamma_\vartheta^2 = \mathrm{Id},$$

cf. §6.4, analogous to the usual Hodge star operator associated to a Riemannian metric. Let $C^\bullet \subset \mathcal{D}'^\bullet(M,E)$ be the finite dimensional space of Pollicott-Ruelle generalized resonant states of $\mathcal{L}_X^\nabla$ for the resonance 0, that is,

$$C^\bullet = \left\{ u \in \mathcal{D}'^\bullet(M,E),\ \mathrm{WF}(u) \subset E_u^*,\ \exists N \in \mathbb{N},\ \left(\mathcal{L}_X^\nabla\right)^N u = 0 \right\},$$

where WF is the Hörmander wavefront set, $E_u^* \subset T^*M$ is the unstable cobundle of X [1], cf. §6.3, and $\mathcal{D}'(M,E)$ denotes the space of E-valued currents. Then ∇ induces a differential on $C^\bullet$ which makes it a finite dimensional cochain complex. Then a result from [DR19b] implies that the complex $(C^\bullet, \nabla)$ is acyclic if we assume that ∇ is. Because Γ_ϑ commutes with $\mathcal{L}_X^\nabla$, it induces a chirality operator on $C^\bullet$. Therefore we can compute the torsion $\tau(C^\bullet, \Gamma_\vartheta)$ of the finite dimensional complex $(C^\bullet, \nabla)$ with

1. the annihilator of $E_u \oplus \mathbb{R}X$ where $E_u \subset TM$ denotes the unstable bundle of the flow

respect to Γ_ϑ, as described in [BK07c] (see §6.2). Then we define the *dynamical torsion* τ_ϑ as the product

$$\tau_\vartheta(\rho)^{(-1)^q} = \pm \underbrace{\tau(C^\bullet,\Gamma_\vartheta)^{(-1)^q}}_{\text{finite dimensional torsion}} \times \underbrace{\lim_{s\to 0} s^{-m(X,\rho)}\zeta_{X,\rho}(s)}_{\text{renormalized zeta function at } s=0} \in \mathbb{C}\setminus 0,$$

where the sign $\pm$ will be given later, $m(X,\rho)$ is the order of $\zeta_{X,\rho}(s)$ at $s=0$ and $q = \frac{\dim(M)-1}{2}$ is the dimension of the unstable bundle of X. Note that the order $m(X,\rho)\in\mathbb{Z}$ is a priori not stable under perturbations of (X,ρ), in fact both terms in the product may not be invariant under small changes of ϑ whereas the dynamical torsion τ_ϑ has interesting invariance properties as we will see below.

2.2.2.3 Statement of the results.

We denote by $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ the set of acyclic representations $\pi_1(M)\to \mathrm{GL}(\mathbb{C}^d)$ and by $\mathcal{A}\subset\mathcal{C}^\infty(M,TM)$ the space of contact forms on M whose Reeb vector field induces an Anosov flow. This is an open subset of the space of contact forms. For any $\vartheta\in\mathcal{A}$, we denote by X_ϑ its Reeb vector field. In the spirit of Ray–Singer's result on the invariance of the analytic torsion with respect to the Riemannian metric [RS71], our first result shows $\tau_\vartheta(\rho)$ is invariant by small perturbations of the contact form $\vartheta\in\mathcal{A}$.

Theorem 2.2.5 (C.–Dang)**.** *Let (M,ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Let $(\vartheta_\tau)_{\tau\in(-\varepsilon,\varepsilon)}$ be a smooth family in $\mathcal{A}$. Then $\partial_\tau \log\tau_{\vartheta_\tau}(\rho)=0$ for any $\rho\in\mathrm{Rep}_{\mathrm{ac}}(M,d)$.*

Remark 2.2.6. In the case where the representation ρ is not acyclic, we can still define $\tau_\vartheta(\rho)$ as an element of the determinant line $\det H^\bullet(M,\rho)$ and this element is invariant under perturbations of $\vartheta\in\mathcal{A}$, cf Remarks 6.4.5 and 6.5.2.

Our second result aims to compare τ_ϑ with Turaev's refined version of the Reidemeister torsion $\tau_{\mathfrak{e},\mathfrak{o}}$, which depends on some choice of Euler structure $\mathfrak{e}$ and orientation $\mathfrak{o}$.

Theorem 2.2.7 (C.–Dang)**.** *Let (M,ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Then $\rho\in\mathrm{Rep}_{\mathrm{ac}}(M,d)\mapsto\tau_\vartheta(\rho)$ is holomorphic*[2] *and there exists an Euler structure $\mathfrak{e}$ such that for any cohomological orientation $\mathfrak{o}$ and any smooth family $(\rho_u)_{u\in(-\varepsilon,\varepsilon)}$ of $\mathrm{Rep}_{\mathrm{ac}}(M,d)$,*

$$\partial_u\log\tau_\vartheta(\rho_u)=\partial_u\log\tau_{\mathfrak{e},\mathfrak{o}}(\rho_u)$$

Moreover, if $\dim M=3$ and $b_1(M)\neq 0$, the map $\rho\mapsto\tau_\vartheta(\rho)/\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is of modulus one on the connected components of $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ containing an acyclic and unitary representation.

In [DGRS20], for ρ acyclic, the authors proved that $0\notin\mathrm{Res}(\mathcal{L}_X^\nabla)$ implies that $X\mapsto\zeta_{X,\rho}(0)$ is locally constant. Then, the equality $|\zeta_{X,\rho}(0)|=\tau_{\mathrm{R}}(\rho)$ was proved indirectly by working near analytic Anosov flows in dimension 3 or near geodesic

2. $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ is a variety over $\mathbb{C}$ see subsection 6.9.2 for the right notion of holomorphicity

flows of hyperbolic 3-manifolds, where the equality is known by the works of Sanchez Morgado and Fried. Whereas in the above theorem, for any contact Anosov flow in any odd dimension, we directly compare the logarithmic derivatives of the dynamical and refined torsions as holomorphic functions on the representation variety : we do not need to work near some vector field X for which the equality $|\zeta_{X,\rho}(0)| = \tau_{\mathrm{R}}(\rho)$ is already known.

Finally, our third result aims to describe how $\partial_u \log \tau_\vartheta(\rho_u)$ depends on the choice of the contact Anosov vector field X_ϑ.

Theorem 2.2.8 (C.–Dang)**.** *Let (M, ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Let $(\rho_u)_{|u|\leqslant\varepsilon}$ be a smooth family in $\mathrm{Rep}_{\mathrm{ac}}(M, d)$. Then for any $\eta \in \mathcal{A}$*

$$\partial_u \log \tau_\eta(\rho_u) = \partial_u \log \tau_\vartheta(\rho_u) + \partial_u \log \underbrace{\det \langle \rho_u, \mathrm{cs}(X_\vartheta, X_\eta)\rangle}_{\text{topological}}$$

where $\mathrm{cs}(X_\vartheta, X_\eta) \in H_1(M, \mathbb{Z})$ is the Chern-Simons class of the pair of vector fields (X_ϑ, X_η).

The underbraced term is topological since it is defined as the pairing of the representation ρ with the Chern–Simons class $\mathrm{cs}(X_\vartheta, X_\eta) \in H_1(M, \mathbb{Z})$ which measures the obstruction to find a homotopy among non singular vector fields connecting X_ϑ and X_η [3]. In particular, if ϑ and η are connected by some path in the space of vector fields without zeros, then $\mathrm{cs}(Y_\eta, X_\vartheta) = 0$ which yields $\det \langle \rho, \mathrm{cs}(X_\vartheta, X_\eta)\rangle = 1$ hence $\partial_u \log \tau_\eta(\rho_u) = \partial_u \log \tau_\vartheta(\rho_u)$ for any acyclic ρ. We refer the reader to subsection 6.7.1 for the definition of Chern-Simons classes.

2.2.3 Related works

Some analogs of our dynamical torsion were introduced by Burghelea–Haller [BH08b] for vector fields which admit a Lyapunov closed 1–form generalizing previous works by Hutchings [Hut02] and Hutchings–Lee [HL99b, HL99a] dealing with Morse–Novikov flows. In that case, the dynamical torsion depends on a choice of Euler structure and is a partially defined function on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$; if $d = 1$, it is shown in [BH08a] that it extends to a rational map on the Zariski closure of $\mathrm{Rep}_{\mathrm{ac}}(M, 1)$ which coincides, up to sign, with Turaev's refined combinatorial torsion (for the same choice of Euler structure). This follows from previous works of Hutchings–Lee [HL99b, HL99a] who introduced some topological invariant involving circle-valued Morse functions. In both works, the considered object has the form

$$\text{Dynamical zeta function at zero} \ \times \ \text{Correction term}$$

where the correction term is the torsion of some finite dimensional complex whose chains are generated by the critical points of the vector field. The chosen Euler structure gives a distinguished basis of the complex and thus a well defined torsion. This is one of the main differences with our work since in the Anosov case, there

3. Note that taking the determinant $\det\langle \rho, \mathrm{cs}(X_\vartheta, X_\eta)\rangle$ does not depend on the choice of representative of $\mathrm{cs}(X_\vartheta, X_\eta)$ in $\pi_1(M)$

are no such choices of distinguished currents in $C^\bullet$. However, as described above, the chirality operator allows us to overcome this problem.

We mention some interesting work of Rumin–Seshadri [RS12] where they relate some dynamical zeta function involving the Reeb flow and some analytic contact torsion on 3–dimensional Seifert CR manifolds. More recently, Spilioti [Spi20], Müller [Mue20] were able to compare the Ruelle zeta function for odd dimensional compact hyperbolic manifolds with some of the complex valued torsions. Finally, for geodesic flows of compact hyperbolic orbisurfaces, Bénard–Frahm–Spilioti [BFS21] were able to show, with the help of the Selberg's trace formula, that $\zeta_{X,\rho}(0)$ coincides (up to sign) with the Turaev torsion, thus generalizing our Theorem 2.2.7 for orbisurfaces.

2.3 Obstacle scattering and periodic orbits

Here we present the results of Chapters 7 and 8, which form Part III ; they contain respectively the articles *Closed billiard trajectories with prescribed bounces* [Chaa] and *Dynamical zeta function for billiards* [CP22] — the latter is written in collaboration with Vesselin Petkov. Let $r \geqslant 3$ be an integer, and $D_1, \dots, D_r \subset \mathbb{R}^d$ a family of smooth, strictly convex obstacles, satisfying the non-eclipse condition

$$\mathrm{conv}(D_i \cup D_j) \cap D_k = \emptyset, \quad i \neq k, \quad j \neq k,$$

where conv is the convex hull. Those obstacles give rise to a *billiard flow*, which generalize the geodesic flow, for which trajectories bounce on the boundary of the obstacles according to Fresnel–Descartes' law. We will denote by $\mathcal{P}_{\mathbf{B}}$ the set of primitive periodic trajectories of the billiard flow. In this setting, we still have the primitive orbit theorem

$$\sharp\{\gamma \in \mathcal{P}_{\mathbf{B}} \ : \ \tau(\gamma) \leqslant t\} \sim \frac{\mathrm{e}^{h_{\mathbf{B}} t}}{h_{\mathbf{B}} t},$$

where $\tau(\gamma)$ is the period of γ and $h_{\mathbf{B}} > 0$ is the entropy of the billiard $\mathbf{B} = \{D_1, \dots D_r\}$.

2.3.1 Constraining the number of bounces

In Chapter 7, we extend Theorem 2.1.4 to the framework of billiard flows. More precisely, assume that $d = 2$ and take another obstacle $D_0 \subset \mathbb{R}^2$ so that the family $D_0, \dots, D_r$ satisfies the non-eclipse condition. For every trajectory $\gamma \in \mathcal{P}$, we denote by $m_0(\gamma)$ the number of reflexions of γ on D_0.

Theorem 2.3.1. *There is $c > 0$ such that for every integer $n > 0$ there holds, as $t \to \infty$,*

$$\sharp\{\gamma \in \mathcal{P} \ : \ \tau(\gamma) \leqslant t,\ m_0(\gamma) = n\} \ \sim \frac{(ct)^n}{n!} \frac{\mathrm{e}^{h_{\mathbf{B}} t}}{h_{\mathbf{B}} t}.$$

This result will be obtained with similar methods that the ones used in Chapter 4. In particular, we also introduce a dynamical scattering operator, and we make use of a recent result by Küster–Schütte–Weich [KSW21] allowing us to see the billiard flow as a smooth flow on a smooth manifold, so that Dyatlov–Guillarmou theory [DG16] can be used to study the resolvent of the billiard flow.

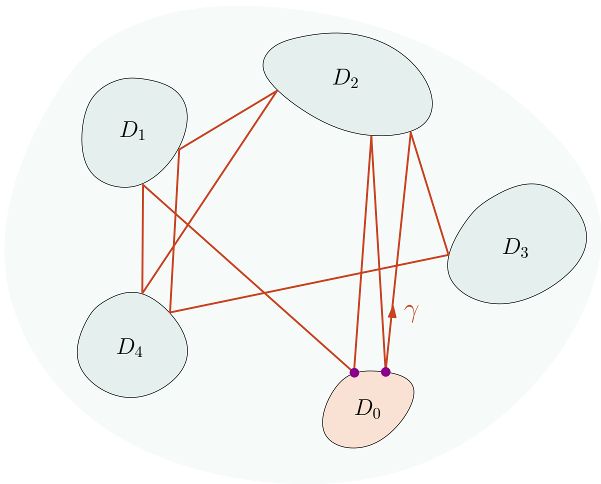

FIGURE 2.3 – A closed billiard trajectory γ with $m_0(\gamma) = 2$.

2.3.2 Dirichlet series and quantum resonances

In Chapter 8, we obtain a meromorphic continuation for certain Dirichlet series linked to the resonances of the Laplacian on $\mathbb{R}^d \setminus \cup_{j=1}^r D_j$. For any positive integer q, we set

$$\eta_q(s) = \sum_{m(\gamma)\in q\mathbb{N}} \frac{\tau^\sharp(\gamma) e^{-s\tau(\gamma)}}{|1 - P_\gamma|^{1/2}}, \quad \mathrm{Re}(s) \gg 1,$$

where the sum runs over all periodic orbits (not necessarily primitive), $m(\gamma)$ is the number of reflexion of γ on the obstacles $D_1, \dots, D_r$, P_γ is the linearized Poincaré map of γ and $|1 - P_\gamma| = |\det(1 - P_\gamma)|$.

Theorem 2.3.2 (C.–Petkov)**.** *The series η_q admits a meromorphic continuation to the whole complex plane, with simple poles and residues in $\mathbb{Z}/q$.*

This theorem is proved by using [KSW21] and [DG16], lifting the billiard flow to a Grasmannian bundle following the work of Faure–Tsujii [FT17] on geodesic flows, and by introducing a q-reflexion bundle, which allows to forget about periodic orbits γ such that $m(\gamma) \notin q\mathbb{Z}$.

In particular, we obtain the meromorphic continuation of the series

$$\eta_{\mathrm{D}}(s) = \sum_\gamma (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma) e^{-s\tau(\gamma)}}{|1 - P_\gamma|^{1/2}}, \quad \mathrm{Re}(s) \gg 1,$$

by writing $\eta_{\mathrm{D}}(s) = 2\eta_2(s) - \eta_1(s)$. The latter series is intimately linked with the quantum resonances $\{\mu_j\} \subset \mathbb{C}$ of Dirichlet Laplacian Δ on $\mathbb{R}^d \setminus \cup_{j=1}^r D_j$, *via* the trace formula of Bardos–Guillot–Ralston [BGR82]. Those resonances are defined as follows. For $\mu \in \mathbb{C}$ with $\mathrm{Im}(\mu) < 0$, the resolvent

$$\mathrm{R}_\Delta(\mu) = (-\Delta - \mu^2)^{-1} : L^2(\Omega) \to L^2(\Omega),$$

where $\Omega = \mathbb{R}^d \setminus D$ and $D = \cup_{j=1}^r D_j$, is well defined. We know since the work of Lax–Phillips [LP67, LP89] that $\mu \mapsto \mathrm{R}_\Delta(\mu)$ admits a meromorphic continuation as a family of operators

$$L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega),$$

for $\mu \in \mathbb{C}$ if d is odd and for μ in a logarithmic cover $\{z \in \mathbb{C} \ : \ -\infty < \arg(z) < \infty\}$ otherwise.

The distribution of those resonances — namely, the existence of a spectral gap — is intimately linked with the decay of the local energy of solutions to the wave equation. Under certain conditions on the topological pressure, such a gap was obtained by Ikawa [Ika88a] and later by Nonnenmacher–Zworski [NZ09] in a more general setting. More recently, in dimension 2, Vacossin [Vac22] showed that the pressure condition could be omitted for a system of convex obstacles.

Lax–Phillips [LP67] conjectured that if $D \subset \mathbb{R}^d$ was a trapping set (in the sense that there exists a trapped trajectory for the billiard flow in $\mathbb{R}^d \setminus D$), then one can find a sequence (μ_{j_k}) of resonances such that $\mathrm{Im}(\mu_{j_k}) \to 0$. Ikawa [Ika82] and Gérard [Gér88] proved that this conjecture is false in the case where D consists in two disjoint convex obstacles. This led Ikawa [Ika88b] to formulate the *modified Lax–Phillips conjecture* (MLPC), as follows.

Conjecture 2.3.1 (Ikawa)**.** *If D is trapping, then there is $\delta > 0$ such that*

$$\sharp\{\mu_j \ : \ \mathrm{Im}(\mu_j) \leqslant \delta\} = \infty.$$

If the dimension d is even, it is implicit that we only consider resonances μ_j such that $\mathrm{Im}(\mu_j) \leqslant \delta$ with $0 < \arg(\mu_j) < \pi$. Ikawa [Ika88b] showed that this conjecture is true as soon as the series η_{D} has a pole — for the Laplacian with Neumann boundary conditions, the same implication is true if we replace η_{D} by η_1; the existence of a pole is then automatic since the coefficients of the series η_1 are positive. If D is a finite union of balls $D_j = B(x_j, \varepsilon)$ centered at $x_j \in \mathbb{R}^d$, Ikawa [Ika88b] proved that the MLPC is true whenever $\varepsilon > 0$ is small enough (depending on the x_j's). Later, Stoyanov [Sto09] extended this result to general obstacles, but also under a smallness condition.

Using the works of Ikawa [Ika88b, Ika90a] and Fried [Fri95], we will show that the MLPC holds for analytic obstacles.

Theorem 2.3.3 (C.–Petkov)**.** *The modified Lax–Phillips conjecture is true for a union of strictly convex real analytic obstacles obstacles, under the non-eclipse condition.*

Première partie

Géodésiques fermées et nombres d'intersection

Chapitre 3

Comptage sur les graphes discrets

Dans ce chapitre, nous donnons la croissance asymptotique du nombre de trajectoires fermées dans un graphe discret fini, quand on impose aux trajectoires de passer un nombre fini de fois à travers une arête donnée. C'est un modèle jouet pour le problème de comptage de géodésiques sur les surfaces que nous allons considérer au chapitre suivant. La méthode présentée pourrait sembler peu naturelle, mais elle illustre parfaitement la stratégie que nous allons adopter pour traiter le cas des surfaces.

Sommaire

3.1 Combinatorial setting

We consider a non-oriented graph G, with vertexes $V = \{v_1, \dots, v_p\}$ and edges $E = \{e_1, \dots, e_q\} \subset V \times V$. By non-oriented, we mean that $(i, j) \in E$ if and only if $(j, i) \in E$ for any $i, j \in \{1, \dots, p\}$. We will write $v_i \sim v_j$ whenever $(i, j) \in E$. A *closed path* in G is a sequence

$$v = (v_{i_1}, e_{j_1}, v_{i_2}, \dots, v_{i_r}, e_{j_r})$$

where $r \geqslant 1$ is some integer, and $e_{j_k} = (i_k, i_{k+1}) \in E$ for any $k \in \mathbb{Z}/r\mathbb{Z}$. A *loop* in G is an equivalence class $[v]$ of a closed path v, where two sequences v and w are identified whenever v is a cyclic permutation of w. The length of a closed trajectory $[v]$ is by definition the integer r and is denoted by $|[v]|$. A loop will be called *primitive* if it is not the multiple of a shorter loop. We denote by $\widetilde{\mathcal{P}}(G)$ (resp. $\mathcal{P}(G)$) the set of loops (resp. primitive loops) of G.

Let A be the adjacency matrix of the graph G, that is, A is the $p \times p$ symmetric matrix defined by

$$A = (a_{ij}) \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if} \quad v_i \sim v_j, \\ 0 & \text{if} \quad \text{not}. \end{cases}$$

We will assume that A is primitive, which means that there is $m \geqslant 1$ such that all the coefficients of A^m are positive. Under this condition, the Perron–Frobenius theorem applies and gives the existence of $r > 0$ such that

$$\mathrm{sp}(A) \subset \bar{D}(0,r) \quad \text{and} \quad \mathrm{sp}(A) \cap \partial D(0,r) = \{r\}, \tag{3.1.1}$$

where $\mathrm{sp}(A)$ is the spectrum of A and $D(0,t) = \{z \in \mathbb{C} \ : \ |z| < t\}$. Moreover r is a simple eigenvalue of A and there is $\mu \in \mathbb{R}^n$ with positive coefficients such that $A\mu = r\mu$. Note also that if $p \geqslant 2$, then necessarily $r > 1$. The following result is well known.

Proposition 3.1.1. *It holds*

$$\sharp\{\omega \in \mathcal{P}(G) \ : \ |\omega| = \ell\} \sim \frac{r^\ell}{\ell}, \quad \ell \to \infty.$$

Proof. The number of closed paths of length ℓ in G is exactly $\mathrm{tr}(A^\ell)$. Moreover, for each loop $\omega \in \widetilde{\mathcal{P}}(G)$, there are exactly $|\omega^\sharp|$ closed paths generating the equivalence class ω, where $\omega^\sharp$ denotes the primitive loop associated to ω. Thus we may write

$$\mathrm{tr}(A^\ell) = \sum_{\substack{\omega \in \widetilde{\mathcal{P}}(G) \\ |\omega| = \ell}} |\omega^\sharp|. \tag{3.1.2}$$

Then it holds

$$\mathrm{tr}(A^\ell) = \sum_{|\omega^\sharp| = \ell} \ell + \sum_{\substack{|\omega^\sharp| < \ell \\ \ell \in |\omega^\sharp|\mathbb{Z}}} |\omega^\sharp|,$$

where the first sum runs over primitive loops of length ℓ while the second runs over the primitive loops $\omega^\sharp$ of length $|\omega^\sharp| < \ell$ and such that ℓ is a multiple of $|\omega^\sharp|$. This last sum is bounded by

$$\sum_{k=0}^{\ell/2} \mathrm{tr}(A^k) \leqslant C r^{\ell/2}$$

for some constant C by (3.1.1) and (3.1.2). Finally we get

$$\sum_{|\omega^\sharp| = \ell} 1 = \mathrm{tr}(A^\ell)/\ell + \mathcal{O}(r^{\ell/2}) \sim r^\ell/\ell$$

as $\ell \to \infty$, which concludes the proof. □

3.2 Imposing a constraint

Next, we fix an edge $e_\star = (i_\star, j_\star) \in V^2$ such that $e \in E$. We also denote $\bar{e}_\star = (j_\star, i_\star)$, and we consider the graph $G_\star$ which is obtained from G by removing the edges $e_\star$ and $\bar{e}_\star$. We assume that the new adjacency matrix $A_\star$ remains primitive. For $n \geqslant 0$ and $\ell \geqslant 1$, we denote by $N(n,\ell)$ the number of loops (resp. primitive loops) in G of length $n+\ell$ passing exactly n times through the edges $e_\star$ or $\bar{e}_\star$. More precisely, for any loop $\omega = [(v_{i_1}, e_{i_1}, \dots, v_{i_{n+\ell}}, e_{i_{n+\ell}})]$, we denote by

$$\mathrm{i}(\omega, e_\star) = \sharp\big\{j \in \mathbb{Z}/(r+n)\mathbb{Z} \ : \ e_{i_j} = e_\star \ \text{ or } \ e_{i_j} = \bar{e}_\star\big\}$$

the number of times ω passes through $e_\star$ or $\bar{e}_\star$, and we set

$$N(n,\ell) = \sharp\{\omega \in \mathcal{P}(G) \ : \ |\omega| = n+\ell, \ \mathrm{i}(\omega, e_\star) = n\}.$$

The purpose of this section is to prove the following result.

Proposition 3.2.1. *There is a constant $c_\star > 0$ such that for any $n \geqslant 0$ it holds*

$$N(n,\ell) \sim \frac{(c_\star \ell)^n}{n!} \frac{r_\star^\ell}{\ell}, \quad \ell \to \infty,$$

where $r_\star \in \,]1, r[$ is the Perron–Frobenius eigenvalue of $A_\star$.

Remark 3.2.2. In fact, we have

$$c_\star = 2\langle v_{i_\star}, \mu_\star\rangle\langle v_{j_\star}, \mu_\star\rangle,$$

where $\mu_\star \in \mathbb{R}^d$ is the unique eigenvector of $A_\star$ associated to the eigenvalue $r_\star$ such that $\mu_\star > 0$ and $\|\mu_\star\|_2 = 1$. Here identified the set of vertexes V with the canonical basis or $\mathbb{R}^n$, by declaring that v_k corresponds the element of $\mathbb{R}^d$ whose coefficients are zero except for the k-th component whose value is 1.

In what follows, we will denote by $\widetilde{\mathcal{P}}(n,\ell)$ (resp. $\mathcal{P}(n,\ell)$) the set of loops (resp. primitive loops) ω of length $n+\ell$ and such that $\mathrm{i}(\omega, e_\star) = n$; we also set

$$\widetilde{\mathcal{P}}(n) = \bigcup_{\ell=0}^{\infty} \widetilde{\mathcal{P}}(n,\ell) \quad \text{and} \quad \mathcal{P}(n) = \bigcup_{\ell=0}^{\infty} \mathcal{P}(n,\ell).$$

Also, if $z^{-1} \notin \mathrm{sp}(A_\star)$, we define

$$a_{km}(z) = \left\langle v_k, (1 - zA_\star)^{-1} v_m \right\rangle, \quad 1 \leqslant k, m \leqslant d.$$

Then the *scattering matrix* $\mathcal{S}_\star(z)$ associated to $e_\star = (i_\star, j_\star)$ is the 2×2 matrix defined by

$$\mathcal{S}_\star(z) = \begin{pmatrix} a_{i_\star i_\star}(z) & a_{i_\star j_\star}(z) \\ a_{j_\star i_\star}(z) & a_{j_\star j_\star}(z) \end{pmatrix}, \quad z^{-1} \notin \mathrm{sp}(A_\star).$$

Finally, we set $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and set $\widetilde{\mathcal{S}}_\star(z) = J\mathcal{S}_\star(z)$.

Proposition 3.2.3 (Trace formula for graphs)**.** *For $|z| < 1/r$ and $n \geqslant 1$ we have the trace formula*

$$\mathrm{tr}\, \widetilde{\mathcal{S}}_\star(z)^n = n z^{-n} \sum_{\omega \in \widetilde{\mathcal{P}}(n)} \frac{|\omega^\sharp|}{|\omega|} z^{|\omega|},$$

where the sum runs over all loops ω passing exactly n times through $e_\star$ or $\bar{e}$, and $|\omega^\sharp|$ is the primitive period of ω.

Proof. For every $k, m \in \{1, \dots, d\}$, $n \geqslant 0$ and $\ell \geqslant 1$ we denote by $\mathcal{C}_{km}(n, \ell)$ the set of trajectories $v = (v_{i_1}, e_{i_1}, \dots, e_{i_{\ell+n}}, v_{i_{\ell+n+1}})$ linking $v_{i_1} = v_k$ to $v_{i_{\ell+n+1}} = v_m$, of length $\ell + n$ (by convention, the path $v = (v_{i_1})$ is of length zero), and passing exactly n times through $e_\star$ or $\bar{e}_\star$, that is,

$$\sharp\{k = 1, \dots, \ell + n \ : \ e_{i_k} = e_\star \ \text{ or } \ e_{i_k} = \bar{e}_\star\} = n.$$

We set $N_{km}(n, \ell) = \sharp\mathcal{C}_{km}(n, \ell)$. Then we claim that for any $n \geqslant 1$ it holds

$$\widetilde{\mathcal{S}}_\star(z)^n = \sum_{\ell=0}^{\infty} z^\ell \begin{pmatrix} N_{i_\star j_\star}(n-1, \ell) & N_{i_\star i_\star}(n-1, \ell) \\ N_{j_\star j_\star}(n-1, \ell) & N_{j_\star i_\star}(n-1, \ell) \end{pmatrix}. \tag{3.2.1}$$

Indeed, for $n = 1$ this follows from the computation

$$\begin{aligned} a_{km}(z) &= \left\langle v_k, (1 - zA_\star)^{-1} v_m \right\rangle \\ &= \sum_{\ell=0}^{\infty} z^\ell \langle v_k, A_\star^\ell v_m \rangle, \end{aligned}$$

and the fact that $\langle v_k, A_\star^\ell v_m \rangle = N_{km}(0, \ell)$ is the number of paths of length ℓ joining v_k to v_m (and not passing through $e_\star$ or $\bar{e}_\star$). Next, assume that (3.2.1) holds for some $n \geqslant 1$. For $m \geqslant 1$ we write

$$\widetilde{\mathcal{S}}_\star(z)^m = \begin{pmatrix} a_{i_\star j_\star}(m, z) & a_{i_\star i_\star}(m, z) \\ a_{j_\star j_\star}(m, z) & a_{j_\star i_\star}(m, z) \end{pmatrix}.$$

Then by (3.2.1) we get

$$\begin{aligned} & a_{i_\star j_\star}(n+1, z) \\ &\qquad = \sum_{\ell=0}^{\infty} z^\ell \big(a_{i_\star j_\star}(z) N_{i_\star j_\star}(n-1, \ell) + a_{i_\star i_\star}(z) N_{j_\star j_\star}(n-1, \ell) \big) \\ &\qquad = \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} z^{\ell+\ell'} \big(N_{i_\star j_\star}(0, \ell') N_{i_\star j_\star}(n-1, \ell) + N_{i_\star i_\star}(0, \ell') N_{j_\star j_\star}(n-1, \ell) \big). \end{aligned}$$

Now by observing that a path $v \in \mathcal{C}_{i_\star j_\star}(n, \beta)$ can be (uniquely) written as a concatenation of the form

$$u^{i_\star j_\star} \cdot \bar{e}_\star \cdot w^{i_\star j_\star} \quad \text{ or } \quad u^{i_\star i_\star} \cdot e_\star \cdot w^{j_\star j_\star},$$

where $u^{km} \in \mathcal{C}_{km}(0, \ell')$, $w^{km} \in \mathcal{C}_{km}(n-1, \ell)$ and $\ell + \ell' = \beta$, we see that

$$N_{i_\star j_\star}(n, \beta) = \sum_{\ell+\ell'=\beta} \left(N_{i_\star j_\star}(0, \ell') N_{i_\star j_\star}(n-1, \ell) + N_{i_\star i_\star}(0, \ell') N_{j_\star j_\star}(n-1, \ell) \right).$$

Thus $a_{i_\star j_\star}(n+1, z) = \sum_{\beta \geqslant 0} z^\beta N_{i_\star j_\star}(n, \beta)$. Similarly one is able to show that $a_{km}(n+1, z)$ coincides with $\sum_{\beta \geqslant 0} z^\beta N_{km}(n, \beta)$ for any $k, m \in \{i_\star, j_\star\}$, and thus we proved by induction that (3.2.1) holds for any $n \geqslant 0$. In particular, we get

$$\operatorname{tr} \widetilde{\mathcal{S}}_\star(z)^n = \sum_{\ell=0}^{\infty} z^\ell \big(N_{i_\star j_\star}(n-1, \ell) + N_{j_\star i_\star}(n-1, \ell) \big). \tag{3.2.2}$$

Now we consider the map $F_{n,\ell} : \mathcal{C}_{i_\star j_\star}(n-1,\ell) \cup \mathcal{C}_{j_\star i_\star}(n-1,\ell) \to \widetilde{\mathcal{P}}(n,\ell)$ which is defined by

$$F_{n,\ell}(u) = \begin{cases} [ue_\star] \text{ if } u \in \mathcal{C}_{i_\star j_\star}(n-1,\ell), \\ [u\bar{e}_\star] \text{ if } u \in \mathcal{C}_{j_\star i_\star}(n-1,\ell). \end{cases}$$

Then $F_{n,\ell}$ is surjective. Moreover, it is not hard to see that for any $\omega \in \widetilde{\mathcal{P}}(n,\ell)$ we have

$$\sharp F_{n,\ell}^{-1}(\{\omega\}) = n\frac{|\omega^\sharp|}{|\omega|}.$$

Therefore, one obtains

$$N_{i_\star j_\star}(n-1,\ell) + N_{j_\star i_\star}(n-1,\ell) = \sum_{\omega \in \widetilde{\mathcal{P}}(n,\ell)} n\frac{|\omega^\sharp|}{|\omega|},$$

and by (3.2.2), the lemma follows. □

3.3 A Tauberian argument

Let $\varepsilon > 0$ small so that $\overline{D}(r_\star,\varepsilon) \cap \mathrm{sp}(A_\star) = \{r_\star\}$, where $D(r_\star,\varepsilon) \subset \mathbb{C}$ is the disk or radius ε centered at $r_\star$. We denote by

$$\Pi_{r_\star} = \frac{1}{2\pi i}\int_{\partial D(r_\star,\varepsilon)} (z - A_\star)^{-1}\mathrm{d}z$$

the spectral projector of $A_\star$ associated to the eigenvalue $r_\star$, which is also given by

$$\Pi_{r_\star} = \langle \mu_\star, \cdot \rangle \mu_\star.$$

Then near $z = 1/r_\star$, we have the development

$$(1 - zA_\star)^{-1} = G(z) + \frac{\Pi_{r_\star}}{1 - zr_\star},$$

where G is holomorphic near $1/r_\star$, and In particular we get, writing $c_k = \langle \mu_\star, v_k \rangle > 0$ for $k = i_\star, j_\star$,

$$\widetilde{S}_\star(z) = H(z) + \frac{1}{1 - zr_\star}\begin{pmatrix} c_{i_\star}c_{j_\star} & c_{i_\star}c_{i_\star} \\ c_{j_\star}c_{j_\star} & c_{j_\star}c_{i_\star} \end{pmatrix}$$

where H is holomorphic near $z = 1/r_\star$. As the matrix $R = \begin{pmatrix} c_{i_\star}c_{j_\star} & c_{i_\star}c_{i_\star} \\ c_{j_\star}c_{j_\star} & c_{j_\star}c_{i_\star} \end{pmatrix}$ is of rank one, we have $\mathrm{tr}(R^n) = \mathrm{tr}(R)^n$ for any n and thus we finally get

$$\mathrm{tr}\, \widetilde{S}_\star(z)^n = \frac{(c_\star)^n}{(1 - zr_\star)^n} + \mathcal{O}\left((1 - zr_\star)^{-n+1}\right), \quad z \to 1/r_\star, \tag{3.3.1}$$

where $c_\star = 2c_{i_\star}c_{j_\star}$.

Proof of Proposition 3.2.1. Proposition 3.2.3 and (3.3.1) give

$$nz^{-n} \sum_{\omega \in \widetilde{\mathcal{P}}(n)} \frac{|\omega^\sharp|}{|\omega|} z^{|\omega|} = \frac{(c_\star)^n}{(1 - zr_\star)^n} + \mathcal{O}\left((1 - zr_\star)^{-n+1}\right), \quad z \to 1/r_\star. \tag{3.3.2}$$

For $|z| < 1/r_\star$ we define $f(z) = \sum_{\ell=0}^\infty a_\ell z^\ell$ where

$$a_\ell = \sum_{\substack{\omega \in \widetilde{\mathcal{P}}(n) \\ |\omega| = \ell}} |\omega^\sharp|, \quad \ell \geqslant 0.$$

Then by (3.3.2) it holds

$$f(z) = \frac{\mathrm{d}}{\mathrm{d}z} \sum_{\omega \in \widetilde{\mathcal{P}}(n)} \frac{|\omega^\sharp|}{|\omega|} z^{|\omega|} = \frac{r_\star^{1-n} (c_\star)^n}{r_\star^{n+1} (r_\star^{-1} - z)^{n+1}} + \mathcal{O}((1 - zr_\star)^{-n}), \quad z \to 1/r_\star. \tag{3.3.3}$$

We will need the following

Lemma 3.3.1. *Let $(a_\ell)_{\ell \geqslant 0}$ be a sequence of complex numbers such that $\sum_\ell a_\ell z^\ell$ converges absolutely for $|z| < r$, for some $r > 0$. Assume that there are $n \geqslant 0$ and $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{C}$ with $\alpha_{n+1} \neq 0$, such that the function $f : \{|z| < r\} \to \mathbb{C}$ defined by*

$$f(z) = \sum_{\ell=0}^\infty a_\ell z^\ell - \sum_{k=1}^{n+1} \frac{\alpha_k}{(r - z)^k}, \quad |z| < r,$$

extends analytically to a disk $\{|z| < r'\}$ where $r' > r$. Then

$$a_\ell \sim \frac{\alpha_{n+1} \ell^n}{n!} r^{\ell - n}, \quad \ell \to \infty.$$

Proof of Lemma 3.3.1. For $|z| < r$ and $k = 2, \ldots, n+1$, we write

$$(r - z)^{-k} = \sum_{\ell=0}^\infty b_{k,\ell} r^{-\ell} z^{\ell - k + 1}$$

where $b_{k,\ell} = \ell(\ell - 1) \cdots (\ell - k + 2)/(k - 1)!$. Then we have $f(z) = \sum_{\ell=0}^\infty A_\ell z^\ell$ where

$$A_\ell = a_\ell - r^{-\ell - 1} - \sum_{k=2}^{n+1} \alpha_k b_{k, \ell + k - 1} r^{-\ell - k + 1}, \quad \ell \geqslant 0.$$

As f is analytic on $\{|z| \leqslant \rho\}$ for $\rho \in \,]r, r'[$, the Cauchy formula yields $A_\ell = \mathcal{O}(\rho^{-\ell})$ as $\ell \to \infty$. In particular $a_\ell \sim \alpha_{n+1} b_{n+1, \ell+n} r^{-\ell - n}$, and noting that

$$b_{n+1, \ell+n} = \frac{(\ell + n) \cdots (\ell + 1)}{n!} \sim \frac{\ell^n}{n!}, \quad \ell \to \infty,$$

we conclude the proof. □

Applying Lemma 3.3.1 with (3.3.3) yields

$$\sum_{\substack{\omega\in\widetilde{\mathcal{P}}(n)\\ |\omega|=\ell}} |\omega^\sharp| \sim \frac{r_\star^{1-n}(c_\star)^n}{r_\star^{n+1}}\frac{\ell^n}{n!}r_\star^{\ell+n} = \frac{(c_\star\ell)^n}{n!}r_\star^{\ell-n}, \quad \ell\to\infty. \tag{3.3.4}$$

On the other hand, we have

$$\sum_{\substack{\omega\in\widetilde{\mathcal{P}}(n)\\ |\omega|=\ell}} |\omega^\sharp| = \sum_{m|\ell}\sum_{\substack{\omega^\sharp\in\mathcal{P}(n)\\ |\omega^\sharp|=m}}\sum_{k=1}^{\ell/m}|\omega^\sharp| = \ell N(n,\ell-n) + \ell\sum_{\substack{m|\ell\\ m\leqslant\ell/2}}\sum_{\substack{\omega^\sharp\in\mathcal{P}(n)\\ |\omega^\sharp|=m}} 1.$$

By (3.3.4) it holds

$$\ell\sum_{\substack{m|\ell\\ m\leqslant\ell/2}}\sum_{\substack{\omega^\sharp\in\mathcal{P}(n)\\ |\omega^\sharp|=m}} 1 = \mathcal{O}(\ell r_\star^{\ell/2}), \quad \ell\to\infty.$$

Thus applying (3.3.4) again we get

$$\ell N(n,\ell-n) \sim \frac{(c_\star\ell)^n}{n!}r_\star^{\ell-n}, \quad \ell\to\infty,$$

which concludes the proof of Proposition 3.2.1. □

Chapitre 4

Comptage des géodésiques fermées sous contraintes d'intersection

Dans ce chapitre, nous explicitons la croissance asymptotique du nombre de géodésiques fermées sur une surface fermée, quand on impose aux géodésiques certaines contraintes d'intersection. Ce chapitre contient l'article *Closed geodesics and intersection numbers* [Chab].

Sommaire

4.1 Introduction

Let (Σ, g) be a closed, oriented, connected, negatively curved Riemannian surface and denote by $\mathcal{P}$ the set of its oriented primitive closed geodesics. For $L > 0$ define

$$N(L) = \sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L\},$$

where for $\gamma \in \mathcal{P}$, we denoted by $\ell(\gamma)$ its length. Then a classical result obtained by Margulis [Mar69] states that

$$N(L) \sim \frac{\mathrm{e}^{hL}}{hL}, \quad L \to \infty,$$

where $h > 0$ is the topological entropy of the geodesic flow of (Σ, g).

In this chapter, we will provide a similar asymptotic result for closed geodesics satisfying certain intersection constraints. Namely, let $\gamma_\star$ be a simple closed geodesic of (Σ, g). For any $\gamma \in \mathcal{P}$, we denote by $\mathrm{i}(\gamma, \gamma_\star)$ the geometric intersection number between γ and $\gamma_\star$ (see §4.2.1), and we set

$$N(n, L) = \sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \mathrm{i}(\gamma, \gamma_\star) = n\}.$$

We first state a result in the case where $\gamma_\star$ is assumed to be not separating, in the sense that $\Sigma \setminus \gamma_\star$ is connected.

Theorem 4.1.1. *Assume that $\gamma_\star$ is not separating. Then there are $c_\star > 0$ and $h_\star \in]0, h[$ such that for any $n \geqslant 1$ it holds*

$$N(n, L) \sim \frac{(c_\star L)^n}{n!} \frac{\mathrm{e}^{h_\star L}}{h_\star L}, \quad L \to \infty. \tag{4.1.1}$$

Let (φ_t) denote the geodesic flow of (Σ, g), acting on the unit tangent bundle $S\Sigma$ of Σ. Then the number $h_\star$ in the above statement is the topological entropy of the flow (φ_t) restricted the trapped set

$$K_\star = \overline{\{(x, v) \in S\Sigma \ : \ \pi(\varphi_t(x, v)) \in \Sigma \setminus \gamma_\star, \ t \in \mathbb{R}\}},$$

where the closure is taken in $S\Sigma$ and $\pi : S\Sigma \to \Sigma$ is the natural projection. Also, we provide in §4.7 a description of the constant $c_\star$ in terms of the Pollicott-Ruelle resonant states of the geodesic flow of the surface with boundary $\Sigma_\star$ obtained by cutting Σ along $\gamma_\star$.

If $\gamma_\star$ is separating then $\mathrm{i}(\gamma, \gamma_\star)$ is even and we have the following result.

Theorem 4.1.2. *Suppose that $\gamma_\star$ separates Σ in two surfaces Σ_1 and Σ_2. Let $h_j \in]0, h[$ denote the entropy of the open system $(\Sigma_j, g|_{\Sigma_j})$ and set $h_\star = \max(h_1, h_2)$. Then there is $c_\star > 0$ such that for each $n \geqslant 1$ we have, as $L \to +\infty$,*

$$N(2n, L) \sim \begin{cases} \dfrac{(c_\star L)^n}{n!} \dfrac{\mathrm{e}^{h_\star L}}{h_\star L} & \text{if } h_1 \neq h_2, \\ 2\dfrac{(c_\star L^2)^n}{(2n)!} \dfrac{\mathrm{e}^{h_\star L}}{h_\star L} & \text{if } \ h_1 = h_2, \end{cases}$$

As before, the number h_j is defined as the topological entropy of the geodesic flow restricted to the trapped set

$$K_j = \overline{\{(x, v) \in S\Sigma \ : \ \pi(\varphi_t(x, v)) \in \Sigma_j \setminus \gamma_\star, \ t \in \mathbb{R}\}},$$

where the closure is taken in $S\Sigma$.

Remark 4.1.3. As explained in the introduction (see §2.1.4), one is more generally able to obtain similar asymptotics results for closed geodesics of which *several* intersection numbers with a *family* of simple curves $\gamma_{\star,1}, \dots, \gamma_{\star,r}$ are prescribed. However, to make the exposition clearer, we will deal in the major part of this chapter with the case $r = 1$. The case $r > 1$ will be obtained later in §4.9 by using identical techniques.

We also mention an equidistribution result. Set

$$\partial_\star = \{(x, v) \in S\Sigma \ : \ x \in \gamma_\star\} \quad \text{and} \quad \Gamma = S\gamma_\star \cup \big\{z \in \partial_\star \ \big| \ \varphi_t(z) \in S\Sigma \setminus \partial_\star, \ t > 0\big\}$$

where $S\gamma_\star = \{(x, v) \in \partial_\star \ : \ v \in T_x\gamma_\star\}$. We define the Scattering map $S : \partial_\star \setminus \Gamma \to \partial_\star$ by

$$S(z) = \varphi_{\ell(z)}(z), \quad \ell(z) = \inf\{t > 0 \ : \ \varphi_t(z) \in \partial_\star\}, \quad z \in \partial_\star \setminus \Gamma.$$

For any $n \in \mathbb{N}_{\geqslant 1}$ we set

$$\Gamma_n = \partial_\star \setminus \big\{z \in \partial_\star \setminus \Gamma \ : \ S^k(z) \in \partial_\star \setminus \Gamma, \ k = 1, \dots, \ n-1\big\}$$

which is a closed set of Lebesgue measure zero, and

$$\ell_n(z) = \ell(z) + \cdots + \ell(S^{n-1}(z)), \quad z \in \partial_\star \setminus \Gamma_n.$$

Theorem 4.1.4. *Assume that $\gamma_\star$ is not separating and let $n \geqslant 1$. For any $f \in C^\infty\big(\partial_\star\big)$ the limit*

$$\lim_{L \to +\infty} \frac{1}{N(n, L)} \sum_{\substack{\gamma \in \mathcal{P} \\ \mathrm{i}(\gamma, \gamma_\star) = n}} \frac{1}{\sharp I_\star(\gamma)} \sum_{z \in I_\star(\gamma)} f(z)$$

exists, where for any $\gamma \in \mathcal{P}$, $I_\star(\gamma) = \{(x, v) \in S\gamma \ : \ x \in \gamma_\star\}$ is the set of incidence vectors of γ along $\gamma_\star$. This formula defines a probability measure μ_n on $\partial_\star$, whose support is contained in Γ_n.

Of course, a similar statement holds even if $\gamma_\star$ is separating though we will not explicitly state it here. As for $c_\star$, we will provide a full description of μ_n in terms of the Pollicott-Ruelle resonant states of the geodesic flow of $(\Sigma_\star, g)$ for the resonance $h_\star$ in §4.7. Here as before $\Sigma_\star$ is the compact surface with boundary obtained by cutting Σ along $\gamma_\star$ (see §4.2.5).

Strategy of proof

A key ingredient used in the proof of Theorems 4.1.1, 4.1.2 and 4.1.4 is the scattering operator $\mathcal{S}(s) : C^\infty(\partial_\star) \to C^\infty(\partial_\star \setminus \Gamma)$ which is defined by

$$\mathcal{S}(s)f(z) = f(S(z))\mathrm{e}^{-s\ell(z)}, \quad z \in \partial_\star \setminus \Gamma, \quad s \in \mathbb{C}.$$

As a first step (which is of independent interest, see Corollary 4.3.3), we prove that for any $\chi \in C_c^\infty(\partial_\star \setminus S\gamma_\star)$, the family $s \mapsto \chi\mathcal{S}(s)\chi$ extends to a meromorphic family of operators $\mathcal{S}(s) : C^\infty(\partial_\star) \to \mathcal{D}'(\partial_\star)$ on the whole complex plane (here $\mathcal{D}'(\partial_\star)$ denotes the space of distributions on $\partial_\star$), whose poles are contained in the set of Pollicott–Ruelle resonances of the geodesic flow of the surface with boundary $(\Sigma_\star, g)$ (see §4.2.6 for the definition of those resonances). In this context, the existence of such resonances follows from the work of Dyatlov–Guillarmou [DG16], and we relate $\mathcal{S}(s)$ with the resolvent $(X + s)^{-1}$ of the geodesic flow (see Proposition 4.3.2). By using the microlocal structure of the resolvent of the geodesic flow provided by [DG16], we are moreover able to prove that the composition $(\chi\mathcal{S}(s)\chi)^n$ is well defined for any $n \geqslant 1$, as well as its super flat trace (meaning that we also look at the action of $\mathcal{S}(s)$ on differential forms, see §B.3.1) which reads

$$\mathrm{tr}_\mathrm{s}^\flat[(\chi\mathcal{S}(s)\chi)^n] = n \sum_{i(\gamma,\gamma_\star)=n} \frac{\ell^\sharp(\gamma)}{\ell(\gamma)} \mathrm{e}^{-s\ell(\gamma)} \prod_{z\in I_\star(\gamma)} \chi^2(z), \tag{4.1.2}$$

where the products runs over all closed geodesics (not necessarily primitive) γ with $\mathrm{i}(\gamma, \gamma_\star) = n$ and $\ell^\sharp(\gamma)$ is the primitive length of γ. This formula will be obtained by using the Atiyah-Bott trace formula [AB67] (though our scattering map S has singularities that we have to deal with). Furthermore, using *a priori* bounds on the growth of $N(n, L)$ (obtained in §4.4 by purely geometrical techniques coming from the theory of CAT(-1) spaces), we prove that $s \mapsto \mathrm{tr}_\mathrm{s}^\flat[(\chi\mathcal{S}(s)\chi)^n]$ has a pole of order n at $s = h_\star$, provided that χ has enough support. For this step, we crucially use the fact that the asymptotics for $N(0, L)$ is already known by [PP83, Dal99], although we could recover it by using the modern techniques introduced in [DG16] without going through the scattering maps. Finally, letting the support of $1 - \chi$ being very close to $S\gamma_\star$, and estimating the growth of geodesics intersecting n times $\gamma_\star$ with at least one small angle, we are able to derive Theorems 4.1.1 and 4.1.2 from a classical Tauberian theorem of Delange [Del54].

We emphasize on the fact that this strategy of proof follows exactly the method used in Chapter 3. We summarize the commonalities in the following tabular :

Toy model	Surfaces
G	$S\Sigma$
$G_\star$	$S\Sigma_\star$
$e_\star$	$S\Sigma\|_{\gamma_\star}$
$(z-A_\star)^{-1}$	$(X+s)^{-1}$
$\mathcal{S}_\star(z)$	$\mathcal{S}(s)$
$\widetilde{\mathcal{S}}_\star(z)$	$\widetilde{\mathcal{S}}(s)$
J	ψ^*
Lemma 3.2.3	(4.1.2)

Organization of the chapter

The chapter is organized as follows. In §4.2 we introduce some geometrical and dynamical tools. In §4.3 we introduce the dynamical scattering operator which is a central object in this paper and we compute its flat trace. In §4.4 we prove a priori bounds on $N(n, L)$. In §4.5 we use a Tauberian argument to estimate certain quantities. In §4.6 we prove Theorem 4.1.1. In §4.7 we prove an equidistribution result. In §4.8 we show that a typical closed geodesic γ satisfies $\mathrm{i}(\gamma, \gamma_\star) \approx I_\star \ell(\gamma)$ for some $I_\star > 0$. Finally in §4.9 we extend the results to the case where we are given more than one closed geodesic.

4.2 Geometrical preliminaries

We recall here some classical geometrical and dynamical notions, and introduce the Pollicott-Ruelle resonances that will arise in our situation. Throughout the whole article, (Σ, g) will denote a closed, connected, oriented Riemannian surface of negative curvature.

4.2.1 Geometric intersection numbers

For any two loops $\alpha, \beta : \mathbb{R}/\mathbb{Z} \to \Sigma$, the *geometric intersection number* between α and β is defined by

$$\mathrm{i}(\alpha, \beta) = \inf_{\alpha' \sim \alpha, \beta' \sim \beta} |\alpha \cap \beta|$$

where the infimum runs over all loops α' and β' freely homotopic to α and β respectively, and

$$|\alpha \cap \beta| = \{(\tau, \tau') \in (\mathbb{R}/\mathbb{Z})^2 \ : \ \alpha(\tau) = \beta(\tau')\}.$$

It is well known that in every non trivial free homotopy class of loops $\mathbf{c}$, there is a unique oriented closed geodesic $\gamma_{\mathbf{c}} \in \mathbf{c}$ which minimizes the length among curves in $\mathbf{c}$. In fact, closed geodesics also minimize intersection numbers, as follows.

Lemma 4.2.1. *Let γ_1, γ_2 be any two non trivial oriented closed geodesics, and assume that γ_1 (resp. γ_2) is not freely homotopic to a power of γ_2 (resp. γ_1). Then it holds*

$$\mathrm{i}(\gamma_1, \gamma_2) = |\gamma_1 \cap \gamma_2|.$$

The above result is rather classical but for reader's convenience we provide a proof in Appendix 4.10.

4.2.2 Structural equations

We recall here some classical facts from [ST76, §7.2] about geometry of surfaces. We have the Liouville one-form α on M defined by

$$\langle \alpha(z), w \rangle = \langle \mathrm{d}_{(x,v)}\pi(w), v\rangle, \quad z = (x,v) \in M, \quad w \in T_{(x,v)}M.$$

Then α is a contact form (that is, $\alpha \wedge \mathrm{d}\alpha$ is a volume form on M_δ) and it turns out that the geodesic vector field X is the Reeb vector field associated to α, that is, it satisfies

$$\iota_X \alpha = 1, \quad \iota_X \mathrm{d}\alpha = 0,$$

where ι denote the interior product. We set $\beta = R_{\pi/2}^* \alpha$ where for $\theta \in \mathbb{R}$, we denoted by $R_\theta : M \to M$ the rotation of angle θ in the fibers (which is defined thanks to the orientation of Σ). Then the volume form vol_g of Σ satisfies [ST76, p. 166]

$$\pi^* \mathrm{vol}_g = \alpha \wedge \beta. \tag{4.2.1}$$

We denote by ψ the connection one-form (see [ST76, Theorem p.169]), that is, the unique one-form on M satisfying

$$\iota_V \psi = 1, \quad \mathrm{d}\alpha = \psi \wedge \beta, \quad \mathrm{d}\beta = \alpha \wedge \psi, \quad \mathrm{d}\psi = -(\kappa \circ \pi)\alpha \wedge \beta, \tag{4.2.2}$$

where V is the vector field generating $(R_\theta)_{\theta \in \mathbb{R}}$ and κ is the Gauss curvature of Σ. Then (α, β, ψ) is a global frame of T^*M. We denote by H the vector field on M such that (X, H, V) is the dual frame of (α, β, ψ). We then have the following commutation relations [ST76, p. 170]

$$[V, X] = H, \quad [V, H] = -X, \quad [X, H] = (\kappa \circ \pi)V. \tag{4.2.3}$$

The orientation of M will be chosen so that (X, H, V) is positively oriented.

4.2.3 The Anosov property

It is known since the work of Anosov [Ano69] that the flow (φ_t) is hyperbolic, that is, for any $z \in M$, there is a $\mathrm{d}\varphi_t$-invariant splitting

$$T_z M = \mathbb{R}X(z) \oplus E_s(z) \oplus E_u(z)$$

which depends continuously on z, and with the following property. For any norm $\|\cdot\|$ on TM, there exist $C, \nu > 0$ such that

$$\|\mathrm{d}\varphi_t(z)v\| \leqslant C\mathrm{e}^{-\nu t}\|v\|, \quad v \in E_s(z), \quad t \geqslant 0, \quad z \in M,$$

and

$$\|\mathrm{d}\varphi_{-t}(z)v\| \leqslant C\mathrm{e}^{-\nu t}\|v\|, \quad v \in E_u(z), \quad t \geqslant 0, \quad z \in M$$

In fact $E_s(z) \oplus E_u(z) = \ker \alpha(z)$ and there exists two continuous functions $r_\pm : M \to \mathbb{R}$ such that $\pm r_\pm > 0$ and

$$E_s(z) = \mathbb{R}(H(z) + r_- V(z)), \quad E_u(z) = \mathbb{R}(H(z) + r_+ V(z)), \quad z \in M.$$

Moreover, the functions $r_\pm$ are differentiable along the flow direction, and they satisfy the Ricatti equation

$$Xr_\pm + r_\pm^2 + \kappa \circ \pi = 0,$$

where κ is the curvature of Σ.

We will denote by $T^*M = E_0^* \oplus E_s^* \oplus E_u^*$ the splitting defined by (here the bundle $\mathbb{R}X$ is denoted by E_0)

$$E_0^*(E_u \oplus E_s) = 0, \quad E_s^*(E_s \oplus E_0) = 0, \quad E_u^*(E_u \oplus E_0) = 0.$$

Then we have $E_0^* = \mathbb{R}\alpha$ and

$$E_s^* = \mathbb{R}(r_-\beta - \psi), \quad E_u^* = \mathbb{R}(r_+\beta - \psi). \tag{4.2.4}$$

Note that this decomposition does not coincide with the usual dual decomposition, but it is motivated by the fact that covectors in E_s^* (resp. E_u^*) are exponentially contracted in the future (resp. in the past) by the symplectic lift Φ_t of φ_t which is defined by

$$\Phi_t(z, \xi) = (\varphi_t(z),\ \mathrm{d}\varphi_t(z)^{-\top} \cdot \xi), \quad (z, \xi) \in T^*M, \quad t \in \mathbb{R}, \tag{4.2.5}$$

where $^{-\top}$ denotes the inverse transpose. We have the following lemma (see [DR20a, §3.2]).

Lemma 4.2.2. *If $t \neq 0$, we have $\iota_V \Phi_t(\beta) \neq 0$ and $\iota_H \Phi_t(\psi) \neq 0$.*

4.2.4 A nice system of coordinates

In what follows we denote

$$\partial_\star = \{(x, v) \in M \ : \ x \in \gamma_\star\} = S\Sigma|_{\gamma_\star}.$$

Lemma 4.2.3. *There exists a tubular neighborhood U of $\partial_\star$ in M and coordinates (τ, ρ, θ) on U with*

$$U \simeq (\mathbb{R}/\ell_\star\mathbb{Z})_\tau \times (-\delta, \delta)_\rho \times (\mathbb{R}/2\pi\mathbb{Z})_\theta,$$

where $\ell_\star$ is the length of $\gamma_\star$, and such that

$$|\rho(z)| = \mathrm{dist}_g(\pi(z), \gamma_\star), \quad S_z\Sigma = \{(\tau(z), \rho(z), \theta) \ : \ \theta \in \mathbb{R}/2\pi\mathbb{Z}\}, \quad z \in U.$$

Moreover in these coordinates, we have, on $\{\rho = 0\}$,

$$X = \cos(\theta)\partial_\tau + \sin(\theta)\partial_\rho, \quad H = -\sin(\theta)\partial_\tau + \cos(\theta)\partial_\rho, \quad V = \partial_\theta,$$

and

$$\alpha = \cos(\theta)\mathrm{d}\tau + \sin(\theta)\mathrm{d}\rho, \quad \beta = -\sin(\theta)\mathrm{d}\tau + \cos(\theta)\mathrm{d}\rho, \quad \psi = \mathrm{d}\theta.$$

Proof. For $\tau \in \mathbb{R}/\ell_\star\mathbb{Z}$ we set $(x_\tau, v_\tau) = \varphi_\tau(\gamma_\star(0), \dot\gamma_\star(0))$. We now define, for $\delta > 0$ small enough,

$$\Psi(\tau, \rho, \theta) = R_{\theta - \pi/2}\varphi_\rho(x_\tau, \nu(x_\tau)), \quad (\tau, \rho, \theta) \in \mathbb{R}/\ell_\star\mathbb{Z} \times (-\delta, \delta) \times \mathbb{R}/2\pi\mathbb{Z},$$

where $R_\eta : S\Sigma \to S\Sigma$ is the rotation of angle η and $\nu(x_\tau) = R_{\pi/2}v_\tau$. Then $\mathrm{d}\Psi(\tau, 0, \theta)$ is injective for any τ, θ. Indeed, we have $\partial_\tau(\pi \circ \Psi)(\tau, 0, \theta) = v_\tau$ and $\partial_\rho(\pi \circ \Psi)(\tau, 0, \theta) = \nu(x_\tau)$. Thus $\mathrm{d}\Psi(\tau, 0, \theta) : \mathbb{R}\partial_\tau \oplus \mathbb{R}\partial_\rho \to T\Sigma$ is injective. Moreover, $\partial_\theta(\pi \circ \Psi)(\tau, 0, \theta) = 0$ and $\partial_\theta\Psi(\tau, 0, \theta) = V(\Psi(\tau, 0, \theta)) \neq 0$. Thus $\mathrm{d}\Psi(\tau, 0, \theta)$ is injective for any τ, θ, and in particular, if $\delta > 0$ is small enough, $\Psi : U \to M$ is an immersion. In particular, since $(\tau, \theta) \mapsto \Psi(\tau, 0, \theta)$ is clearly injective, we obtain that $\Psi|_U$ is a diffeomorphism onto its image provided that δ is chosen small enough.

Because $V = \partial_\theta$ and $\iota_V\alpha = \iota_V\beta = 0$, we may write $\alpha(\tau, 0, \theta) = a(\tau, \theta)\mathrm{d}\tau + b(\tau, \theta)\mathrm{d}\rho$ and $\beta(\tau, 0, \theta) = a'(\tau, \theta)\mathrm{d}\tau + b'(\tau, \theta)\mathrm{d}\rho$ for some smooth functions a, a', b, b'. Now since $\mathrm{d}\alpha = \psi \wedge \beta$ we obtain $\mathcal{L}_V\alpha = \iota_V\mathrm{d}\alpha = \beta$, and similarly $\mathcal{L}_V\beta = -\alpha$. Thus we obtain $a' = \partial_\theta a$, $b' = \partial_\theta b$ and

$$\partial_\theta^2 a + a = 0, \quad \partial_\theta^2 b + b = 0.$$

In consequence we have $a(\tau, \theta) = a_1(\tau)\cos\theta + a_2(\tau)\sin\theta$ and $b(\tau, \theta) = b_1(\tau)\cos\theta + b_2(\tau)\sin\theta$ for some smooth functions a_1, a_2, b_1, b_2. Moreover, by definition of the coordinates (τ, ρ, θ), one has

$$X(\tau, 0, 0) = \partial_\tau \quad \text{and} \quad X(\tau, 0, \pi/2) = \partial_\rho. \tag{4.2.6}$$

Therefore $a_1 = b_2 = 1$ and $a_2 = b_1 = 0$. We thus get the desired formulas for α and β. Now writing $\psi = a''\mathrm{d}\tau + b''\mathrm{d}\rho + \mathrm{d}\theta$ and using $\mathcal{L}_V\psi = 0$, we obtain $\partial_\theta a'' = \partial_\theta b'' = 0$. As $\iota_X\psi = 0$ we obtain $a'' = b'' = 0$ by (4.2.6). The formulae for X, H, V follow. □

Remark 4.2.4. If $\tilde{\partial} = \{\rho = 0\}$, we get for any $z = (\tau, 0, \theta) \in \partial$

$$T_z\tilde{\partial} = \mathbb{R}V(z) \oplus \mathbb{R}(\cos(\theta)X(z) - \sin(\theta)H(z)), \quad N_z^*\tilde{\partial} = \mathbb{R}(\sin(\theta)\alpha(z) + \cos(\theta)\beta(z)).$$

4.2.5 Cutting the surface along $\gamma_\star$

As mentioned in the introduction, we may see $\Sigma \setminus \gamma_\star$ as the interior of a compact surface $\Sigma_\star$ with boundary consisting of two copies of $\gamma_\star$. By gluing two copies of the annulus U obtained in the preceding subsection on each component of the boundary of $\Sigma_\star$, we construct a slightly larger surface $\Sigma_\delta \supset \Sigma_\star$ whose boundary is identified with the boundary of U (see Figure 4.1).

Lemma 4.2.5. *The surface Σ_δ has strictly convex boundary, in the sense that the second fundamental form of the boundary $\partial\Sigma_\delta$ with respect to its outward normal pointing vector is strictly negative.*

Proof. In the coordinates defined (τ, ρ) given by Lemma 4.2.3, the metric g has the form

$$\mathrm{d}\rho^2 + f(\tau, \rho)\mathrm{d}\tau^2, \tag{4.2.7}$$

for some $f > 0$ satisfying $\partial_\rho f(\tau, 0) = 0$. Indeed, if ∇ is the Levi-Civita connexion, one has

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\langle\partial_\rho, \partial_\tau\rangle = \langle\nabla_{\partial_\rho}\partial_\rho, \partial_\tau\rangle + \langle\partial_\rho, \nabla_{\partial_\rho}\partial_\tau\rangle \;\; = \langle\partial_\rho, \nabla_{\partial_\tau}\partial_\rho\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\partial_\rho, \partial_\rho\rangle = 0,$$

since $\nabla_{\partial_\rho}\partial_\rho = 0$ (indeed, $\rho \mapsto (\tau, \rho)$ is a geodesic curve). Thus $\langle\partial_\tau, \partial_\rho\rangle = \langle\partial_\tau, \partial_\rho\rangle|_{\rho=0} = 0$, and in particular g has the form (4.2.7) with $f(\tau, \rho) = \langle\partial_\tau, \partial_\tau\rangle$, and we have

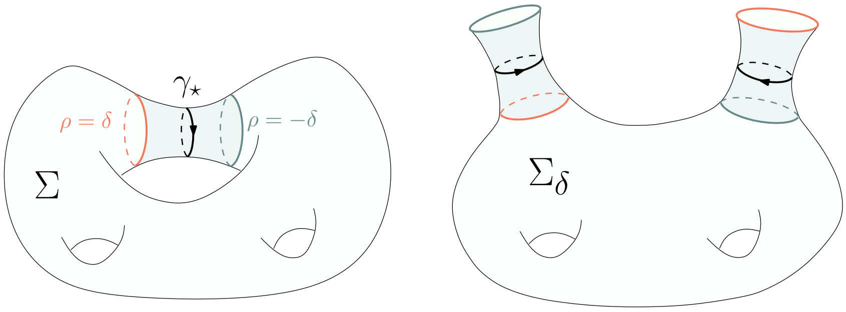

FIGURE 4.1 – The surfaces Σ (on the left) and Σ_δ (on the right), in the case where $\gamma_\star$ is not separating. In Σ, the darker region corresponds to the neighborhood $\pi(U)$ of $\gamma_\star$.

$\partial_\rho f(\tau, 0) = \partial_\rho \langle \partial_\tau, \partial_\tau \rangle = 2\partial_\tau \langle \partial_\rho, \partial_\tau \rangle|_{\rho=0} = 0$ (indeed, since $\tau \mapsto (\tau, 0)$ is a geodesic curve, we have $\nabla_{\partial_\tau} \partial_\tau = 0$ on $\{\rho = 0\}$). In those coordinates, the scalar curvature reads

$$\kappa(\tau, \rho) = -\partial_\rho^2 f(\tau, \rho)/f(\tau, \rho).$$

As $\kappa < 0$ we get $\partial_\rho^2 f > 0$, which gives $\pm \partial_\rho f > 0$ on $\{\pm \rho > 0\}$. The second fundamental form of $\partial \Sigma_\delta$ with respect to ∂_ρ is defined by

$$\langle \nabla_{\partial_\tau} \partial_\tau, \partial_\rho \rangle = -\partial_\rho f(\tau, \rho)/2,$$

which concludes the proof, since ∂_ρ is outward pointing (resp. inward pointing) on $\{\rho = \delta\}$ (resp. $\{\rho = -\delta\}$). □

Lemma 4.2.6. *In the coordinates given by Lemma 4.2.3, we have*

$$\pm X^2 \rho > 0 \quad \text{on} \quad \{\pm \rho > 0\}.$$

Proof. Using the fact that in the coordinates (τ, ρ) the metric g has the form (4.2.7), we get that the Christoffel symbols of g are given by

$$\Gamma^\rho_{\rho\rho} = \Gamma^\rho_{\tau\rho} = 0, \quad \Gamma^\rho_{\tau\tau} = -\partial_\rho f/2.$$

In particular, if $t \mapsto (\tau(t), \rho(t))$ is a geodesic path, we get

$$\ddot{\rho}(t) - \partial_\rho f(\tau(t), \rho(t))/2 = 0.$$

Because $\partial_\rho f(\tau, 0) = 0$ and $-\partial_\rho^2 f/f = \kappa < 0$ we obtain that $\pm \partial_\rho f > 0$ whenever $\pm \rho > 0$. This concludes the proof. □

4.2.6 The resolvent of the geodesic flow for open systems

In what follows, we denote by $\Omega^\bullet(M_\delta)$ the set of differential forms on M_δ and by $\Omega^\bullet_c(M_\delta)$ the elements of $\Omega^\bullet(M_\delta)$ whose support is contained in the interior of M_δ. Here $M_\delta = S\Sigma_\delta$ is the unit tangent bundle of Σ_δ. The set of currents on M_δ, denoted by

$\mathcal{D}'^\bullet(M_\delta)$ is defined as the topological dual of $\Omega_c^\bullet(M_\delta)$. Note that we have an inclusion $\Omega^\bullet(M_\delta) \hookrightarrow \mathcal{D}'^\bullet(M_\delta)$ via the pairing

$$\langle u, v\rangle = \int_{M_\delta} u \wedge v, \quad u, v \in \Omega^\bullet(M_\delta).$$

The geodesic flow φ on M induces a flow on $M_\delta = S\Sigma_\delta$ which we still denote by φ. We set

$$\partial_\pm M_\delta = \{(x,v) \in \partial M_\delta \ : \ \pm\langle v, \nu_\delta(x)\rangle > 0\}, \quad \partial_0 M_\delta = \{(x,v) \in \partial M_\delta \ : \ \pm\langle v, \nu_\delta(x)\rangle = 0\},$$

where $\nu_\delta(x)$ is the unit vector orthogonal to $\partial\Sigma_\delta$, based at x, and pointing outward. Next, define

$$\ell_{\pm,\delta}(z) = \inf\{t > 0 \ : \ \varphi_{\pm t}(z) \in \partial M_\delta\}, \quad z \in \mathrm{int}(M_\delta) \cup \partial_\mp M_\delta,$$

and $\ell_{\pm,\delta}(z) = 0$ for $z \in \partial_\pm M_\delta \cup \partial_0 M_\delta$, where $\mathrm{int}(M_\delta)$ denotes the interior of M_δ. The numbers $\ell_{\pm,\delta}(z)$ are the first exit times of z in the future and in the past. We also set

$$\Gamma_{\pm,\delta} = \{z \in M_\delta \ : \ \ell_\mp(z) = +\infty\}, \quad K_\delta = \Gamma_\delta^+ \cap \Gamma_\delta^-$$

and we define the operators $R_{\pm,\delta}(s)$ by

$$R_{\pm,\delta}(s)\omega(z) = \pm \int_0^{\ell_{\mp,\delta}(z)} \varphi_{\mp t}^* \omega(z) \mathrm{e}^{-ts} \mathrm{d}t, \quad z \in M_\delta, \quad \omega \in \Omega_c^\bullet(M_\delta), \tag{4.2.8}$$

which are well defined as operators $\Omega_c^\bullet(M_\delta) \to C(M_\delta, \wedge^\bullet T^* M_\delta)$ whenever $\mathrm{Re}(s) \gg 1$, where $C(M_\delta, \wedge^\bullet T^* M_\delta)$ denotes the space of continuous differential forms on M_δ. Note that our convention of $R_{\pm,\delta}(s)$ differs from that of [Gui17]. The operator $R_{+,\delta}(s)$ (resp. $R_{-,\delta}(s)$) is the resolvent of $\mathcal{L}_X$ in the future (resp. in the past) for the spectral parameter s. More precisely we have

$$(\mathcal{L}_X \pm s) R_{\pm,\delta}(s) = \mathrm{Id}_{\Omega_c^\bullet(M_\delta)}, \tag{4.2.9}$$

and for any $(u,v) \in \Omega_c^\bullet(M_\delta \setminus \Gamma_{-,\delta}) \times \Omega_c^\bullet(M_\delta \setminus \Gamma_{+,\delta})$ it holds

$$\int_{M_\delta} (R_{+,\delta}(s)u) \wedge v = -\int_{M_\delta} u \wedge R_{-,\delta}(s)v. \tag{4.2.10}$$

Indeed, for such u, v, there is $L > 0$ such that

$$\mathrm{supp}(u) \subset \{\ell_{+,\delta} \leqslant L\} \quad \text{and} \quad \mathrm{supp}(v) \subset \{\ell_{-,\delta} \leqslant L\}. \tag{4.2.11}$$

In particular, the forms $R_{+,\delta}(s)u$ and $R_{-,\delta}(s)v$ are smooth up to the boundary of M_δ. Indeed, (4.2.11) implies that for any $z \in M_\delta$ and $t \in [0, \ell_{-,\delta}(z)]$ we have

$$\varphi_{-t}^* u(z) \neq 0 \quad \Longrightarrow \quad t \leqslant L.$$

Therefore one gets for any $z \in M_\delta$

$$R_{+,\delta}(s)u(z) = \int_0^{\ell_{-,\delta}(z)} \varphi_{-t}^* u(z) \mathrm{e}^{-ts} \mathrm{d}t = \int_0^{\min(\ell_{-,\delta}(z), L+1)} \varphi_{-t}^* u(z) \mathrm{e}^{-ts} \mathrm{d}t,$$

and thus $R_{+,\delta}u$ is smooth since $\varphi_{-t}^* u(z) = 0$ if $L \leqslant t \leqslant \ell_{-,\delta}(z)$. Similarly, $R_{-,\delta}(s)v$ is smooth. Finally, note that we have the inclusions

$$\mathrm{supp}(R_{+,\delta}(s)u) \cap \partial M_\delta \subset \partial_+ M_\delta, \quad \mathrm{supp}(R_{-,\delta}(s)v) \cap \partial M_\delta \subset \partial_- M_\delta.$$

In particular, Stokes' formula and (4.2.9) imply (4.2.10).

Because the boundary of Σ_δ is strictly convex, it follows from [DG16, Proposition 6.1] that the family of operators $R_\pm(s)$ extends to a meromorphic family of operators

$$R_{\pm,\delta}(s) : \Omega_c^\bullet(M_\delta) \to \mathcal{D}'^\bullet(M_\delta),$$

satisfying

$$\mathrm{WF}'(R_{\pm,\delta}(s)) \subset \Delta(T^* M_\delta) \cup \Upsilon_{\pm,\delta} \cup (E_{\pm,\delta}^* \times E_{\mp,\delta}^*), \tag{4.2.12}$$

where $\Delta(T^* M_\delta)$ is the diagonal in $T^* M_\delta \times T^* M_\delta$,

$$\Upsilon_{\pm,\delta} = \{(\Phi_t(z,\xi),(z,\xi)) \in T^*(M_\delta \times M_\delta) \ : \ 0 \leqslant \pm t \leqslant \ell_{\pm,\delta}(z), \ \langle X(z), \xi \rangle = 0\},$$

and where

$$E_{+,\delta}^* = E_u^*|_{\Gamma_\delta^+}, \quad E_{-,\delta}^* = E_s^*|_{\Gamma_\delta^-}.$$

Here, we denoted

$$\mathrm{WF}'(R_{\pm,\delta}(s)) = \{(z,\xi,z',\xi') \in T^*(M_\delta \times M_\delta) \ : \ (z,\xi,z',-\xi') \in \mathrm{WF}(R_{\pm,\delta}(s))\},$$

where WF is the classical Hörmander wavefront set [Hör90, §8]. In fact, by (4.2.12), we mean that $s \mapsto R_\pm(s)$ is meromorphic as a map $\mathbb{C} \to \mathcal{D}'_{\mathbf{\Gamma}'_\pm}(M_\delta \times M_\delta)$ (we identify $R_\pm(s)$ and its Schwartz kernel) where $\mathbf{\Gamma}_\pm$ is given by the right hand side of (4.2.12), $\mathbf{\Gamma}'_\pm = \{(z,\xi,z',-\xi') \ : \ (z,\xi,z',-\xi') \in \mathbf{\Gamma}_\pm\}$, and where

$$\mathcal{D}'_{\mathbf{\Gamma}'_\pm}(M_\delta \times M_\delta) = \{R \in \mathcal{D}'(M_\delta \times M_\delta) \ : \ \mathrm{WF}(R) \subset \mathbf{\Gamma}'_\pm\}$$

is endowed with its natural topology (see [Hör90, Definition 8.2.2]).

Near any $s_0 \in \mathbb{C}$, we have the development

$$R_{\pm,\delta}(s) = Y_{\pm,\delta}(s) + \sum_{j=1}^{J(s_0)} \frac{(X \pm s_0)^{j-1} \Pi_{\pm,\delta}(s_0)}{(s-s_0)^j},$$

where $Y_{\pm,\delta}(s)$ is holomorphic near $s = s_0$, and $\Pi_{\pm,\delta}(s_0)$ is a finite rank projector satisfying

$$\mathrm{WF}'(\Pi_{\pm,\delta}(s_0)) \subset E_{\pm,\delta}^* \times E_{\mp,\delta}^*, \quad \mathrm{supp}(\Pi_{\pm,\delta}(s_0)) \subset \Gamma_\delta^\pm \times \Gamma_\delta^\mp,$$

where we identified $\Pi_{\pm,\delta}(s_0)$ and its Schwartz kernel.

4.2.7 Restriction of the resolvent on the geodesic boundary

For any $\varepsilon > 0$, define the open sets

$$A_{\pm,\varepsilon} = \{\ell_{\pm,\delta} > \varepsilon\} \cap \{\ell_{\mp,\delta} > 0\} \subset \mathrm{int}(M_\delta),$$

and notice that if ε is small we have $M_{\delta/2} \subset A_{\pm,\varepsilon}$. Then we have diffeomorphisms $\varphi_{\pm\varepsilon} : A_{\pm,\varepsilon} \to A_{\mp,\varepsilon}$ which induce maps

$$\varphi^*_{\pm\varepsilon} : \mathcal{D}'^{\bullet}(A_{\mp,\varepsilon}) \to \mathcal{D}'^{\bullet}(A_{\pm,\varepsilon}).$$

Using a slight abuse of notation, we will still denote by $\varphi^*_{\pm\varepsilon} : \mathcal{D}'^{\bullet}(M_\delta) \to \mathcal{D}'^{\bullet}(A_{\pm,\varepsilon})$ the composition of $\varphi^*_{\pm\varepsilon}$ with the inclusion $\mathcal{D}'^{\bullet}(M_\delta) \hookrightarrow \mathcal{D}'^{\bullet}(A_{\mp,\varepsilon})$ (which is given by the restriction). Let

$$\partial = \partial(S\Sigma_\star) = \{(x,v) \in M_\delta \ : \ x \in \gamma_\star \sqcup \gamma_\star\},$$

and $\partial_0 = S\gamma_\star \sqcup S\gamma_\star \subset \partial$.

Lemma 4.2.7. *For any $\varepsilon > 0$ small enough, we have*

$$\mathrm{WF}(\varphi^*_{\mp\varepsilon} R_{\pm,\delta}(s)) \cap N^*(\partial \times \partial) = \emptyset,$$

where

$$N^*(\partial \times \partial) = \{(z',\xi',z,\xi) \in T^*(M_\delta \times M_\delta) \ : \ \langle \xi', \ T_{z'}\partial\rangle = \langle \xi, T_z\partial\rangle = 0\}.$$

Proof. We prove the statement for $R_{+,\delta}(s)$. By (4.2.12) and multiplicativity of wavefront sets (see [Hör90, Theorem 8.2.14]), we have

$$\mathrm{WF}'(\varphi^*_{-\varepsilon} R_{+,\delta}(s)) \subset \Delta_\varepsilon \cup \Upsilon^\varepsilon_{+,\delta} \cup (E^*_{+,\delta} \times E^*_{-,\delta}), \tag{4.2.13}$$

where

$$\Delta_\varepsilon = \{(\Phi_\varepsilon(z,\xi),(z,\xi)) \ : \ (z,\xi) \in T^*M_\delta\}$$

and

$$\Upsilon^\varepsilon_{+,\delta} = \{(\Phi_t(z,\xi),(z,\xi)) \ : \ \varepsilon \leqslant t \leqslant \ell_{+,\delta}(z), \ \langle X(z),\xi\rangle = 0\}.$$

Now assume that there is $\Xi = (z',\xi',z,\xi)$ lying in

$$N^*(\partial \times \partial) \cap \left(\Delta_\varepsilon \cup \Upsilon^\varepsilon_{+,\delta} \cup (E^*_{+,\delta} \times E^*_{-,\delta})\right).$$

If $\Xi \in \Delta_\varepsilon$, then necessarily we have $z, z' \in \partial_0$, because $\varphi_\varepsilon(\partial \setminus \partial_0) \cap \partial = \emptyset$ whenever $\varepsilon > 0$ is smaller than the injectivity radius of the manifold [1]. We thus have $\xi \in N^*_z\partial = \mathbb{R}\beta(z)$ by Remark 4.2.4 ; now $\Phi_\varepsilon(\beta(z))$ does not lie in $\mathbb{R}\beta(\varphi_\varepsilon(z))$ by Lemma 4.2.2, and therefore $\xi = 0$.

If $\Xi \in \Upsilon^\varepsilon_{+,\delta}$, then there is $T \geqslant \varepsilon$ such that $\Phi_T(z,\xi) = (z',\xi')$ with $\langle \xi, X(z)\rangle = 0$. However by Remark 4.2.4, if $(z,\xi) \in N^*_z\partial$ and $\langle \xi, X(z)\rangle = 0$ then $z \in \partial_0$. Thus by what precedes, we obtain $\xi = 0$.

1. Indeed, let $x \in \partial\Sigma$. If $(x,v) \in \partial \setminus \partial_0$ satisfies that $(y,w) = \varphi_\varepsilon(x,v) \in \partial$, then the exponential map at x is not injective on the closed ball $B(0,\varepsilon) \subset T_x\Sigma$ of radius ε since we have $\pi(\varphi_{\varepsilon'}(x,v')) = y$ for some $v' \in S_x\Sigma$ tangent to $\partial\Sigma$ and some $\varepsilon' \in [0,\varepsilon]$, as it follows from the fact that $\partial\Sigma$ is totally geodesic.

Finally, (4.2.4) and Remark 4.2.4 imply that $N^*\partial \cap E^*_{\pm,\delta} \subset \{0\}$. Thus we showed that $\mathrm{WF}'(\varphi^*_{-\varepsilon} R_{+,\delta}(s)) \cap N^*(\partial \times \partial) = \emptyset$, which is equivalent to the conclusion of the lemma[2]. □

Remark 4.2.8. This estimate together with [Hör90, Theorem 8.2.4], imply that the operator $\iota^* \iota_X \varphi^*_{\mp\varepsilon} R_{+,\delta}(s)\iota_*$ is well defined and satisfies

$$\mathrm{WF}\left(\iota^* \iota_X \varphi^*_{\mp\varepsilon} R_{+,\delta}(s)\iota_*\right) \subset \mathrm{d}(\iota \times \iota)^\top\, \mathrm{WF}\left(\varphi^*_{\mp\varepsilon} R_{+,\delta}(s)\right)$$

where $\iota : \partial \hookrightarrow M_\delta$ and $\iota \times \iota : \partial \times \partial \hookrightarrow M_\delta \times M_\delta$ are the inclusions. Indeed, the Schwartz kernel of $\iota^* \iota_X \varphi^*_{\mp\varepsilon} R_{+,\delta}(s)\iota_*$ coincides with the pullback by $\iota \times \iota$ of the kernel of $\iota_X \varphi^*_{\mp\varepsilon} R_{+,\delta}(s)$. It also follows from [Hör90, Theorem 8.2.14] that the operator $\iota^* \iota_X \varphi^*_{\mp\varepsilon} R_{+,\delta}(s)$ maps

$$\mathcal{D}'^{\bullet}_{N^*\partial}(M_\delta) \to \mathcal{D}'^{\bullet}(\partial)$$

continuously.

Here the pushforward $\iota_* : \Omega^\bullet(\partial) \to \mathcal{D}'^{\bullet+1}(M_\delta)$ is defined as follows. If $u \in \Omega^k(\partial)$, we define the current $\iota_* u \in \mathcal{D}'^{k+1}(M_\delta)$ by

$$\langle \iota_* u, v\rangle = \int_\partial u \wedge \iota^* v, \quad v \in \Omega^{n-k-1}(M_\delta).$$

4.3 The scattering operator

In this section we introduce the dynamical scattering operator $\mathcal{S}_\pm(s)$ associated to our problem. By relating the scattering operator to the resolvent described above, we are able to compute its wavefront set. In consequence we obtain that the composition $(\chi \mathcal{S}_\pm(s))^n$ is well defined for $\chi \in C^\infty_c(\partial \setminus \partial_0)$, and we give a formula for its flat trace.

For each $x \in \partial\Sigma_\star$, let $\nu(x)$ be the normal outward pointing vector to the boundary of $\Sigma_\star$, and set

$$\partial_\pm = \{(x, v) \in \partial \ : \ \pm\langle \nu(x), v\rangle_g > 0\}.$$

4.3.1 First definitions

We define the exit times in the future and in the past by

$$\ell_\pm(z) = \inf\{t > 0 \ : \ \varphi_{\pm t}(z) \in \partial\}, \quad z \in M \setminus (\partial_\pm \cup \partial_0),$$

and we declare that $\ell_\pm(z) = \infty$ whenever $z \in \partial_\pm \cup \partial_0$. Then we set

$$\Gamma_\pm = \{z \in M \ : \ \ell_\mp(z) = +\infty\}.$$

The set Γ_+ (resp. Γ_-) is the set of points of M which are trapped in the past (resp. in the future). The scattering map $S_\pm : \partial_\mp \setminus \Gamma_\mp \to \partial_\pm \setminus \Gamma_\pm$ is defined by

$$S_\pm(z) = \varphi_{\pm \ell_\pm(z)}(z), \quad z \in \partial_\mp \setminus \Gamma_\mp,$$

2. Indeed, since the set $\{(z, \xi, z', \xi') \ : \ (z, \xi, z', -\xi') \in N^*(\partial \times \partial)\}$ coincides with $N^*(\partial \times \partial)$, we may use WF or WF′ indifferently.

and satisfies $S_\pm \circ S_\mp = \mathrm{Id}_{\partial_\pm \setminus \Gamma_\pm}$. For $s \in \mathbb{C}$, the scattering operator

$$\mathcal{S}_\pm(s) : \Omega^\bullet_c(\partial_\mp \setminus \Gamma_\mp) \to \Omega^\bullet_c(\partial_\pm \setminus \Gamma_\pm)$$

is given by

$$\mathcal{S}_\pm(s)\omega = (S^*_\mp \omega)\mathrm{e}^{-s\ell_\mp(\cdot)}, \quad \omega \in \Omega^\bullet_c(\partial_\mp \setminus \Gamma_\mp).$$

Remark 4.3.1. If $\mathrm{Re}(s)$ is large enough, $\mathcal{S}_\pm(s)$ extends as a map

$$C^0(\partial, \wedge^\bullet T^*\partial) \to C^0(\partial, \wedge^\bullet T^*\partial)$$

(here $C^0(\partial, \wedge^\bullet T^*\partial)$ is the space of continuous forms on ∂), by declaring that

$$\mathcal{S}_\pm(s)\omega(z) = S^*_\mp \omega(z)\mathrm{e}^{-s\ell_\mp(z)} \quad \text{if } z \in \partial_\pm \setminus \Gamma_\pm$$

and $\mathcal{S}_\pm(s)\omega(z) = 0$ otherwise. Indeed, by Lemma 4.3.9 below and (4.3.16) there is $C > 0$ such that

$$\|S^*_\mp \omega(z)\| \leqslant C\mathrm{e}^{C\ell_\mp(z)}\|\omega\|_\infty, \quad z \in \partial_\pm \setminus \Gamma_\pm, \quad \omega \in \Omega^\bullet(M),$$

where $\|\omega\|_\infty$ is the uniform norm on $C^0(M, \wedge^\bullet T^*M)$.

4.3.2 The scattering operator via the resolvent

In this paragraph we will see that $\mathcal{S}_\pm(s)$ can be computed in terms of the resolvent. More precisely, we have the following result.

Proposition 4.3.2. *For any* $\mathrm{Re}(s)$ *large enough we have*

$$\mathcal{S}_\pm(s) = (-1)^N \mathrm{e}^{\pm\varepsilon s}\iota^*\iota_X\varphi^*_{\mp\varepsilon}R_{\pm,\delta}(s)\iota_*$$

as maps $\Omega^\bullet_c(\partial \setminus \partial_0) \to \mathcal{D}'^\bullet(\partial)$, *where* $N : \Omega^\bullet(\partial) \to \mathbb{N}$ *is the degree operator, that is,* $N(w) = k$ *if* w *is a* k*-form.*

As a consequence of this proposition and Remark 4.2.8 together with the continuity of the pullback [Hör90, Theorem 8.2.4]

$$(\iota \times \iota)^* : \mathcal{D}'^\bullet_{\mathbf{\Gamma}_{\pm,\varepsilon}}(M_\delta \times M_\delta) \to \mathcal{D}'^\bullet(\partial \times \partial),$$

where $\mathbf{\Gamma}_{\pm,\varepsilon}$ is the right hand side of (4.2.13), we get the

Corollary 4.3.3. *The scattering operator* $s \mapsto \mathcal{S}_\pm(s) : \Omega^\bullet(\partial \setminus \partial_0) \to \mathcal{D}'^\bullet(\partial)$ *extends as a meromorphic family of* $s \in \mathbb{C}$ *with poles of finite rank, with poles contained in the set of Pollicott-Ruelle resonances of* $\mathcal{L}_X$*, that is, the set of poles of* $s \mapsto R_{\pm,\delta}(s)$*.*

Before proving Proposition 4.3.2, we start by an intermediate result.

Lemma 4.3.4. *We have* $\mathcal{S}_\pm(s) = (-1)^N \mathrm{e}^{\pm\varepsilon s}\iota^*\iota_X\varphi^*_{\mp\varepsilon}R_{\pm,\delta}(s)\iota_*$ *as maps*

$$\Omega^\bullet_c(\partial_\mp \setminus \Gamma_\mp) \to \mathcal{D}'^\bullet(\partial_\pm \setminus \Gamma_\pm).$$

Remark 4.3.5. (i) Note that Proposition 4.3.2 is not a direct consequence of Lemma 4.3.4. Indeed, the operator $\mathcal{Q}_{\varepsilon,\pm}(s) = (-1)^N \mathrm{e}^{\pm\varepsilon s}\iota^*\iota_X\varphi_{\mp\varepsilon}^* R_{\pm,\delta}(s)\iota_*$ could hide some singularities near $\Gamma_\pm$; Proposition 4.3.2 tells us that is it not the case, at least far from ∂_0.

(ii) A consequence of Proposition 4.3.2 is that $\mathcal{Q}_{\varepsilon,\pm}(s)$ is identically zero on $\partial_\pm$ (in the sense that $\mathcal{Q}_{\varepsilon,\pm}(s)u = 0$ whenever $\mathrm{supp}(u) \subset \partial_\pm$), as it is the case for $\mathcal{S}_\pm(s)$. This can be seen directly from using the fact that

$$\mathrm{supp}(\varphi_{\mp\varepsilon}^* R_{\pm,\delta}(s)\iota_* u) \subset \{\varphi_t(z) \ : \ z \in \mathrm{supp}(u),\ \varepsilon \leqslant \pm t \leqslant \ell_{\pm,\delta}(z)\}.$$

Proof. Let $u \in \Omega_c^\bullet(\partial_- \setminus \Gamma_-)$, and $U' \subset \partial_-$ be a neighborhood of $\mathrm{supp}\, u$ such that $\overline{U}'$ does not intersect ∂_0. Let $\varepsilon > 0$ small such that

$$z \in \partial_- \implies \ell_+(z) > \varepsilon.$$

The existence of such an ε follows from the fact that for each $x \in \partial\Sigma$, the exponential map $\exp_x : T_x\Sigma \to \Sigma$ is injective on $B(0,\varepsilon) \subset T_x\Sigma$ whenever $\varepsilon > 0$ is small enough (independent of x). Note also that for every $z \in \partial_-$, we have

$$\pi(\varphi_t(z)) \in \Sigma_\delta \setminus \Sigma_\star, \quad -\ell_{-,\delta}(z) < t < 0,$$

as it follows from Lemma 4.2.6. Next, let us set

$$U = \{(t,z) \in \mathbb{R} \times U' \ : \ -\ell_{-,\delta}(z) < t < \varepsilon\}.$$

Then U is diffeomorphic to a tubular neighborhood of U' in M_δ via $(t,z) \mapsto \varphi_t(z)$ [3]. Let $\chi \in C^\infty(\mathbb{R})$ such that $\chi \equiv 1$ near $]-\infty, 0]$ and $\chi \equiv 0$ on $]\varepsilon/2, +\infty[$. Set, in the above coordinates,

$$\psi(t,z) = \chi(t)\mathrm{e}^{-ts}u(z) \in \wedge^\bullet T^*_{(t,z)}M_\delta,$$

where we see $u(z)$ as a form in $T^*_{(t,z)}M$ by declaring $\iota_{\partial_t}u(z) = 0$. We extend ψ by 0 on M and we set

$$\phi = \psi - R_{+,\delta}(s)(\mathcal{L}_X + s)\psi.$$

Then ϕ is smooth by (4.2.8) since $\mathrm{supp}\,\psi \cap \Gamma_- = \emptyset$. Moreover $(\mathcal{L}_X + s)\phi = 0$, and we have

$$\phi|_{\partial_-} = u, \quad \phi|_{\partial_+} = \mathcal{S}_+(s)u,$$

where $\mathcal{S}_+(s) = \mathcal{S}_+(s)|_{\Omega_c^\bullet(\partial_-\setminus\Gamma_-)}$. Let $h \in \Omega_c^\bullet(M_\delta \setminus \Gamma_{+,\delta})$, so that $R_{-,\delta}(s)h$ is smooth

3. Indeed, the map $G : (t,z) \mapsto \varphi_t(z)$ is clearly smooth on U. By lemma 4.2.6, we have that $t \mapsto \rho(\varphi_t(z))$ is strictly increasing for $z \in \partial_-$. Therefore by unicity of the integral curves of X we see that G is injective. The inverse of G is given by $G^{-1}(z') = (t(z'), z(z'))$ where $t(z') = \inf\{t \geqslant 0 \ : \ \varphi_t(z') \in \partial\}$ and $z(z') = \varphi_{-t(z')}(z')$, which is smooth on $G(U)$ by the implicit function theorem.

(see the discussion following (4.2.10)). We have, by (4.2.9) and (4.2.10),

$$\begin{aligned}\int_{M_\delta} \phi \wedge h &= \int_{M_\delta} \psi \wedge h - \int_{M_\delta} R_{+,\delta}(s)(\mathcal{L}_X + s)\psi \wedge h \\ &= \int_{M_\delta} \psi \wedge h + \int_{M_\delta} (\mathcal{L}_X + s)\psi \wedge R_{-,\delta}(s)h \\ &= \int_{M_\delta} \psi \wedge h - \int_{M_\delta} \psi \wedge (\mathcal{L}_X - s)R_{-,\delta}(s)h + \int_{\partial M_\delta} \iota_X \left(\psi \wedge R_{-,\delta}(s)h\right) \\ &= \int_{\partial M_\delta} \iota_X \left(\psi \wedge R_{-,\delta}(s)h\right) \\ &= (-1)^{\deg \psi} \int_{\partial_{-,\delta}} \psi \wedge \iota_X R_{-,\delta}(s)h,\end{aligned}$$

since $\iota_X \psi = 0$ and ψ has no support near $\partial_{+,\delta}$. Now we let $\Phi : \partial_- \to \partial_{-,\delta}$ be defined by $\Phi(z) = \varphi_{-\ell_{-,\delta}(z)}(z)$. Assume that the support of h does not intersect U. Then a change of variable gives

$$\Phi^*(\iota_X R_{-,\delta}(s)h)|_{\partial_{-,\delta}} = \iota_X R_{-,\delta}(s)h\mathrm{e}^{-s\ell_{-,\delta}(\cdot)},$$

As we have $\Phi^*(\psi|_{\partial_{-,\delta}}) = (\psi|_{\partial_-})\mathrm{e}^{+s\ell_{-,\delta}(\cdot)} = u\mathrm{e}^{+s\ell_{-,\delta}(\cdot)}$ by definition of ψ, we obtain

$$\int_{M_\delta} \phi \wedge h = (-1)^{\deg u} \int_{\partial_-} u \wedge \iota^*(\iota_X R_{-,\delta}(s)h). \tag{4.3.1}$$

Now because $(\mathcal{L}_X - s)R_{-,\delta}(s)h = h$, we get $(\mathcal{L}_X - s)R_{-,\delta}(s)h = 0$ near U and thus $\varphi_\varepsilon^* R_{-,\delta}(s)h = \mathrm{e}^{\varepsilon s}R_{-,\delta}(s)h$ near U. Let $v \in \Omega_c^\bullet(\partial_+ \setminus \Gamma_+)$; then $\overline{U} \cap \mathrm{supp}(v) = \emptyset$ (because $\mathrm{supp}(v) \subset \partial_+ \setminus \Gamma_+$). As $\mathrm{WF}(\iota_* v) \subset N^*\partial$, we may find $h_n \in \Omega_c^\bullet(M_\delta \setminus \Gamma_{+,\delta})$, $n \in \mathbb{N}$, such that $h_n \to \iota_* v$ in $\mathcal{D}'^\bullet_{N^*\partial}(M_\delta)$, and with the property that $\mathrm{supp}(h_n) \cap \overline{U} = \emptyset$ [4]. Then applying (4.3.1) to $h = h_n$ and letting $n \to \infty$ yields [5]

$$\int_{\partial_+} (\mathcal{S}_+(s)u) \wedge v = (-1)^{\deg u}\mathrm{e}^{-\varepsilon s} \int_{\partial_-} u \wedge \iota^* \iota_X \varphi_\varepsilon^* R_{-,\delta}(s)\iota_* v,$$

because $\phi|_{\partial_+} = \mathcal{S}_+(s)u$. Since $\int_{\partial_+} \mathcal{S}_+(s)u \wedge v = \int_{\partial_-} u \wedge \mathcal{S}_-(s)v$, we obtain

$$\mathcal{S}_-(s) = (-1)^{\deg u}\mathrm{e}^{-\varepsilon s}\iota^*\iota_X \varphi_\varepsilon^* R_{-,\delta}(s)\iota_*$$

as maps $\Omega_c^\bullet(\partial_+ \setminus \Gamma_+) \to \Omega_c^\bullet(\partial_- \setminus \Gamma_-)$. We can replace X by $-X$ to obtain the desired formula for $\mathcal{S}_+(s)$, which concludes. □

Proof of Proposition 4.3.2. Let $u \in \Omega^\bullet(\partial \setminus \partial_0)$ and write $u = u(\tau, \theta) \in T^*_{(\tau,\theta)}\partial$. Let $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\int_{\mathbb{R}} \chi = 1$, $\chi(0) \neq 0$, $\chi \equiv 0$ on $\mathbb{R} \setminus (-\delta/2, \delta/2)$, and $\chi > 0$ on $(-\delta/2, \delta/2)$. For $n \in \mathbb{N}_{\geqslant 1}$ we set $\chi_n = n\chi(n\cdot)$, so that χ_n converges to the Dirac measure on $\mathbb{R}$ as $n \to +\infty$. We define $u_n \in \Omega_c^\bullet(M_\delta)$ in the (τ, ρ, θ) coordinates by

$$u_n = \chi_n(\rho)u(\tau, \theta) \wedge \mathrm{d}\rho.$$

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4. For example, we may take $h_n(\rho, \tau, \theta) = \chi_n(\rho)v(\tau, \theta) \wedge \mathrm{d}\rho$ where $\chi_n \in C_c^\infty(]-\delta, \delta[)$ converges to the Dirac measure.

5. Here we use that $\iota^*\iota_X\varphi_\varepsilon^* R_{-,\delta}(s)h_n \to \iota^*\iota_X\varphi_\varepsilon^* R_{-,\delta}(s)\iota_* v$ in $\mathcal{D}'^\bullet(\partial)$ as $n \to \infty$ by Remark 4.2.8 since $h_n \to \iota_* v$ in $\mathcal{D}'^\bullet_{N^*\partial}(M_\delta)$.

Then $u_n \to (-1)^N \iota_* u$ in $\mathcal{D}'_{N^*\partial}(M_\delta)$ since $\partial = \{\rho = 0\}$. In particular, setting

$$f_n = \iota^* \varphi^*_{-\varepsilon} \iota_X R_{+,\delta}(s) u_n, \quad n \geqslant 1,$$

Remark 4.2.8 gives that $f_n \to (-1)^N \iota^* \varphi^*_{-\varepsilon} \iota_X R_{+,\delta}(s) \iota_* u$ in $\mathcal{D}'^{\bullet}(\partial)$. Moreover, if $\mathrm{Re}(s)$ is large enough, then for any $n \in \mathbb{N}$, we have $(-1)^N \iota^* \varphi^*_{-\varepsilon} \iota_X R_{+,\delta}(s) u_n \in C^0(M_\delta, \wedge^\bullet T^* M_\delta)$ and thus $f_n \in C^0(\partial, \wedge^\bullet T^* \partial)$. Then we claim that $f_n \to \mathcal{S}_+(s) u$ in $\mathcal{D}'^{\bullet}(\partial \setminus \partial_0)$ when $n \to +\infty$, where we recall that

$$\mathcal{S}_+(s) u(z) = \begin{cases} S_-^* u(z) \mathrm{e}^{-s\ell_-(z)} & \text{if} \quad z \in \partial_+ \setminus \Gamma_+, \\ 0 & \text{if} \quad \text{not}. \end{cases}$$

Let $F = \{|\rho| \leqslant \delta/2\}$. Since the neighborhood $\{|\rho| < \delta/2\}$ is strictly convex, there exists $L > 0$ such that for any $z \in F$ and $T > 0$ such that $\varphi_{-T}(z) \in F$, we have

$$\Big(\forall t \in \,]0, T[\,, \ \varphi_{-t}(z) \notin F\Big) \quad \Longrightarrow \quad T \geqslant L. \tag{4.3.2}$$

Next, take $z \in \partial_+ \setminus \Gamma_+$. Then the set $\{t \in [\varepsilon, \ell_{-,\delta}(z)] \ : \ \varphi_{-t}(z) \in F\}$ is a finite union of closed intervals, say

$$\{t \geqslant \varepsilon \ : \ \varphi_t(z) \in F\} = \bigcup_{k=0}^{K(z)} [a_k(z), b_k(z)],$$

with $a_k(z) \leqslant b_k(z) \leqslant +\infty$ and $b_k(z) < a_{k+1}(z)$ for every k. We set $\rho(t) = \rho(\varphi_{-t}(z))$ for any $t \geqslant 0$, and we take any smooth norm $\|\cdot\|$ on $\wedge^\bullet T^* M_\delta$. Note that $u_n = \chi_n(\rho) u_1$. Moreover, if $z \in M_\delta$ and $t < \ell_{-\delta}(z)$, we have

$$\|\varphi^*_{-t} u_1(z)\| \leqslant C \|u_1(\varphi_{-t} z)\| \exp(C|t|) \tag{4.3.3}$$

for some $C > 0$. Let $\theta_0 > 0$ small and $h \in C^\infty(M_\delta, [0,1])$ such that $h = 1$ on $\operatorname{supp} u_1$ and

$$h(\tau, \rho, \theta) = 0, \quad \operatorname{dist}(\theta, \pi\mathbb{Z}) < \theta_0 \tag{4.3.4}$$

(such a h exists if θ_0 is small enough since $u \in \Omega^\bullet(\partial \setminus \partial_0)$). Then there is $c = c(\theta_0) > 0$ such that $|X\rho| \geqslant c$ on $\operatorname{supp} h$, as it follows from Lemma 4.2.3. In particular if $\mathrm{Re}(s) > C$ we have by (4.3.3) and (4.3.4)

$$\begin{aligned} \|f_n(z)\| &\leqslant \int_\varepsilon^{\ell_{-,\delta}(z)} (\chi_n \circ \rho)(\varphi_{-t}(z)) \|\varphi^*_{-t}(\iota_X u_1)(z)\| \mathrm{e}^{-ts} \mathrm{d}t \\ &\leqslant C \|u\|_\infty \sum_{k=0}^{K(z)} \mathrm{e}^{(C-s)a_k(z)} \int_{a_k(z)}^{b_k(z)} \chi_n(\rho(t)) h(\varphi_{-t}(z)) \mathrm{d}t \\ &\leqslant C c^{-1} \|u\|_\infty \sum_{k=0}^{K(z)} \mathrm{e}^{(C-s)a_k(z)} \int_{a_k(z)}^{b_k(z)} \chi_n(\rho(t)) |X\rho(\varphi_{-t}(z))| \mathrm{d}t. \end{aligned}$$

Of course, for $t < \ell_{-,\delta}(z)$, we have $X\rho(\varphi_{-t}(z)) = \rho'(t)$. Moreover by Lemma 4.2.6 we have $\pm X^2 \rho > 0$ if $\pm \rho > 0$. Thus we may separate each interval $[a_k(z), b_k(z)]$ into two subintervals on which $|\rho'| > 0$ and change variables to get

$$\int_{a_k(z)}^{b_k(z)} \chi_n(\rho(t)) |\rho'(t)| \mathrm{d}t \leqslant 2 \int_{\mathbb{R}} \chi_n(\rho) \mathrm{d}\rho \leqslant 2.$$

By (4.3.2), we have $a_k(z) \geqslant kL$ for any k. Therefore we obtain

$$\|f_n(z)\| \leqslant \frac{2\|u\|_\infty}{1-\mathrm{e}^{(C-\mathrm{Re}(s))L}}, \quad z \in \partial_+ \setminus \Gamma_+, \quad n \geqslant 1. \tag{4.3.5}$$

Moreover, if $z \in \partial_-$, we have that $t \mapsto \rho(\varphi_{-t}(z))$ is strictly increasing for any $z \in \partial_-$ by Lemma 4.2.6. Thus we may reproduce the argument made above to obtain that (4.3.5) also holds for $z \in \partial_-$. Finally, it is shown in [Gui17, §2.4] that $\mathrm{Leb}(\Gamma_+ \cap \partial_+) = 0$ [6]. In particular, since each f_n is a continuous, (4.3.5) holds for any $z \in (\partial_+ \cup \partial_-) \setminus \Gamma_+ = \partial$.

Next, let $v \in \Omega^\bullet(\partial)$. By Lemma 4.2.6, the set $\{\varphi_{-t}(z) \ : \ t \geqslant \varepsilon\}$ is included in $\{\rho \geqslant \rho(\varphi_{-\varepsilon}(z))\}$ for any $z \in \partial_-$. In particular, as $\mathrm{supp}(u_n) \to \partial$ when $n \to \infty$, we have $f_n(z) \to 0$ for $z \in \partial_-$. By dominated convergence we get as $n \to \infty$

$$\int_{\partial_-} f_n \wedge v \to 0.$$

Next, let $\eta > 0$, and $\chi_\pm \in C_c^\infty(\partial_\pm \setminus \Gamma_\pm)$ such that

$$\chi_- \equiv 1 \text{ on } \mathrm{supp}(\chi_+ \circ S_+) \quad \text{and} \quad \mathrm{vol}(\mathrm{supp}(1-\chi_+)) < \eta. \tag{4.3.6}$$

Such functions exist as $\mathrm{Leb}(\Gamma_+ \cap \partial) = 0$. We have

$$\int_{\partial_+} f_n \wedge v = \int_{\partial_+} \chi_+ f_n \wedge v + \int_{\partial_+} (1-\chi_+) f_n \wedge v.$$

Note that on $\mathrm{supp}\,\chi_+$, we have $f_n = \tilde{f}_n$ where $\tilde{f}_n$ is defined exactly as f_n, replacing u by $\tilde{u} = \chi_- u \in \Omega^\bullet(\partial_- \setminus \Gamma_-)$. By Lemma 4.3.4 we have $\mathcal{Q}_{\varepsilon,+}(s)\tilde{u} = \mathcal{S}_+(s)\tilde{u}$, and since $\tilde{f}_n \to \mathcal{Q}_{\varepsilon,+}(s)\tilde{u}$ we have

$$\int_{\partial_+} \chi_+ f_n \wedge v = \int_{\partial_+} \chi_+ \tilde{f}_n \wedge v \to \int_{\partial_+} \chi_+ \mathcal{S}_+(s)\tilde{u} \wedge v = \int_{\partial_+} \chi_+ \mathcal{S}_+(s) u \wedge v$$

where we used that $\mathcal{S}_+(s)u = \mathcal{S}_+(s)\tilde{u}$ on $\mathrm{supp}\,\chi_+$. On the other hand, as the forms f_n are uniformly bounded by (4.3.5) and the discussion below, there is $C > 0$ such that for any $n \geqslant 1$

$$\left| \int_{\partial_+} (1-\chi_+)\mathcal{S}_+(s) u \wedge v \right| < C\eta \quad \text{and} \quad \left| \int_{\partial_+} (1-\chi_+) f_n \wedge v \right| < C\eta,$$

where we used the second part of (4.3.6). Summarizing the above facts, we obtain that for $n \geqslant 1$ big enough, one has

$$\left| \int_{\partial} f_n \wedge v - \int_{\partial} \mathcal{S}_+(s) u \wedge v \right| \leqslant 4C\eta.$$

Thus $f_n \to \mathcal{S}_+(s)u$ in $\mathcal{D}'^\bullet(\partial)$, which concludes the proof. □

6. Actually, [Gui17, §2.4] says that $\mathrm{Leb}(\Gamma_{+,\delta} \cap \partial_{+,\delta}) = 0$. However the map $J_\delta : z \mapsto \varphi_{\ell_{+,\delta}(z)}(z)$ realizes a local diffeomorphism $\partial_+ \to J_\delta(\partial_{+,\delta})$, and we have $J_\delta(\Gamma_+) \subset \Gamma_{+,\delta}$.

4.3.3 Composing the scattering maps

Recall that ∂ has two connected components $\partial^{(1)}$ and $\partial^{(2)}$ that we can identify in a natural way. We denote by $\psi : \partial \to \partial$ the map exchanging those components via this identification (in particular $\psi(\partial_\pm) = \partial_\mp$), and we set

$$\tilde{\mathcal{S}}_\pm(s) = \psi^* \circ \mathcal{S}_\pm(s).$$

Also we denote by $\Psi = T^*\partial \to T^*\partial$ the symplectic lift of ψ to $T^*\partial$, that is

$$\Psi(z,\xi) = (\psi(z), \mathrm{d}\psi_z^{-\top}\xi), \quad (z,\xi) \in T^*\partial.$$

Lemma 4.3.6. *Let $\chi \in C_c^\infty(\partial \backslash \partial_0)$. Then for any $n \geqslant 1$, the composition $\left(\chi\tilde{\mathcal{S}}_\pm(s)\chi\right)^n$, which is well defined $C^0(\partial, \wedge^\bullet T^*\partial) \to C^0(\partial, \wedge^\bullet T^*\partial)$ for $\mathrm{Re}(s)$ large and holomorphic with respect to s by Remark 4.3.1, admits a meromorphic continuation as a family of operators $\Omega^\bullet(\partial) \to \mathcal{D}'^\bullet(\partial)$.*

Proof. We prove the lemma for $\mathcal{S}_+(s)$. First, assume that $n = 2$. According to [Hör90, Theorem 8.2.14], it suffices to show that $A_1 \cap B_1 = \emptyset$, where for $n \geqslant 1$ we set

$$\begin{aligned} A_n &= \left\{(z,\xi) \ : \ \exists z' \in \partial, \ (z',0,z,\xi) \in \mathrm{WF}'\left((\chi\tilde{\mathcal{S}}_\pm(s))^n\right)\right\}, \\ B_n &= \left\{(z,\xi) \ : \ \exists z' \in \partial, \ (z,\xi,z',0) \in \mathrm{WF}\left((\chi\tilde{\mathcal{S}}_\pm(s))^n\right)\right\}. \end{aligned} \tag{4.3.7}$$

By Proposition 4.3.2, we have

$$\mathrm{WF}'(\chi\mathcal{S}_+(s)\chi)|_{\mathrm{supp}(\chi\times\chi)} \subset \mathrm{d}(\iota\times\iota)^\top\left(\Delta_\varepsilon \cup \Upsilon^\varepsilon_{+,\delta} \cup (E^*_{+,\delta} \times E^*_{\mp,\delta})\right), \tag{4.3.8}$$

where Δ_ε and $\Upsilon^\varepsilon_{+,\delta}$ are defined in the proof of Lemma 4.2.7. Note that in the coordinates of Lemma 4.2.3, we have $\iota(z) = (\tau, 0, \theta) \in \partial$ for any $z = (\tau,\theta) \in \partial$ and thus

$$\mathrm{d}\iota^\top(z,\eta) = \eta_\tau \mathrm{d}\tau + \eta_\theta \mathrm{d}\theta, \quad \eta = \eta_\tau\mathrm{d}\tau + \eta_\rho \mathrm{d}\rho + \eta_\theta\mathrm{d}\theta \in T_z^*M.$$

As χ is supported far from ∂_0, we have $(\varphi_\varepsilon(z'), z') \notin \partial\times\partial$ for any $z' \in \operatorname{supp}\chi$ (see for example Lemma 4.2.6), and for any $\eta \in T^*_{z'}M_\delta$ such that $\langle X(z'), \eta\rangle = 0$, we have

$$\mathrm{d}\iota^\top(z',\eta) = 0 \implies \eta = 0 \tag{4.3.9}$$

by Lemma 4.2.3 since $\partial_0 = \{(\tau,0,\theta) \ : \ \theta \in \pi\mathbb{Z}\}$. This implies that A_1 is contained in $E^*_{-,\partial}$ while B_1 is contained in $\Psi(E^*_{+,\partial})$, where $E^*_{+,\partial} = (\mathrm{d}\iota)^\top(E^*_{+,\delta})$. Now we claim that $\Psi(E^*_{+,\partial}) \cap E^*_{-,\partial} \subset \{0\}$ far from ∂_0. By Lemma 4.2.3 and §4.2.3 one has, for any $z = (\tau,0,\theta) \in \partial^{(j)} \cap \Gamma_\pm$,

$$E^*_{+,\partial}(z) = \mathbb{R}(\mathrm{d}\iota)_z^\top(r_+(z)\beta(z) - \psi(z)) = \mathbb{R}(-\sin(\theta)r_+(z)\mathrm{d}\tau - \mathrm{d}\theta),$$

since $\iota(\tau,\theta) = (\tau,0,\theta)$. Then $r_+(\psi(z)) \neq r_-(z)$ for all z. Indeed, the contrary would mean that $E_s(z') \cap E_u(z') \neq \{0\}$ for some $z' \in M$ (represented by both z and $\psi(z)$ in M_δ), which is not possible. Now we have $\sin(\theta) \neq 0$ for $z \notin \partial_0$. As a consequence (4.3.7) is true, since $\operatorname{supp}\chi \cap \partial_0 = \emptyset$. This concludes the case $n = 2$, and by [Hör90, Theorem 8.2.14] we also have the bound

$$\mathrm{WF}'\big((\chi\tilde{\mathcal{S}}_+(s)\chi)^2\big) \subset \Big(\mathrm{WF}'(\chi\tilde{\mathcal{S}}_+(s)\chi) \circ \mathrm{WF}'(\chi\tilde{\mathcal{S}}_+(s)\chi)\Big) \cup (B_1 \times \underline{0}) \cup (\underline{0} \times A_1),$$

where $\underline{0}$ denote the zero section in $T^*\partial$, with $A_1 \subset E^*_{-,\partial}$ and $B_1 \subset \Psi(E^*_{+,\partial})$. Note that if we set

$$E^*_{s,\partial_\pm} = \mathrm{d}\iota^\top(E^*_s|_{\partial_\pm}) \quad \text{and} \quad E^*_{u,\partial_\pm} = \mathrm{d}\iota^\top(E^*_u|_{\partial_\pm}),$$

we have $A_1 \subset E^*_{s,\partial_-}$ and $B_1 \subset \Psi(E^*_{u,\partial_+}) = E^*_{u,\partial_-}$.

Next, we proceed by induction and we assume that for some $n \geqslant 2$, the composition $(\chi\tilde{\mathcal{S}}_\pm(s))^n$ is well defined with the bound

$$\begin{aligned} &\mathrm{WF}'\big((\chi\tilde{\mathcal{S}}_+(s))^n\big) \\ &\quad\subset \Big(\mathrm{WF}'\big(\chi\tilde{\mathcal{S}}_+(s)\chi)^{n-1}\big) \circ \mathrm{WF}'\big(\chi\tilde{\mathcal{S}}_+(s)\chi\big)\Big) \cup (B_{n-1} \times \underline{0}) \cup (\underline{0} \times A_1), \end{aligned} \tag{4.3.10}$$

and that $A_{n-1} \subset E^*_{s,\partial_-}$ and $B_{n-1} \subset E^*_{u,\partial_-}$. This formula implies that the set A_n is included in

$$\Big\{(z,\xi) \in T^*\partial \ : \exists z', z'' \in \partial,\ (z', 0, z'', -\eta) \in \mathrm{WF}\big((\chi\tilde{\mathcal{S}}_+(s)\chi)^{n-1}\big) \\ \text{and } (z'', \eta, z, \xi) \in \mathrm{WF}(\chi\tilde{\mathcal{S}}_+(s)\chi)\Big\} \cup A_1.$$

We have $A_{n-1} \subset E^*_{s,\partial_-}$, and note that $\Psi(E^*_{+,\partial}) \subset E^*_{u,\partial_-}$ and $E^*_{u,\partial_-} \cap E^*_{s,\partial_-} = \{0\}$. Moreover, as mentioned above, we have $\varphi_\varepsilon(z') \notin \partial$ whenever $z' \in \mathrm{supp}(\chi)$. Thus we obtain by (4.3.8)

$$A_n \subset \Big\{(z,\xi) \ : \ (z'', \eta, z, \xi) \in \mathrm{d}(\iota \times \iota)^\top(\Upsilon^\varepsilon_{+,\delta}) \text{ for some } \eta \in \Psi(E^*_{s,\partial_-})\Big\} \cup A_1.$$

Now suppose $(z'', \eta, z, \xi) \in \mathrm{d}(\iota \times \iota)^\top(\Upsilon^\varepsilon_{+,\delta})$ with $z'', z \in \mathrm{supp}\,\chi$. Note that we have $\Psi(E^*_{s,\partial_-}) = E^*_{s,\partial_+}$ and thus, if $\eta \in \Psi(E^*_{s,\partial_-}) \cap \mathrm{d}\iota(z'')^\top \ker X(z'')$, then $\eta = \mathrm{d}\iota(z'')^\top \tilde{\eta}$ for some $\tilde{\eta} \in E^*_s(z'')$ by (4.3.9). Since E^*_s is preserved by Φ_{-t}, we obtain $(z,\xi) \in \mathrm{d}\iota^\top(E^*_s)$. In particular, this yields $A_n \subset E^*_{s,\partial_-}$. Reversing the roles of $(\chi\tilde{\mathcal{S}}_+(s))^{n-1}$ and $\chi\tilde{\mathcal{S}}_+(s)$ in (4.3.10), we get that B_n is included in

$$\Big\{(z,\xi) \in T^*\partial \ : \exists z', z'' \in \partial,\ (z, \xi, z', -\eta) \in \mathrm{WF}(\chi\tilde{\mathcal{S}}_+(s)\chi) \\ \text{and } (z', \eta, z'', 0) \in \mathrm{WF}\big((\chi\tilde{\mathcal{S}}_+(s)\chi)^{n-1}\big)\Big\} \cup B_1.$$

Proceeding as above, one gets $B_n \subset E^*_{u,\partial_-}$. Finally, we have $B_n \cap A_1 = \emptyset$, since $E^*_{u,\partial_-} \cap E^*_{s,\partial_-}$ on $\mathrm{supp}\,\chi$ by (4.3.9). As a consequence, we obtain that the composition $\big(\chi\tilde{\mathcal{S}}_+(s)\chi\big)^{n+1} = \big(\chi\tilde{\mathcal{S}}_+(s)\chi\big)^n \circ \big(\chi\tilde{\mathcal{S}}_+(s)\chi\big)$ is well defined by [Hör90, Theorem 8.2.14], and that (4.3.10) holds with n replaced by $n+1$. This concludes the proof. □

Remark 4.3.7. Using inductively (4.3.10), one can actually show that the set $\mathrm{WF}'\big((\chi\tilde{\mathcal{S}}_+(s)\chi)^n\big)$ is contained in $\mathrm{d}(\hat{\iota} \times \hat{\iota})^\top \tilde{\mathbf{\Gamma}}_{\varepsilon,+}$, where

$$\tilde{\mathbf{\Gamma}}_{\varepsilon,+} = \Big\{\Big(\hat{\Phi}_t(z,\xi), (z,\xi)\Big) \ : z, \hat{\varphi}_t(z) \in S\Sigma|_{\gamma_\star} \cap \hat{\iota}(\mathrm{supp}\,\chi), \\ \langle X(z), \xi\rangle = 0,\ t \geqslant \varepsilon\Big\} \cup (E^*_u \times E^*_s)\,|_{\mathrm{supp}(\chi\times\chi)}.$$

Here (and only here), in order to avoid confusion, we denoted by $\hat{\varphi}$ (resp. $\hat{\Phi}_t$) the complete geodesic flow on $M = S\Sigma$ (resp. the symplectic lift of the geodesic flow on T^*M), and by $\hat{\iota} : \partial \to S\Sigma|_{\gamma_\star} \hookrightarrow M$ the identification of both components of ∂.

4.3.4 The flat trace of the scattering operator

Let $A : \Omega^\bullet(\partial) \to \mathcal{D}'^\bullet(\partial)$ be an operator such that $\mathrm{WF}'(A) \cap \Delta(T^*\partial) = \emptyset$, where $\Delta(T^*\partial)$ is the diagonal in $T^*(\partial \times \partial)$. Then by [Hör90, Theorem 8.2.4] the pull-back $\iota_\Delta^* K_A$ is well defined, where $\iota_\Delta : z \mapsto (z,z)$ is the diagonal inclusion and $K_A \in \mathcal{D}'^3(\partial \times \partial)$ is the Schwartz kernel of A, defined by

$$\int_\partial A(u) \wedge v = \int_{\partial\times\partial} K_A \wedge \pi_1^* u \wedge \pi_2^* v, \quad u, v \in \Omega^\bullet(\partial),$$

where $\pi_j : \partial \times \partial \to \partial$ is the projection on the j-th factor ($j = 1, 2$). We then define the (super) flat trace of A by

$$\mathrm{tr}_\mathrm{s}^\flat A = \langle \iota_\Delta^* K_A, 1\rangle.$$

In fact, it is not hard to see that

$$\mathrm{tr}_\mathrm{s}^\flat(A) = \sum_{k=0}^{2} (-1)^k \mathrm{tr}^\flat(A_k), \tag{4.3.11}$$

where $\mathrm{tr}^\flat$ is the transversal trace of Attiyah-Bott [AB67] and A_k is the operator

$$A_k : C^\infty\big(\partial, \wedge^k T^*\partial\big) \to \mathcal{D}'\big(\partial, \wedge^k T^*\partial\big)$$

induced by A on the space of k-forms (see also [DZ16, §2.4] for an introduction to the flat trace).

The purpose of this section is to compute the flat trace of $\mathcal{S}_\pm(s)$. In what follows, for any closed geodesic $\gamma : \mathbb{R}/\ell\mathbb{Z} \to \Sigma$, we will denote

$$I_\star(\gamma) = \{z \in S\Sigma|_{\gamma_\star} \ : \ z = (\gamma(\tau), \dot\gamma(\tau)) \text{ for some } \tau \in \mathbb{R}/\ell\mathbb{Z}\}$$

the set of incidence vectors of γ along $\gamma_\star$, and

$$I_{\star,\pm}(\gamma) = p_\star^{-1}(I_\star(\gamma)) \cap \partial_\mp$$

where $p_\star : S\Sigma_\star \to S\Sigma$ is the natural projection.

Proposition 4.3.8. *Let $\chi \in C_c^\infty(\partial \setminus \partial_0)$. For any $n \geqslant 1$, the operator $(\chi\tilde{\mathcal{S}}_\pm(s))^n$ has a well defined flat trace and for $\mathrm{Re}(s)$ big enough we have*

$$-\,\mathrm{tr}_\mathrm{s}^\flat\big((\chi\tilde{\mathcal{S}}_\pm(s)\chi)^n\big) = n \sum_{i(\gamma,\gamma_\star)=n} \frac{\ell^\sharp(\gamma)}{\ell(\gamma)} \mathrm{e}^{-s\ell(\gamma)} \left(\prod_{z\in I_{\star,\pm}(\gamma)} \chi^2(z)\right)^{\ell(\gamma)/\ell^\sharp(\gamma)}, \tag{4.3.12}$$

where the sum runs over all closed geodesics γ of (Σ, g) (not necessarily primitive) such that $\mathrm{i}(\gamma,\gamma_\star) = n$. Here $\ell(\gamma)$ is the length of γ and $\ell^\sharp(\gamma)$ its primitive length.

This formula should be compared with the formula

$$\mathrm{tr}_\mathrm{s}^\flat\big((\chi f^*\chi)^n\big) = \sum_{\gamma\in\mathrm{Per}_n(f)} m^\sharp(\gamma)\,\mathrm{sgn}(\det(1-P_\gamma)) \left(\prod_{z\in\gamma} \chi^2(z)\right)^{n/m^\sharp(\gamma)}$$

which is valid for any smooth Anosov diffeomorphism $f : Z \to Z$ of a closed manifold Z and $\chi \in C^\infty(Z)$. Here $f^* : C^\infty(Z) \to C^\infty(Z)$ is the pull-back operator, $\mathrm{Per}_n(f)$ is the set of n-periodic orbits of f, $m^\sharp(\gamma)$ is the minimal period of γ and P_γ is the linearized Poincaré map of γ (that is, $P_\gamma = \mathrm{d}f^n(z)$ for $z \in \gamma$). Note that the above sum is finite, unlike the sum in (4.3.12). This is due to the fact that $S_\pm$ is singular at $\Gamma_\pm$, which allows $S_\pm$ to have an infinite number of n-periodic points.

Proof. The proof that the intersection

$$\mathrm{WF}'\big((\chi\tilde{\mathcal{S}}_\pm(s)\chi)^n\big) \cap \Delta(T^*\partial) \tag{4.3.13}$$

is empty follows from the estimate given in Remark 4.3.7, since $E_u^* \cap E_s^* = \{0\}$ and $\mathrm{d}\hat{\iota}(z)^\top$ is injective $\ker X(\hat{\iota}(z)) \to T_z^*\partial$ for any $z \in \mathrm{supp}(\chi)$.

For any $n \geqslant 1$ we define the set $\tilde{\Gamma}_\pm^n \subset \partial$ by

$$\complement\tilde{\Gamma}_\pm^n = \{z \in \partial \ : \ (\tilde{S}_\pm)^k(z) \text{ is well defined for } k = 1, \dots, n\},$$

where $\tilde{S} = \psi \circ S$. Equivalently, we have

$$\tilde{\Gamma}_\pm^1 = \Gamma_\pm \quad \text{and} \quad \tilde{\Gamma}_\pm^{n+1} = \tilde{\Gamma}_\pm^n \cap (\tilde{S}_\mp)^n\big(\Gamma_\pm \setminus \tilde{\Gamma}_\mp^n\big)$$

for $n \geqslant 1$. Also we set

$$\tilde{\ell}_{\pm,n}(z) = \ell_\pm(z) + \ell_\pm(\tilde{S}_\pm(z)) + \cdots + \ell_\pm(\tilde{S}_\pm^{n-1}(z)), \quad z \in \complement\tilde{\Gamma}_\pm^n, \tag{4.3.14}$$

where $\ell_\pm(z) = \inf\{t > 0 \ : \ \varphi_{\pm t}(z) \in \partial\}$, with the convention that $\tilde{\ell}_{\pm,n}(z) = +\infty$ if $z \in \tilde{\Gamma}_\pm^n$. We will need the following

Lemma 4.3.9. *Let $n \geqslant 1$. For any $k \geqslant 1$, there exists $C_{k,n} > 0$ such that*

$$\|\mathrm{d}^k\ell_{\pm,n}(z)\| \leqslant C_{k,n}\exp(C_{k,n}\ell_{\pm,n}(z)), \quad z \in \complement\tilde{\Gamma}_\pm^n.$$

Proof. By induction on n, using (4.3.14) and the fact that $S_\pm(\complement\tilde{\Gamma}_\pm^n) = \complement\tilde{\Gamma}_\pm^{n-1}$, we see that the lemma reduces to proving the estimate

$$\|\mathrm{d}^k\ell_\pm(z)\| \leqslant C_k\exp(C_k\ell_\pm(z)), \quad z \in \complement\tilde{\Gamma}_\pm^1. \tag{4.3.15}$$

In what follows, C_k is a constant depending only on k, which may change at each line. First, notice that $\|\mathrm{d}^k\varphi_t(z)\| \leqslant C_k\mathrm{e}^{C_k|t|}$ for any $t \in \mathbb{R}$ and $z \in M_\delta$ such that $\varphi_t(z) \in M_\delta$, for some constant C_k (see for example [Bon15, Proposition A.4.1]). Moreover, we have

$$\mathrm{d}S_\pm(z) = \mathrm{d}[\varphi_{\ell_\pm(z)}](z) + X(S_\pm(z))\mathrm{d}\ell_\pm(z), \quad z \in \complement\tilde{\Gamma}_\pm^1.$$

By induction we obtain that for any k

$$\|\mathrm{d}^kS_\pm(z)\| \leqslant C_k\exp(C_k\ell_\pm(z)) + C_k\sum_{j=1}^k\|\mathrm{d}^j\ell_\pm(z)\|^{m_j}, \quad m_j \in \mathbb{N}, \quad j = 1, \dots, k, \tag{4.3.16}$$

for any $z \in \complement\tilde{\Gamma}^1_\pm$. Let (τ, ρ, θ) be the coordinates defined near ∂ given by Lemma 4.2.3. Then $\rho(S_\pm(z)) = 0$ for $z \in \tilde{\Gamma}^\pm_1$ and thus

$$(X\rho)(S_\pm(z))\mathrm{d}\ell_\pm(z) = -\mathrm{d}\rho(S_\pm(z)) \circ \mathrm{d}[\varphi_{\ell_\pm(z)}](z), \quad z \in \complement\tilde{\Gamma}^1_\pm. \tag{4.3.17}$$

Let $z \notin \tilde{\Gamma}^\pm_1$; Lemma 4.2.3 gives

$$(X\rho)(S_\pm(z)) = \sin\big(\theta(S_\pm(z))\big). \tag{4.3.18}$$

Set $z' = S_\pm(z)$, and write $(\tau(t), \rho(t)) = \pi(\varphi_{\mp t}(z'))$, so that $\rho(0) = 0$. By the proof of Lemma 4.2.6, we have that $t \mapsto |\rho(t)|$ is strictly increasing (indeed $z \notin \tilde{\Gamma}^\pm_1$ and thus $\dot\rho(0) = \pm X\rho(z') \neq 0$) and whenever $|\rho(t)| \leqslant \delta/2$ it holds

$$\ddot\rho(t) = G(\tau(t), \rho(t)) \tag{4.3.19}$$

for some smooth function $G \in C^\infty((\mathbb{R}/\ell_\star\mathbb{Z})_\tau \times [-\delta/2, \delta/2]_\rho)$ satisfying $G(\tau, 0) = 0$ and $\partial_\rho G(\tau, \rho) > 0$. If $D = \sup|\partial_\rho G|$, we have $|G(\tau, \rho)| \leqslant D|\rho|$ and thus $|\ddot\rho(t)| \leqslant D|\rho(t)|$, with $\rho(0) = \ddot\rho(0) = 0$ and $\dot\rho(0) = \pm X\rho(S_\pm(z))$. By comparing the solution of (4.3.19) with the solutions of $\ddot y(t) = Dy(t)$, we obtain

$$|\rho(t)| \leqslant |X\rho(z')|\,\mathrm{sh}(Dt).$$

In particular we have $|\rho(t)| < \delta/2$ whenever $|X\rho(S_\pm(z))|\,\mathrm{sh}(Dt) < \delta/2$, and thus $\mathrm{sh}(D\ell_\mp(z')) \geqslant \delta/2|X\rho(z')|$. By (4.3.18), we conclude that there is $C > 0$ such that

$$\big|\sin\big(\theta(S_\pm(z))\big)\big| \geqslant C\exp(-C\ell_\pm(z)), \quad z \in \complement\tilde{\Gamma}^1_\pm. \tag{4.3.20}$$

We therefore obtain for any $z \in \tilde{\Gamma}^\pm_1$,

$$\begin{aligned}\|\mathrm{d}\ell_\pm(z)\| &\leqslant C^{-1}\exp(C\ell_\pm(z))\|\mathrm{d}\rho(S_\pm(z))\|\|\mathrm{d}[\varphi_{\ell_\pm(z)}](z)\| \\ &\leqslant C\mathrm{e}^{C\ell_\pm(z)}.\end{aligned}$$

Now, using repetively (4.3.16), (4.3.17) and (4.3.20), we obtain (4.3.15) by induction on k. □

Consider $\widetilde\chi \in C^\infty(\mathbb{R}, [0,1])$ such that $\widetilde\chi \equiv 1$ on $]-\infty, 1]$ and $\widetilde\chi \equiv 0$ on $[2, +\infty[$, and set $\widetilde\chi_L(z) = \widetilde\chi(\ell_{\pm,n}(z) - L)$ for $z \in \partial$. Then $\widetilde\chi_L \in C^\infty_c(\partial \setminus \tilde{\Gamma}^n_\pm)$ and by (7.2.3) we see that the Atiyah-Bott trace formula [AB67, Corollary 5.4] reads in our case

$$\langle \iota^*_\Delta K_{\chi,\pm,n}(s), \widetilde\chi_L\rangle = -\sum_{(\tilde S_\mp)^n(z)=z} \mathrm{e}^{-s\ell_{\pm,n}(z)}\widetilde\chi_L(z)\prod_{k=0}^{n-1}\chi^2\big((\tilde S_\mp)^k(z)\big), \tag{4.3.21}$$

where $K_{\chi,\pm,n}(s)$ is the Schwartz kernel of $(\chi\mathcal{S}_\pm(s))^n$. Indeed, a simple computation (for example in the spirit of [DZ16, Appendix B] [7]) shows that for any diffeomorphism $f : \partial \to \partial$ with isolated nondegenerate fixed points, it holds

$$\mathrm{tr}^\flat(F_k) = \sum_{f(z)=z}\frac{\mathrm{tr}\wedge^k\mathrm{d}f(z)}{|\det(1-\mathrm{d}f(z))|} \tag{4.3.22}$$

7. Actually in the aforementioned reference the authors deal with flows, but the diffeomorphism case is even simpler.

where $F_k : \Omega^k(\partial) \to \Omega^k(\partial)$ is defined by $F_k\omega = f^*\omega$ and $\wedge^k \mathrm{d}f(z)$ is the map induced by $\mathrm{d}f(z)$ on $\wedge^k T_z^*\partial$. Since $\sum_k (-1)^k \mathrm{tr}(\wedge^k \mathrm{d}f(z)) = \det(1-\mathrm{d}f(z))$ it holds

$$\mathrm{tr}_s^\flat(F) = \sum_k (-1)^k \mathrm{tr}^\flat(F_k) = \sum_{f(z)=z} \operatorname{sgn}\det(1-\mathrm{d}f(z)). \tag{4.3.23}$$

Now note that $\widetilde{\chi}_L(\chi\tilde{\mathcal{S}}_\pm(s)\chi)^n$ is by definition the operator given by

$$\omega \;\mapsto\; \widetilde{\chi}_L(\cdot)\left(\prod_{k=1}^n \left(\chi\circ\left(\tilde{S}_\mp\right)^k\right)\left(\chi\circ\left(\tilde{S}_\mp\right)^{k-1}\right)\right)\mathrm{e}^{-s\ell_{\pm,n}(\cdot)}\left(\tilde{S}_\mp\right)^{n*}w. \tag{4.3.24}$$

Moreover, $\operatorname{sgn}\det\left(1-\mathrm{d}\left(\tilde{S}_\mp\right)^n(z)\right) = -1$ for any z such that $\left(\tilde{S}_\mp\right)^n(z) = z$. Indeed, for such a z, $\mathrm{d}\left(\tilde{S}_\mp\right)^n(z)$ is conjugated to the linearized Poincaré map

$$P_z = \mathrm{d}(\varphi_{\ell_{\pm,n}(z)})(z)|_{E^u(z)\oplus E^s(z)},$$

which satisfies $\det(1-P_z) < 0$ as the matrix of P_z in the decomposition $E^u(z)\oplus E^s(z)$ reads $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ for some $\lambda > 1$ (since φ_t preserves the volume form $\alpha\wedge\mathrm{d}\alpha$). Finally, by (4.3.13), the pairing in the left hand side of (4.3.21) is well defined ; morever the proof of (4.3.22) can be revisited for the operator (4.3.24) thanks to the introduction of our cutoff functions $\tilde{\chi}_L$ and χ, yielding (4.3.21).

As $L \to +\infty$, the right hand side of (4.3.21) converges to

$$n\sum_{i(\gamma,\gamma_\star)=n} \frac{\ell^\sharp(\gamma)}{\ell(\gamma)}\mathrm{e}^{-s\ell(\gamma)}\left(\prod_{z\in I_{\star,\pm}(\gamma)}\chi^2(z)\right)^{\ell(\gamma)/\ell^\sharp(\gamma)},$$

since for any closed geodesic $\gamma : \mathbb{R}/\mathbb{Z} \to \Sigma$ such that $\mathrm{i}(\gamma,\gamma_\star) = n$ we have

$$\sharp\{z\in\partial \;:\; z = (\gamma(\tau),\gamma'(\tau)) \text{ for some } \tau\} = n\ell^\sharp(\gamma)/\ell(\gamma).$$

Note that the sum converges whenever $\mathrm{Re}(s)$ is large enough by Margulis' asymptotic formula given in the introduction. It remains to see that $\langle i_\Delta^* K_{\chi,\pm,n}(s), 1-\widetilde{\chi}_L\rangle \to 0$ as $L \to +\infty$. Note that Lemma 4.3.9 gives

$$\left\|\mathrm{d}^k\widetilde{\chi}_L\right\| \leqslant C_k\mathrm{e}^{C_k L}. \tag{4.3.25}$$

By Remark 4.3.1, if $s_0 > 0$ is large enough, one has $\mathcal{S}_\pm(s_0) : \Omega^\bullet(\partial) \to C^0(\partial, \wedge^\bullet T^*\partial)$. Also for any $s\in\mathbb{C}$ with $\mathrm{Re}(s) > 0$ we have

$$\mathcal{S}_\pm(s_0+s)w = (\mathcal{S}_\pm(s_0)w)\mathrm{e}^{-s\ell_\pm(\cdot)}, \quad w\in\Omega^\bullet(\partial). \tag{4.3.26}$$

Let $N\in\mathbb{N}$ such that $\iota_\Delta^* K_{\chi,\pm,n}(s_0)$ extends as a continuous linear form on $C^N(\partial)$. Then applying Lemma 4.3.9, we see that if $\mathrm{Re}(s)$ is large enough, the function $\exp(-s\ell_{\pm,n}(\cdot))$ lies in $C^N(\partial)$. Thus the product $\mathrm{e}^{-s\ell_{\pm,n}(\cdot)}\iota_\Delta^* K_{\chi,\pm,n}(s_0)$ is well defined and by (4.3.25) we have

$$\begin{aligned}\left|\left\langle \mathrm{e}^{-s\ell_{\pm,n}(\cdot)}\iota_\Delta^* K_{\chi,\pm,n}(s_0), (1-\widetilde{\chi}_L)\right\rangle\right| &= \left|\left\langle \iota_\Delta^* K_{\chi,\pm,n}(s_0), (1-\widetilde{\chi}_L)\mathrm{e}^{-s\ell_{\pm,n}(\cdot)}\right\rangle\right| \\ &\leqslant C\left\|(1-\widetilde{\chi}_L)\mathrm{e}^{-s\ell_{\pm,n}(\cdot)}\right\|_{C^N(\partial)} \\ &\leqslant C_N\mathrm{e}^{(C_N-\mathrm{Re}(s))L},\end{aligned}$$

since $\ell_{\pm,n} \geqslant L$ on $\operatorname{supp}(1-\widetilde{\chi}_L)$. Therefore, to obtain that $\langle i_\Delta^* K_{\chi,\pm,n}(s_0+s), 1-\widetilde{\chi}_L\rangle \to 0$ as $L \to +\infty$, it suffices to show that

$$\mathrm{e}^{-s\ell_{\pm,n}(\cdot)} \iota_\Delta^* K_{\chi,\pm,n}(s_0) = \iota_\Delta^* K_{\chi,\pm,n}(s_0+s).$$

This equality is a consequence of (4.3.26) and Lemma 4.11.1, since we can take s arbitrarily large to make □

Recall from Remark 4.3.7 that $s \mapsto (\chi \tilde{\mathcal{S}}_\pm(s)\chi)^n$ admits a meromorphic continuation in $\mathcal{D}'^3_{\mathbf{\Gamma}'_{\varepsilon,\pm}}(\partial \times \partial)$ where $\mathbf{\Gamma}'_{\varepsilon,\pm}$ does not intersect the conormal to the diagonal in $\partial \times \partial$. In particular, we have the

Corollary 4.3.10. *The function $s \mapsto \eta_{\pm,\chi,n}(s)$ defined for* $\operatorname{Re}(s) \gg 1$ *by the right hand side of (4.3.12) extends to a meromorphic function on the whole complex plane.*

To prove Theorem 4.1.1, we wish to use a standard Tauberian argument near the first pole of $\eta_{\pm,\chi,n}$ to obtain the growth of $N(n, L)$. Indeed, it is known (see §4.5) that $s \mapsto R_{\pm,\delta}(s)$ has a pole at $s = h_\star$. However since $\eta_{\pm,\chi,n}$ is given by the trace of the restriction to ∂ of $R_{\pm,\delta}$, it is not clear a priori that $\eta_{\pm,\chi,n}$ will have the right behavior at $s = h_\star$. However in the next section we obtain some priori bounds on $N(n, L)$; this will imply that $\eta_{\pm,\chi,n}$ has indeed a pole at $s = h_\star$ of order n.

4.4 A priori bounds on the growth of geodesics with fixed intersection number with $\gamma_\star$

The purpose of this section is to get *a priori* bounds on $N(1, L)$ (and $N(2, L)$ in the case where $\gamma_\star$ is separating), using Parry-Pollicott's bound for Axiom A flows [PP83].

Choose some point $x_\star \in \gamma_\star$. Let g be the genus of Σ and $(a_1, b_1, \ldots, a_\mathrm{g}, b_\mathrm{g})$ be a basis of generators of Σ, so that the fundamental group of Σ is the finitely presented group given by

$$\pi_1(\Sigma) = \langle a_1, b_1, \ldots, a_\mathrm{g}, b_\mathrm{g},\ [a_1, b_1] \cdots [a_\mathrm{g}, b_\mathrm{g}] = 1\rangle, \tag{4.4.1}$$

where we set $\pi_1(\Sigma) = \pi_1(\Sigma, x_\star)$ for some choice of $x_\star \in \gamma_\star$ (see Figure 4.4.1 for the case $\gamma_\star$ is not separating, and Figure 4.4 otherwise).

4.4.1 The case $\gamma_\star$ is not separating

Up to applying a diffeomorphism to Σ, we may assume that $\gamma_\star$ is represented by $a_\mathrm{g} \in \pi_1(\Sigma)$. The cutted surface $\Sigma_\star$ is a topological surface of genus $\mathrm{g}-1$ with 2 punctures and the fundamental group $\pi_1(\Sigma_\star) = \pi_1(\Sigma_\star, x_\star)$ [8] is the free group given by $\langle a_1, b_1, \ldots, a_\mathrm{g}\rangle$, as it follows from the fact that $\Sigma_\star$ is homotopically equivalent to a connected sum of $2\mathrm{g}-1$ circles. We refer to Figure 4.4.1 for a picture of the generators and the choice of $x_\star$. By the presentation of $\pi_1(\Sigma)$ given above, we have

$$b_\mathrm{g} a_\mathrm{g} b_\mathrm{g}^{-1} = a'_\mathrm{g} \quad \text{where} \quad a'_\mathrm{g} = [a_1, b_1] \cdots [a_{\mathrm{g}-1}, b_{\mathrm{g}-1}] a_\mathrm{g}, \tag{4.4.2}$$

8. Here, in order not to burden the notations, we still denote by $x_\star \in \Sigma_\star$ a lift of $x_\star \in \Sigma$ by the natural map $q_\star : \Sigma_\star \to \Sigma$, see Figure 4.4.1.

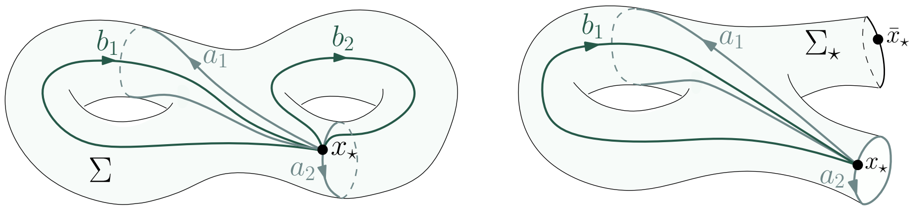

FIGURE 4.2 – The generators $a_1, b_1, \dots, a_g, b_g$ of $\pi_1(\Sigma)$ (on the left) and the generators $a_1, b_1, \dots, a_g$ of $\pi_1(\Sigma_\star)$ (on the right) when $g = 2$. Here $\gamma_\star$ is assumed not separating and is represented by a_2 in $\pi_1(\Sigma)$.

and note that a'_g also defines an element of $\pi_1(\Sigma_\star)$.

Lemma 4.4.1. *The map $q_\star : \Sigma_\star \to \Sigma$ given by the identification of the boundary components of Σ induces a map $q_{\star,*} : \pi_1(\Sigma_\star) \to \pi_1(\Sigma)$, which is injective.*

Proof. Let $\langle a_g \rangle$ (resp. $\langle a'_g \rangle$) be the infinite cyclic subgroup of $\pi_1(\Sigma_\star)$ generated by a_g (resp. a'_g). Then by (4.4.1) and (4.4.2), the group $\pi_1(\Sigma)$ is the HNN extension $\pi_1(\Sigma_\star)*_\phi$ of $\pi_1(\Sigma_\star)$ with respect to the the isomorphism $\phi : \langle a'_g \rangle \to \langle a_g \rangle$ given by $\phi(a'_g) = a_g$, that is, $\pi_1(\Sigma_\star)*_\phi$ is the finitely presented group defined by

$$\pi_1(\Sigma_\star)*_\phi = \langle a_1, b_1, \dots, a_g, t \ : \ t^{-1} a'_g t = a_g \rangle,$$

see [LS01, §IV.2]. Now the map $q_{\star,*} : \pi_1(\Sigma_\star) \to \pi_1(\Sigma)$ coincides with the natural map $\pi_1(\Sigma_\star) \to \pi_1(\Sigma_\star)*_\phi$, and this map is injective by [LS01, Theorem IV.2.1]. □

We may see the cutted surface $\Sigma_\star$ as the convex core of a complete, non compact, negatively curved surface, with funnels. Indeed, by Lemma 4.4.1, the group $\pi_1(\Sigma_\star)$ can be thought as a subgroup of $\pi_1(\Sigma)$, and the convex core of the infinite surface $\Sigma^e_\star = \pi_1(\Sigma_\star)\backslash\widetilde{\Sigma}$ is canonically isometric to $\Sigma_\star$ (here $\widetilde{\Sigma}$ is a universal cover of Σ). Another way to obtain this is by gluing two arbitrary funnels, as follows. Recall that near each connected component of the boundary $\partial\Sigma_\star \subset \Sigma_\delta$, we have coordinates $(\tau, \rho) \in \mathbb{R}/\ell_\star\mathbb{Z}_\tau \times [-\delta, \delta]_\rho$ given by Lemma 4.2.3 for which $\partial\Sigma_\star = \{\rho = 0\}$ and $\partial\Sigma_\delta = \{\rho = \delta\}$. In those coordinates, the metric has the form $\mathrm{d}\rho^2 + f(\tau, \rho)\mathrm{d}\tau^2$ for some smooth function f satisfying $\partial_\rho f(\tau, 0) = 0$ and $\kappa(\tau, \rho) = -\partial^2_\rho f(\tau, \rho)/f(\tau, \rho)$. Then we arbitrarily extend f to a smooth function on $(\mathbb{R}/\ell_\star\mathbb{Z})_\tau \times [-\delta, +\infty[$ so that for some constants $c, C > 0$ it holds

$$c \leqslant \partial^2_\rho f / f \leqslant C.$$

By gluing the funnels $(\mathbb{R}/\ell_\star\mathbb{Z}) \times [0, \infty[$ and $\Sigma_\star$ along the corresponding connected components, we obtain a complete negatively curved surface $\Sigma^e_\star$, whose metric in the funnels is given by $\mathrm{d}\rho^2 + f(\tau, \rho)\mathrm{d}\tau^2$. We will again denote by (φ_t) the geodesic flow on the unit tangent bundle $S\Sigma^e_\star$ of $\Sigma^e_\star$.

Let $\widetilde{\Sigma}_\star$ denote the universal cover of $\Sigma^e_\star$ and let $\tilde{x}_\star \in \widetilde{\Sigma}_\star$ such that $\pi(\tilde{x}_\star) = x_\star$ where $\pi : \widetilde{\Sigma}_\star \to \Sigma^e_\star$ is the natural projection. Then $\pi_1(\Sigma^e_\star, x_\star) = \pi_1(\Sigma_\star)$ acts on $\widetilde{\Sigma}_\star$ by deck transformations so that $\Sigma^e_\star \simeq \pi_1(\Sigma_\star)\backslash\widetilde{\Sigma}_\star$. Moreover, Lemma 4.2.6 implies that the recurrent set of the geodesic flow on $S\Sigma_\star^e$ is compact and included in $S\Sigma_\star$; thus $\pi_1(\Sigma_\star)$ is convex-cocompact in the sense of [Dal99]. The aforementioned lemma also implies that every closed geodesic in $\Sigma_\star^e$ which is not contained in $\partial\Sigma_\star$ is actually contained in the interior of $\Sigma_\star$.

It is well known that there is a one-to-one correspondance between oriented closed geodesics on $\Sigma_\star^e$ (all of them belonging to $\Sigma_\star$) and the set of free homotopy classes of loops in $\Sigma_\star^e$. The latter set is itself in one-to-one correspondance with the set of conjugacy classes of $\pi_1(\Sigma_\star)$. We set

$$\ell_\star(w) = \mathrm{dist}(\tilde{x}_\star, w\tilde{x}_\star), \quad w \in \pi_1(\Sigma_\star),$$

where the distance comes from the metric $\pi^* g$ on $\widetilde{\Sigma}_\star$. For any $w \in \pi_1(\Sigma_\star)$, we denote by $[w]$ the associated conjugacy class of $\pi_1(\Sigma_\star)$. Note that if $\gamma_{[w]}$ denotes the unique geodesic in the free homotopy class of w (which is represented by the conjugacy class $[w]$), we have $\ell(\gamma_{[w]}) \leqslant \ell_\star(w)$. We also denote by

$$\mathrm{wl}(w) = \min\left\{n \geqslant 0 \ : \ w = \alpha_1 \cdots \alpha_n, \ \alpha_j \in \mathscr{L}_{\mathrm{g}} \setminus \{b_{\mathrm{g}}, b_{\mathrm{g}}^{-1}\}\right\} \tag{4.4.3}$$

the word length of an element $w \in \pi_1(\Sigma_\star)$, where $\mathscr{L}_{\mathrm{g}} = \bigcup_{k=1}^{\mathrm{g}} \left\{a_k, a_k^{-1}, b_k, b_k^{-1}\right\}$. We will say that a word $\alpha_1 \cdots \alpha_k$ with $\alpha_j \in \mathscr{L}_{\mathrm{g}}$ is reduced if $\alpha_j \neq (\alpha_{j+1})^{-1}$ for any $j = 1, \ldots, k-1$. As $\pi_1(\Sigma_\star)$ is free, for each $w \in \pi_1(\Sigma_\star)$, there is exactly one reduced word $\alpha_1 \cdots \alpha_n$ such that $n = \mathrm{wl}(w)$, see [LS01, p.4]. It follows from the Milnor-Švarc lemma [BH13, Proposition I.8.19] that for some constant $D > 0$ we have

$$\frac{1}{D}\mathrm{wl}(w) - D \leqslant \ell_\star(w) \leqslant D\mathrm{wl}(w) + D, \quad w \in \pi_1(\Sigma_\star). \tag{4.4.4}$$

Also, as $\pi_1(\Sigma_\star)$ is convex co-compact we have the classical orbital counting (see [Rob03, §1.F and Corollaire 2])

$$\sharp\{w \in \pi_1(\Sigma_\star) \ : \ \ell_\star(w) \leqslant L\} \sim A\mathrm{e}^{h_\star L}, \quad L \to \infty \tag{4.4.5}$$

for some $A > 0$, where $h_\star > 0$ is the topological entropy of the geodesic flow of $(\Sigma_\star^e, g)$ restricted to the trapped set

$$K_\star^e = \{(x, v) \in S\Sigma_\star^e \ : \ \varphi_t(x, v) \in S\Sigma_\star, \ t \in \mathbb{R}\}.$$

In fact, $h_\star > 0$ also coincides with the entropy of the geodesic flow of (Σ, g) restricted to the trapped set $K_\star$ mentioned in the introduction,

$$K_\star = \overline{\{(x, v) \in S\Sigma \ : \ \pi(\varphi_t(x, v)) \in \Sigma \setminus \gamma_\star, \ t \in \mathbb{R}\}},$$

where the closure is taken in $S\Sigma$, and we have $K_\star^e = p_\star^{-1}(K_\star)$ where $p_\star : S\Sigma_\star \to S\Sigma$ is the natural map given by the identification of both components of $\partial S\Sigma_\star$.

4.4.1.1 Lower bound

In this paragraph we will prove the

Proposition 4.4.2. *If $\gamma_\star$ is not separating, then there is $C > 0$ such that for any L large enough,*

$$N(1, L) \geqslant C\mathrm{e}^{h_\star L}/L.$$

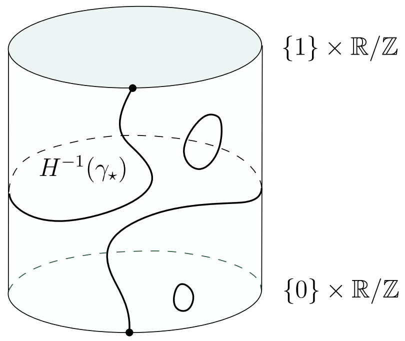

FIGURE 4.3 – Proof of Lemma 4.4.3. The path linking $(0,[0]) \in \{0\} \times \mathbb{R}/\mathbb{Z}$ to $(1,[0])$ is the image of F.

Note that the bound given in Theorem 4.1.1 is actually $N(1,L) \sim c_\star e^{h_\star L}$. We could obtain a better bound with the methods presented in the paragraph 4.4.2 below which deals with the not separating case; however Proposition 4.4.2 will be sufficient for our purposes.

Lemma 4.4.3. *Take $w, w' \in \pi_1(\Sigma_\star)$. Then $[wb_g] = [w'b_g]$ as conjugacy classes of $\pi_1(\Sigma)$ if and only if $w = a_g^n w' a_g'^{-n}$ in $\pi_1(\Sigma_\star)$ for some $n \in \mathbb{Z}$.*

Proof. If $w = a_g^n w' b_g a_g^{-n} b_g^{-1}$, then clearly wb_g and $w'b_g$ are conjugated in $\pi_1(\Sigma, x_\star)$. Reciprocally, assume that $[wb_g] = [w'b_g]$. We may find smooth paths γ and γ' representing respectively the elements wb_g and $w'b_g$, with $\mathrm{i}(\gamma,\gamma_\star) = \mathrm{i}(\gamma',\gamma_\star) = 1$ and such that the intersections $\gamma \cap \gamma_\star$ and $\gamma' \cap \gamma_\star$ are transversal. As $[wb_g] = [w'b_g]$, the loops γ and γ' lie in the same free homotopy class. Thus, there is a smooth homotopy $H : [0,1] \times \mathbb{R}/\mathbb{Z} \to \Sigma$ such that $H(0,\cdot) = \gamma$ and $H(1,\cdot) = \gamma'$. We may assume that H is transversal to $\gamma_\star$ (see for example [GP10, Corollary p.73]) in the sense that

$$\mathrm{d}H(s,\tau)\big(T_{(s,\tau)}([0,1] \times \mathbb{R}/\mathbb{Z})\big) + T_{H(s,\tau)}\gamma_\star = T_{H(s,\tau)}\Sigma, \quad H(s,\tau) \in \gamma_\star.$$

In particular, $H^{-1}(\gamma_\star)$ is a smooth submanifold of $[0,1] \times \mathbb{R}/\mathbb{Z}$. As γ and γ' intersect transversally $\gamma_\star$ exactly once, we have $H^{-1}(\gamma_\star) \cap (\{j\} \times \mathbb{R}/\mathbb{Z}) = \{j\} \times \{[0]\}$ for $j = 0,1$ (here $[0]$ is sent to $x_\star$ by both γ and γ'). Thus, necessarily, there exists an embedding $F : [0,1] \to [0,1] \times \mathbb{R}/\mathbb{Z}$ such that $\mathrm{Im}(F) \subset H^{-1}(\gamma_\star)$ and $F(j) = (j,[0])$ for $j = 0,1$ (see Figure 4.3). Write $F = (S,T)$, and define

$$\widetilde{H}(s,t) = H(S(s), [T(s) + t]), \quad (s,t) \in [0,1] \times [0,1].$$

It is immediate to check that $\widetilde{H}$ realizes an homotopy between γ and γ', and we have $\widetilde{H}(s,0) = H(F(s)) \in \gamma_\star$ for any $s \in [0,1]$. For any s, let us denote by c_s the path $[0,1] \ni u \mapsto \widetilde{H}(su,0)$ which links $x_\star$ to $H(S(s),[T(s)])$ within $\gamma_\star$. The the continuous family of paths $s \mapsto \gamma_s$, where γ_s is given by the concatenation $c_s^{-1}\widetilde{H}(s,\cdot)c_s$, realizes a continuous interpolation between $\gamma_0 = \gamma$ and $\gamma_1 = c_1^{-1}\gamma' c_1$. As $S(1) = 1$ and $T(1) = [0]$ we have $c_1(0) = c_1(1) = x_\star$, and since $c_1(u) \in \gamma_\star$ for each $u \in [0,1]$ we get $c_1 = a_g^{-n}$ for some $n \in \mathbb{Z}$. This yields $wb_g = a_g^n w' b_g a_g^{-n}$ in $\pi_1(\Sigma)$, and thus $w = a_g^n w' a_g'^{-n}$ where the equality stands in $\pi_1(\Sigma)$. By lemma 4.4.1, this equality actually holds in $\pi_1(\Sigma_\star)$, which concludes the proof. □

Proof of Proposition 4.4.2. In what follows, C is a constant that may change at each line. For any $w \in \pi_1(\Sigma_\star)$ and $n \in \mathbb{Z}$ we have by (4.4.4)

$$\ell_\star(a_g^n w a_g'^{-n}) \geqslant \frac{1}{D}\mathrm{wl}(a_g^n w a_g'^{-n}) - D. \tag{4.4.6}$$

Let w' be the unique reduce word such that $w' = w a_g'^{-n}$. Then write $w' = a_g^{-k} w''$ for some w'' where $|k|$ is maximal, and note that necessarily $|k| \leqslant \mathrm{wl}(w) + 1$, since $a_g' = [a_1, b_1] \cdots [a_{g-1}, b_{g-1}] a_g$. Then

$$\mathrm{wl}(a_g^n w a_g'^{-n}) = |n| - |k| + \mathrm{wl}(w'') = |n| - 2|k| + \mathrm{wl}(w') \geqslant |n| - 2(\mathrm{wl}(w)+1) + \mathrm{wl}(w').$$

Now the triangle inequality for wl gives $(4(g-1)+1)|n| = \mathrm{wl}(a_g'^{-n}) \leqslant \mathrm{wl}(w') + \mathrm{wl}(w^{-1})$ and thus we obtain $\mathrm{wl}(a_g^n w a_g'^{-n}) \geqslant C|n| - C\mathrm{wl}(w) - C$ for each n. Injecting this in (4.4.6) yields (for some different C)

$$\ell_\star(a_g^n w a_g'^{-n}) \geqslant C|n| - C\mathrm{wl}(w) - C, \quad n \in \mathbb{Z}.$$

In particular, for any L and w such that $\ell_\star(w) \leqslant L$, we have

$$\left|\left\{n \in \mathbb{Z} \ : \ \ell_\star(a_g^n w a_g'^{-n}) \leqslant L\right\}\right| \leqslant CL + C. \tag{4.4.7}$$

Now for $w \in \pi_1(\Sigma_\star)$ set $\mathcal{C}_w = \{a_g^n w a_g'^{-n} \ : \ n \in \mathbb{Z}\} \subset \pi_1(\Sigma_\star)$ and denote by $\mathscr{C}$ the set $\{\mathcal{C}_w \ : \ w \in \pi_1(\Sigma_\star)\}$. For $\mathcal{C} \in \mathscr{C}$ we set $\ell_\star(\mathcal{C}) = \inf_{w \in \mathcal{C}} \ell_\star(w)$. Then by Lemma 4.4.3, we have a well defined and injective map

$$\{\mathcal{C} \in \mathscr{C} \ : \ \ell_\star(\mathcal{C}) \leqslant L\} \to \{\gamma \in \mathcal{P}_1 \ : \ \ell(\gamma) \leqslant L + C\}, \quad \mathcal{C}_w \mapsto [w b_g],$$

where $\mathcal{P}_1$ denotes the set of primitive geodesics γ such that $\mathrm{i}(\gamma, \gamma_\star) = 1$ [9]. In particular we get with (4.4.7) and (7.4.2)

$$\begin{aligned} N(1, L) &\geqslant |\{\mathcal{C} \in \mathscr{C} \ : \ \ell_\star(\mathcal{C}) \leqslant L - C\}| \\ &\geqslant \frac{1}{CL + C} \sum_{\substack{\mathcal{C} \in \mathscr{C} \\ \ell_\star(\mathcal{C}) \leqslant L - C}} |\{w \in \mathcal{C} \ : \ \ell_\star(w) \leqslant L - C\}| \\ &= \frac{1}{CL + C} |\{w \in \pi_1(\Sigma_\star) \ : \ \ell_\star(w) \leqslant L - C\}| \\ &\geqslant \frac{1}{CL + C} \exp(h_\star(L - C)), \end{aligned} \tag{4.4.8}$$

where the equality in the third line comes from the fact that $\pi_1(\Sigma_\star)$ is the disjoint union of the subsets $\mathcal{C}$ with $\mathcal{C} \in \mathscr{C}$. This completes the proof. □

9. Each class $[w b_g]$ defines a geodesic in $\mathcal{P}_1$. Indeed, it follows from Lemma 4.2.1 that $\mathrm{i}([w b_g], \gamma_\star) \leqslant 1$. On the other hand the absolute value of the algebraic intersection number between $w b_g$ and a_g is 1, and this implies that there is at least one intersection point between $[w b_g]$ and $\gamma_\star$, since the algebraic intersection number is preserved by free homotopies.

4.4.1.2 Upper bound

Each $\gamma \in \mathcal{P}_1$ with $\ell(\gamma) \leqslant L$ lies in the free homotopy class of $w' b_{\mathrm{g}}^{\pm 1}$ for some $w' \in \pi_1(\Sigma_\star, x_\star)$ and $\ell_\star(w) \leqslant L + C$. In particular (7.4.2) gives the bound

$$N(1, L) \leqslant C \exp(h_\star L)$$

for large L. Now let $\gamma \in \mathcal{P}_2$ with $\ell(\gamma) \leqslant L$. Then we may find a deformation of the loop γ into a loop γ' which is represented by the conjugacy class of $w b_{\mathrm{g}}^{\pm 1} w' b_{\mathrm{g}}^{\pm 1}$ in $\pi_1(\Sigma)$, for some $w, w' \in \pi_1(\Sigma_\star)$. This deformation can be made so that $\ell_\star(w) + \ell_\star(w') \leqslant L + C$. Thus we get

$$\begin{aligned} N(2, L) &\leqslant C \sum_{\substack{w, w' \in \pi_1(\Sigma_\star) \\ \ell_\star(w) + \ell_\star(w') \leqslant L + C}} 1 \\ &\leqslant \sum_{k=0}^{L+C} C \exp(h_\star k) C \exp(h_\star(L + C - k)) \\ &\leqslant C' L \exp(h_\star L). \end{aligned}$$

Iterating this process we finally get, for large L,

$$N(n, L) \leqslant C L^{n-1} \exp(h_\star L).$$

4.4.2 The case $\gamma_\star$ is separating

In this paragraph we assume $\gamma_\star$ is separating, and we write $\Sigma \setminus \gamma_\star = \Sigma_1 \sqcup \Sigma_2$ where the surfaces Σ_j are connected. Up to applying a diffeomorphism to Σ, we may assume that $\gamma_\star$ represents the class

$$[a_1, b_1] \cdots [a_{\mathrm{g}_1}, b_{\mathrm{g}_1}] = [a_{\mathrm{g}}, b_{\mathrm{g}}]^{-1} \cdots [a_{\mathrm{g}_1+1}, b_{\mathrm{g}_1+1}]^{-1} \in \pi_1(\Sigma) \tag{4.4.9}$$

(see Figure 4.4). Here g_1 is the genus of the surface Σ_1, and the genus g_2 of Σ_2 satisfies $\mathrm{g}_1 + \mathrm{g}_2 = \mathrm{g}$.

We set $\pi_1(\Sigma) = \pi_1(\Sigma, x_\star)$ and $\pi_1(\Sigma_j) = \pi_1(\Sigma_j, x_\star)$ for $j = 1, 2$ (we see Σ_j as a compact surface with boundary $\gamma_\star$ so that $x_\star$ lives on both surfaces). Then $\pi_1(\Sigma_1)$ (resp. $\pi_1(\Sigma_2)$) is the free group generated by $a_1, b_1, \ldots, a_{\mathrm{g}_1}, b_{\mathrm{g}_1}$ (resp. $a_{\mathrm{g}_1+1}, b_{\mathrm{g}_1+1}, \ldots, a_{\mathrm{g}}, b_{\mathrm{g}}$), and we denote by $w_{\star,1}$ and $w_{\star,2}$ the two natural words given by (4.4.9) representing $\gamma_\star$ in $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$, respectively. Note that we have a well defined map

$$\begin{array}{ccc} \pi_1(\Sigma_1) \times \pi_1(\Sigma_2) & \longrightarrow & \pi_1(\Sigma) \\ (w_1, w_2) & \longmapsto & w_2 w_1 \end{array}$$

given by the composition of two curves.

Lemma 4.4.4. *For $j = 1, 2$, the map $q_{j,*} : \pi_1(\Sigma_j) \to \pi_1(\Sigma)$ induced by the inclusion $\Sigma_j \hookrightarrow \Sigma$ is injective.*

Proof. For $j = 1, 2$ let $\langle w_{\star,j} \rangle$ be the infinite cyclic group of $\pi_1(\Sigma_j)$ generated by $w_{\star,j}$, and let $\phi : \langle w_{\star,1} \rangle \to \langle w_{\star,2} \rangle$ be the isomorphism given by $\phi(w_{\star,1}) = w_{\star,2}$. By (4.4.1),

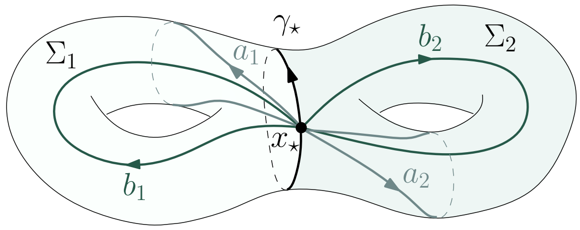

FIGURE 4.4 – The generators $a_1, b_1, \dots, a_g, b_g$ of $\pi_1(\Sigma)$. Here $\gamma_\star$ is assumed separating and $g_1 = g_2 = 1$.

the group $\pi_1(\Sigma)$ is the free product with amalgamation $\pi_1(\Sigma_1) *_\phi \pi_1(\Sigma_2)$, that is, the finitely presented group given by

$$\pi_1(\Sigma_1) *_\phi \pi_1(\Sigma_2) = \{a_1, b_1, \dots, a_g, b_g \ : \ w_{\star,1} = \phi(w_{\star,1})\},$$

see [LS01, §IV.2]. With this representation the map $q_{j,*}$ coincides with the natural map $\pi_1(\Sigma_j) \to \pi_1(\Sigma_1) *_\phi \pi_1(\Sigma_2)$, and this map is injective by [LS01, Theorem IV.2.6]. This completes the proof. □

For any $w \in \pi_1(\Sigma)$, we will denote by $[w]$ its conjugacy class, and by γ_w the unique geodesic of Σ such that γ_w is isotopic to any curve in w (in fact we will often identify $[w]$ and γ_w). Let $(\widetilde{\Sigma}, \tilde{g})$ be the universal cover of (Σ, g), and choose $\tilde{x}_\star \in \widetilde{\Sigma}$ some lift of $x_\star$. Then $\pi_1(\Sigma)$ acts as deck transformations on $\widetilde{\Sigma}$ and we will denote

$$\ell_\star(w) = \mathrm{dist}_{\widetilde{\Sigma}}(\tilde{x}_\star, w\tilde{x}_\star), \quad w \in \pi_1(\Sigma).$$

As in the preceding subsection, we have the orbital counting

$$\sharp\{w_j \in \pi_1(\Sigma_j) \ : \ \ell_\star(w_j) \leqslant L\} \sim A_j \mathrm{e}^{h_j L}, \quad L \to \infty, \quad j = 1, 2, \tag{4.4.10}$$

for some $A_1, A_2 > 0$, where $h_j > 0$ is the topological entropy of the geodesic flow restricted to the trapped set

$$K_j = \overline{\left\{(x, v) \in S\Sigma_j^\circ \ : \ \varphi_t(x, v) \in S\Sigma_j^\circ, \ t \in \mathbb{R}\right\}}$$

where $\Sigma_j^\circ = \Sigma_j \setminus \partial\Sigma_j$ for $j = 1, 2$.

4.4.2.1 Lower bound

Unlike the case $\gamma_\star$ not separating, we will need a better lower bound. Namely, we prove here the following result.

Proposition 4.4.5. *Assume that $\gamma_\star$ is separating, and that $h_1 = h_2 = h_\star$. Then there is $C > 0$ such that for L large enough,*

$$N(2, L) \geqslant \frac{C L \mathrm{e}^{h_\star L}}{\log(L)^4}.$$

If $h_1 \neq h_2$ we have for L large enough, if $h_\star = \max(h_1, h_2)$,

$$N(2, L) \geqslant \frac{C \mathrm{e}^{h_\star L}}{\log(L)^2}.$$

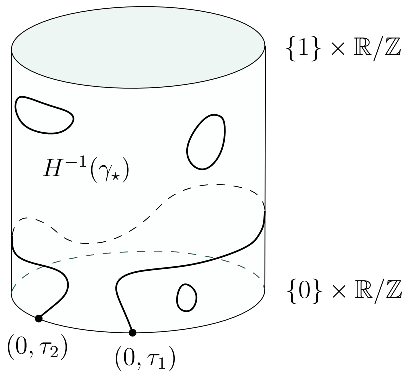

FIGURE 4.5 – Proof of Lemma 4.4.6. The path linking $(0,\tau_1)$ to $(0,\tau_2)$ is the image of F.

The strategy to prove Proposition 4.4.5 is the following. We wish to construct enough closed geodesics intersecting $\gamma_\star$ exactly twice by considering conjugacy classes of the form $[w_2w_1]$ where $w_j \in \pi_1(\Sigma_j)$ for $j = 1, 2$. Lemma 4.4.6 below will tell us that if w_j is not a power of $w_{\star,j}$ for $j = 1, 2$, then the closed geodesic representing $[w_2w_1]$ indeed intersects $\gamma_\star$ exactly twice. Next, in Lemma 4.4.7, we describe the injectivity defect of the map $(w_1, w_2) \mapsto [w_2w_1]$. Finally in Proposition 4.4.8, we show that this injectivity defect is not too harmuful in the sense that there are not too much $w_j, w'_j \in \pi_1(\Sigma_j)$ such that $[w_2w_1] = [w'_2w'_1]$. This will allow us to obtain the desired bound with a logarithmic loss.

Lemma 4.4.6. *For two elements $w_j \in \pi_1(\Sigma_j)$, $j = 1, 2$, we have $\mathrm{i}(\gamma_{w_2w_1}, \gamma_\star) = 2$ except if $w_j = w_{\star,j}^k$ in $\pi_1(\Sigma_j)$ for some $k \in \mathbb{Z}$ and $j \in \{1, 2\}$, in which case $\mathrm{i}(\gamma_{w_2w_1}, \gamma_\star) = 0$.*

Proof. Let $\gamma : \mathbb{R}/\mathbb{Z} \to \Sigma$ be a smooth curve in the free homotopy class of w_2w_1 such that $\{\tau \in \mathbb{R}/\mathbb{Z} \ : \ \gamma(\tau) \in \gamma_\star\} = \{\tau_1, \tau_2\}$ for some $\tau_1 \neq \tau_2 \in \mathbb{R}/\mathbb{Z}$. We may also choose γ so that $\gamma|_{[\tau_1,\tau_2]}$ (resp. $\gamma|_{[\tau_2,\tau_1]}$) is homotopic to some representative $\gamma_1 : [0, 1] \to \Sigma_1$ of w_1 (resp. some representative $\gamma_2 : [0, 1] \to \Sigma_2$ of w_2) relatively to $\gamma_\star$, meaning that there is a homotopy between $\gamma|_{[\tau_1,\tau_2]}$ and γ_1 with endpoints (not necessarily fixed) in $\gamma_\star$. Here $[\tau_1, \tau_2] \subset \mathbb{R}/\mathbb{Z}$ is the interval linking τ_1 and τ_2 in the counterclockwise direction.

As $\gamma_{w_2w_1}$ minimizes the quantity $\mathrm{i}(\gamma, \gamma_\star)$ for $\gamma \in [\gamma_{w_2w_1}]$ (see Lemma 4.2.1) we have either $\mathrm{i}(\gamma_{w_2w_1}, \gamma_\star) = 0$ or $\mathrm{i}(\gamma_{w_2w_1}, \gamma_\star) = 2$. If $\mathrm{i}(\gamma_{w_2w_1}, \gamma_\star) = 0$ then there exists a homotopy $H : [0, 1] \times \mathbb{R}/\mathbb{Z} \to \Sigma$ such that $H(0, \cdot) = \gamma$ and $H(1, \cdot) = \gamma$, so that $H(1, \tau) \notin \gamma_\star$ for any τ. As in the proof of Lemma 4.4.3 we may assume that H is transversal to $\gamma_\star$, in the sense that

$$\mathrm{d}H(s,\tau)\big(T_{(s,\tau)}([0,1] \times \mathbb{R}/\mathbb{Z})\big) + T_{H(s,\tau)}\gamma_\star = T_{H(s,\tau)}\Sigma, \quad H(s,\tau) \in \gamma_\star,$$

so that the preimage

$$H^{-1}(\gamma_\star) \subset [0,1] \times \mathbb{R}/\mathbb{Z}$$

is an embedded submanifold of $[0, 1] \times \mathbb{R}/\mathbb{Z}$ (see Figure 4.5). As $H^{-1}(\gamma_\star) \cap \{s =$

$0\} = \{\tau_1, \tau_2\}$ and $H^{-1}(\gamma_\star) \cap \{s = 1\} = \emptyset$ it follows that there is an embedding $F : [0,1] \to [0,1] \times \mathbb{R}/\mathbb{Z}$ such that $F(0) = (0, \tau_1)$, $F(1) = (0, \tau_2)$ and

$$F(t) \in H^{-1}(\gamma_\star), \quad t \in [0,1].$$

As F is an embedding, we have that F is homotopic (by an homotopy which preserves the endpoints) either to $J_{[\tau_1,\tau_2]}$ or to $J_{[\tau_2,\tau_1]}$, where $J_{[\tau,\tau']} : [0,1] \to [0,1] \times \mathbb{R}/\mathbb{Z}$ is the natural map that sends $[0,1]$ to $\{0\} \times [\tau, \tau']$. We may assume without loss of generality that $F \sim J_{[\tau_1,\tau_2]}$. In particular, writing $F = (S,T)$, the map T is homotopic to $I_{[\tau_1,\tau_2]} = p_2 \circ J_{[\tau_1,\tau_2]}$, where $p_2 : [0,1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the projection over the second factor. This means that there is $G : [0,1] \times [0,1] \to \mathbb{R}/\mathbb{Z}$ such that for any $s, t \in [0,1]$,

$$G(s,0) = \tau_1, \quad G(s,1) = \tau_2, \quad G(0,t) = \tau_1 + t(\tau_2 - \tau_1), \quad G(1,t) = T(t).$$

Now we set $\tilde{H}(s,t) = H(sS(t), G(s,t))$ for $s,t \in [0,1]$. Then

$$\tilde{H}(0,t) = \gamma(\tau_1 + t(\tau_2 - \tau_1)), \quad \tilde{H}(1,t) = (H \circ F)(t), \quad t \in [0,1],$$

and

$$\tilde{H}(s,0) = H(0,\tau_1) = x_1, \quad \tilde{H}(s,1) = H(0,\tau_2) = x_2, \quad s \in [0,1].$$

We conclude that $t \mapsto \gamma|_{[\tau_1,\tau_2]}(\tau_1 + t(\tau_2 - \tau_1))$, and thus γ_1, is homotopic (relatively to $\gamma_\star$) to some curve contained in $\gamma_\star$. Thus $w_1 = w_\star^k$ for some $k \in \mathbb{Z}$, in $\pi_1(\Sigma)$. As the inclusion $\pi_1(\Sigma_j) \to \pi_1(\Sigma)$ is injective by Lemma 4.4.4, the lemma follows. □

Now, we need to understand when the geodesics given by $[w_2 w_1]$ and $[w_2' w_1']$ are the same. This is the purpose of the following

Lemma 4.4.7. *Take $w_j, w_j' \in \pi_1(\Sigma_j)$, $j = 1,2$ such that $\mathrm{i}(\gamma_{[w_2 w_1]}, \gamma_\star) = 2$. Then it holds $[w_2 w_1] = [w_2' w_1']$ as conjugacy classes of $\pi_1(\Sigma)$ if and only if there are $p, q \in \mathbb{Z}$ such that*

$$w_2 = w_{\star,2}^p w_2' w_{\star,2}^q, \quad w_1 = w_{\star,1}^{-q} w_1' w_{\star,1}^{-p}. \tag{4.4.11}$$

Proof. Again, let $\gamma : \mathbb{R}/\mathbb{Z} \to \Sigma$ be a smooth curve intersecting transversely $\gamma_\star$ such that $\{\tau \in \mathbb{R}/\mathbb{Z} \ : \ \gamma(\tau) \in \gamma_\star\} = \{\tau_1, \tau_2\}$ for some $\tau_1 \neq \tau_2 \in \mathbb{R}/\mathbb{Z}$, with $\gamma([\tau_1, \tau_2]) \subset \Sigma_1$ and $\gamma([\tau_2, \tau_1]) \subset \Sigma_2$. Let $x_j = \gamma(\tau_j)$ for $j = 1,2$ and chose arbitrary paths c_j contained in $\gamma_\star$ linking x_j to $x_\star$. Note that all the preceding choices can be made so that the curve $\gamma_1 = c_2 \gamma|_{[\tau_1,\tau_2]} c_1^{-1}$ (resp. $\gamma_2 = c_1 \gamma|_{[\tau_2,\tau_1]} c_2^{-1}$) represents $w_\star^p w_1 w_\star^q$ (resp. $w_\star^{-q} w_2 w_\star^{-p}$) for some $p, q \in \mathbb{Z}$. We may proceed in the same way to obtain $\gamma', \tau_1', \tau_2', c_1', c_2', p', q'$ so that the same properties hold with w_1, w_2 replaced by w_1', w_2'. By hypothesis, we have that γ is freely homotopic to γ'. Thus we may find a smooth map $H : [0,1] \times \mathbb{R}/\mathbb{Z} \to \Sigma$ such that $H(0,\cdot) = \gamma$ and $H(1,\cdot) = \gamma'$. As in Lemma 4.4.6, H may be chosen to be transversal to $\gamma_\star$, so that

$$H^{-1}(\gamma_\star) \subset [0,1] \times \mathbb{R}/\mathbb{Z}$$

is a finite union of smooth embedded submanifolds of $[0,1] \times \mathbb{R}/\mathbb{Z}$. Let $(x, \rho) : \Sigma \to \mathbb{R}/\mathbb{Z} \times (-\varepsilon, \varepsilon)$ be coordinates near $\gamma_\star$ such that $\{\rho = 0\} = \gamma_\star$ and $|\rho| = \mathrm{dist}(\gamma_\star, \cdot)$ and such that $\{(-1)^{j-1}\rho \geqslant 0\} \subset \Sigma_j$. As $H^{-1}(\gamma_\star) \cap \{s = 0\} = \{\tau_1, \tau_2\}$ and $H^{-1}(\gamma_\star) \cap \{s = 1\} = \{\tau_1', \tau_2'\}$, we have two smooth embeddings $F_1, F_2 : [0,1] \to [0,1] \times \mathbb{R}/\mathbb{Z}$ such that $F_j([0,1]) \subset H^{-1}(\gamma_\star)$ and $F_j(0) = (0, \tau_j)$ for $j = 1,2$, with $F_j(1) = \tau_1'$ or τ_2'

(indeed we have $\mathrm{i}(\gamma,\gamma_\star)=2$ and thus there is a path in $H^{-1}(\gamma_\star)$ linking $\{s=0\}$ to $\{s=1\}$, since otherwise we could proceed as in the proof of Lemma 4.4.6 to obtain that $\mathrm{i}(\gamma,\gamma_\star)=0$). In fact we have $F_1(1)=(1,\tau_1')$ and $F_2(1)=(1,\tau_2')$ (we shall prove it later). Set $F_j=(S_j,T_j)$, and

$$\widetilde{H}(s,t)=H\big((1-t)S_1(s)+tS_2(s),\ T_1(s)+t(T_2(s)-T_1(s))\big),\quad s,t\in[0,1].$$

Then it holds

$$\widetilde{H}(0,t)=\gamma(\tau_1+t(\tau_2-\tau_1)),\quad \widetilde{H}(1,t)=\gamma'(\tau_1'+t(\tau_2'-\tau_1')),$$

and

$$\widetilde{H}(s,0)=H(S_1(s),T_1(s)),\quad \widetilde{H}(s,1)=H(S_2(s),T_2(s)),\quad s\in[0,1].$$

For $j=1,2$ let $c_j(s),\ s\in[0,1]$, be paths contained in $\gamma_\star$ depending continuously on s and linking $T_j(s)$ to $x_\star$, such that $c_j(0)=c_j$. Then the construction of $\widetilde{H}$ shows that

$$c_2(0)\gamma|_{[\tau_1,\tau_2]}c_1(0)^{-1}\sim c_2(1)\gamma'|_{[\tau_1',\tau_2']}c_1(1)^{-1},$$

and reversing the role of τ_1 and τ_2 in the constructions made above,

$$c_1(0)\gamma|_{[\tau_2,\tau_1]}c_2(0)^{-1}\sim c_1(1)\gamma'|_{[\tau_2',\tau_1']}c_2(1)^{-1}.$$

Thus we obtain

$$w_\star^p w_1 w_\star^q = c_2(1)c_2'^{-1}w_\star^{p'}w_1'w_\star^{q'}c_1'c_1(1)^{-1}$$

and

$$w_\star^{-q}w_2w_\star^{-p}=c_1(1)c_1'^{-1}w_\star^{-q'}w_2w_\star^{-p'}c_2'c_2(1)^{-1},$$

which is the conclusion of Lemma 4.4.7 as the paths $c_1(1)c_1'^{-1}$ and $c_2(1)c_2'^{-1}$ are contained in $\gamma_\star$ (and again, the inclusions $\pi_1(\Sigma_j)\to\pi_1(\Sigma)$, $j=1,2$, are injective).

Thus it remains to show that $F_j(1)=(1,\tau_j')$ for $j=1,2$. We extend ρ into a smooth function $\rho:\Sigma\to\mathbb{R}$ such that $(-1)^{j-1}\rho>0$ on $\Sigma_j\setminus\gamma_\star$. Now there exists a continuous path $G:[0,1]\to([0,1]\times\mathbb{R}/\mathbb{Z})\setminus H^{-1}(\gamma_\star)$ such that

$$G(0)\in\{0\}\times\,]\tau_1,\tau_2[\quad\text{and}\quad G(1)\in\{1\}\times(\mathbb{R}/\mathbb{Z}\setminus\{\tau_1',\tau_2'\})$$

(indeed, otherwise it would mean that there is a continuous path in $[0,1]\times\mathbb{R}/\mathbb{Z}$ linking $(0,\tau_1)$ to $(0,\tau_2)$, which would imply, as in Lemma 4.4.6, that $\mathrm{i}(\gamma,\gamma_\star)=0$). In particular, we have $\rho\circ H\circ G>0$ since $\rho(H(0,\tau))>0$ for $\tau\in\,]\tau_1,\tau_2[$. Thus necessarily $G(1)\in\{1\}\times]\tau_1',\tau_2'[$ since $\rho(H(1,\tau))<0$ for $\tau\in\,]\tau_2',\tau_1'[$. Now, as $\mathrm{Im}(F_1)\cap\mathrm{Im}(F_2)=\emptyset$ (again, if the intersection was not empty we could find a path linking $(0,\tau_1)$ to $(0,\tau_2)$), we have that $G(1)$ lies in $]T_1(1),T_2(1)[$. Since $(\rho\circ H\circ G)(1)>0$, it follows that $T_1(1)=\tau_1'$ and $T_2(1)=\tau_2'$. The lemma is proven. □

The above lemma motivates the next result.

Proposition 4.4.8. *There is a constant $C>0$ such that the following holds. For any $w\in\pi_1(\Sigma_j)$ such that w is not a power of $w_{\star,j}$. Then there are $p_w,q_w\in\mathbb{Z}$ such that if $w'=w_{\star,j}^{p_w}ww_{\star,j}^{q_w}$ it holds*

$$\ell_\star(w_{\star,j}^p w' w_{\star,j}^q)\geqslant(|p|+|q|)\ell(\gamma_\star)+\ell_\star(w')-C,\quad p,q\in\mathbb{Z}.\tag{4.4.12}$$

In what follows, for any $x, y \in \widetilde{\Sigma}$, we will denote by $[x, y]$ the unique geodesic segment joining x and y. Before starting the proof of Proposition 4.4.8, we state a classical result valid in negatively curved spaces.

Lemma 4.4.9. *For each $\delta > 0$ there exists a constant $C > 0$ such that the following holds. For any sequence of geodesic segments $[x_0, x_1], [x_1, x_2], [x_2, x_3]$ in $\widetilde{\Sigma}$ such that $\operatorname{dist}(x_1, x_2) \geqslant \delta$ and such that the angle between $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ is equal to $\pm\pi/2$ for $j = 1, 2$ it holds*

$$\operatorname{dist}(x_0, x_3) \geqslant \operatorname{dist}(x_0, x_1) + \operatorname{dist}(x_1, x_2) + \operatorname{dist}(x_2, x_3) - C. \tag{4.4.13}$$

Proof. We will need the following intermediate result.

Fact 4.4.10. *For any $\varepsilon > 0$, there is $C > 0$ such that for any pairwise distinct points $x, y, z \in \widetilde{\Sigma}$ such that the absolute value of the angle (taken in $]-\pi, \pi]$) between $[x, y]$ and $[y, z]$ is not smaller than ε, we have*

$$\operatorname{dist}(x, z) \geqslant \operatorname{dist}(x, y) + \operatorname{dist}(y, z) - C.$$

Proof of Fact 4.4.10. We prove the result by comparing $\widetilde{\Sigma}$ with a model space of constant curvature, as follows. Let $a = \operatorname{dist}(x, y)$, $b = \operatorname{dist}(y, z)$, $c = \operatorname{dist}(x, z)$, and $\gamma = \angle([x, y], [y, z])$. Let $\widetilde{\Sigma}_k$ be simply connected complete Riemannian surface with constant curvature $-k^2 < 0$, such that $\kappa \leqslant -k^2$ everywhere for some $k > 0$ (recall that κ is the curvature of Σ). Consider any points $\bar{x}, \bar{y}, \bar{z} \in \widetilde{\Sigma}_k$ such that

$$\operatorname{dist}_k(\bar{x}, \bar{y}) = a, \quad \operatorname{dist}_k(\bar{y}, \bar{z}) = b \quad \text{and} \quad \angle([\bar{x}, \bar{y}], [\bar{y}, \bar{z}]) = \gamma,$$

where dist_k is the distance in $\widetilde{\Sigma}_k$, and set $\bar{c} = \operatorname{dist}_k(x, z)$. Then by a classical trigonometric formula for spaces of constant negative curvature (see [BH13, I.2.7]),

$$\operatorname{ch}(k\bar{c}) = \operatorname{ch}(ka)\operatorname{ch}(kb) - \operatorname{sh}(ka)\operatorname{sh}(kb)\cos(\gamma).$$

As $\gamma \in]-\pi, \pi] \setminus]-\varepsilon, \varepsilon[$, we have $\cos(\gamma) \leqslant 1 - \eta$ for some $\eta \in]0, 1[$ depending on ε. Thus

$$\operatorname{ch}(k\bar{c}) \geqslant \eta \operatorname{ch}(ka)\operatorname{ch}(kb).$$

Using $\exp(t)/2 \leqslant \operatorname{ch}(t) \leqslant \exp(t)$ for $t \geqslant 0$ one gets

$$\bar{c} \geqslant a + b + \frac{\log(\eta/4)}{k}.$$

As the scalar curvature of $\widetilde{\Sigma}$ is everywhere not greater than $-k^2$, the space $\widetilde{\Sigma}$ is a $\mathrm{CAT}(-k^2)$ space (see [BH13, Theorem II.4.1]). In particular by comparison one obtains $c \geqslant \bar{c}$ (see [BH13, Proposition II.1.7]), which concludes the proof. □

We are now in position to prove Lemma 4.4.9. Let x_0, x_1, x_2, x_3 as in the statement. For $j = 0, 1, 2$ we set $d_j = \operatorname{dist}(x_j, x_{j+1})$. We first assume one of the numbers d_0 or d_2 is not greater than δ, say $d_0 \leqslant \delta$. Then Fact 4.4.10 (applied with $x = x_1$, $y = x_2$ and $z = x_3$) yields $\operatorname{dist}(x_1, x_3) \geqslant d_1 + d_2 - C$ and thus

$$\operatorname{dist}(x_0, x_3) \geqslant \operatorname{dist}(x_1, x_3) - \operatorname{dist}(x_0, x_1) \geqslant d_1 + d_2 + C - a_0 \geqslant d_0 + d_1 + d_2 + C - 2\delta.$$

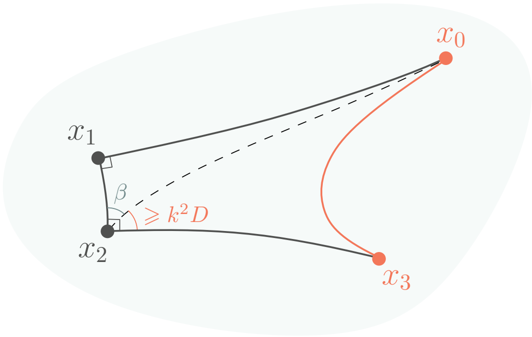

FIGURE 4.6 – Proof of Lemma 4.4.9.

Therefore we may assume that $d_0, d_2 \geqslant \delta$. Applying Fact 4.4.10 for the points x_0, x_1 and x_2 yields

$$\mathrm{dist}(x_0, x_2) \geqslant d_0 + d_1 - C. \tag{4.4.14}$$

For any pairwise distinct $x, y, z \in \widetilde{\Sigma}$, we denote by $\Delta(x, y, z)$ the triangle generated by x, y, z. Then as $d_0, d_1 \geqslant \delta$, the triangle $\Delta(x_0, x_1, x_2)$ contains some triangle $\Delta(x, y, z)$ with a right angle at y and $\mathrm{dist}(x, y) = \mathrm{dist}(y, z) = \delta$ (namely, $y = x_1$, $x \in [x_1, x_0]$ and $z \in [x_1, x_2]$). Clearly the area $|\Delta(x, y, z)|$ of $\Delta(x, y, z)$ is bounded from below by some constant $D > 0$ depending only on $\delta > 0$ (indeed, it suffices to verify this property for x, y, z lying in a compact set given by a finite union of fundamental domains of $\widetilde{\Sigma}$). Therefore, $|\Delta(x_0, x_1, x_2)| \geqslant D$. Let α and β be the angles of $\Delta(x_0, x_1, x_2)$ at x_0 and x_2, respectively (see Figure 4.6). Let $\tilde{\mu}_g$ bet the Riemannian measure of $\widetilde{\Sigma}$, and $\tilde{\kappa}$ its scalar curvature. Then, by the Gauss-Bonnet formula [Lee97, Theorem 9.3], it holds

$$\int_{\Delta(x_0, x_1, x_2)} \tilde{\kappa} \, \mathrm{d}\tilde{\mu}_g + \pi/2 + (\pi - \alpha) + (\pi - \beta) = 2\pi.$$

This gives

$$\beta \leqslant \pi/2 - \alpha - k^2 |\Delta(x_0, x_1, x_2)| \leqslant \pi/2 - k^2 D.$$

Therefore the angle between $[x_0, x_2]$ and $[x_2, x_3]$ is not smaller than $k^2 D$. In particular, we may apply Fact 4.4.10 to get $\mathrm{dist}(x_0, x_3) \geqslant \mathrm{dist}(x_0, x_2) + d_2 - C$ for some C depending only on $k^2 D$. Combining this with (4.4.14), we conclude the proof. □

Proof of Proposition 4.4.8. We fix $j \in \{1, 2\}$ and denote $w_\star = w_{\star,j}$ for simplicity. Let $w \in \pi_1(\Sigma_j)$ such that $w \neq w_\star^k$ for any k. Then w is not the trivial element and thus it is hyperbolic. Recall that $(\widetilde{\Sigma}, \tilde{g})$ is the universal cover of (Σ, g) and that $\pi_1(\Sigma)$ act by deck transformations on $\widetilde{\Sigma}$. For any $u \in \pi_1(\Sigma) \setminus \{1\}$, we denote by

$$u_\pm = \lim_{k \to +\infty} u^{\pm k}(z)$$

the two distinct fixed points of u in the boundary at infinity $\partial_\infty \widetilde{\Sigma}$ of $\widetilde{\Sigma}$ (here z denotes any point in $\widetilde{\Sigma}$). We also denote by A_u the translation axis of u, that is, the unique complete geodesic of $(\widetilde{\Sigma}, \tilde{g})$ converging towards u_+ (resp. u_-) in the future (resp. in the past). Note that $A_{ww_\star w^{-1}} = wA_{w_\star}$. As the conjugacy classes $[ww_\star w^{-1}]$ and $[w_\star]$ both represent the geodesic $\gamma_\star$, we have either $A_{w_\star} = wA_{w_\star}$ or $A_{w_\star} \cap wA_{w_\star} = \emptyset$.

Since w is not a power of $w_\star$, we necessarily have $A_{w_\star} \cap wA_{w_\star} = \emptyset$. Write $\gamma_\star = \{\varphi_s(z_\star) \ : \ t \in [0, \ell(\gamma_\star)]\}$ for some $z_\star = (x_\star, v_\star) \in M$. By hyperbolicity of the geodesic flow, there is $D > 0$ such that the following holds. For any $z \in M$ such that $\inf_{s\in\mathbb{R}} \mathrm{dist}_M(z, \varphi_s(z_\star)) \leqslant D$, it holds

$$\varphi_{\ell(\gamma_\star)}(z) = z \qquad \Longrightarrow \qquad z = \varphi_s(z_\star) \text{ for some } s \in \mathbb{R}. \tag{4.4.15}$$

This fact implies

$$\mathrm{dist}(A_{w_\star}, wA_{w_\star}) \geqslant D. \tag{4.4.16}$$

Let $\tilde{x} \in A_{w_\star}$ and $\tilde{y} \in wA_{w_\star}$ be the unique points such that $\mathrm{dist}(\tilde{x}, \tilde{y}) = \mathrm{dist}(A_{w_\star}, wA_{w_\star})$,

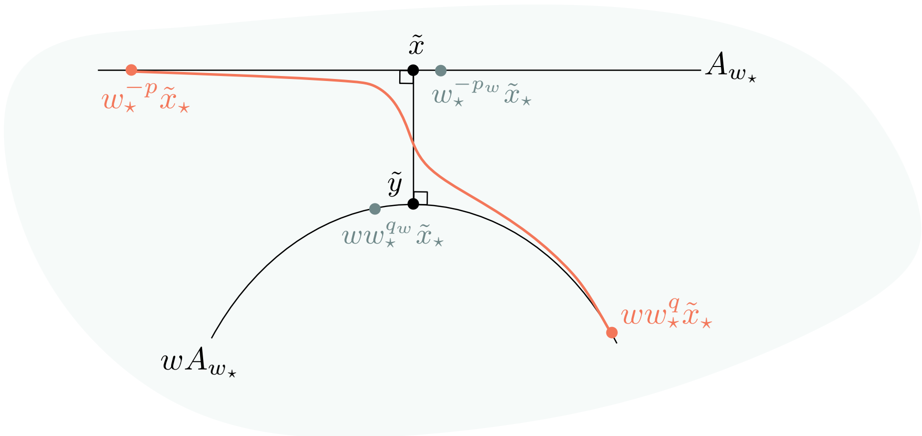

FIGURE 4.7 – Proof of Proposition 4.4.8.

and take $p, q \in \mathbb{Z}$. By (4.4.16), we may apply Lemma 4.4.9 with the sequence of geodesic segments $[w_\star^{-p}\tilde{x}_\star, \tilde{x}]$, $[\tilde{x}, \tilde{y}]$, $[\tilde{y}, ww_\star^q\tilde{x}_\star]$ to obtain

$$\mathrm{dist}(ww_\star^q\tilde{x}_\star, w_\star^{-p}\tilde{x}_\star) \geqslant \mathrm{dist}(ww_\star^q\tilde{x}_\star, \tilde{y}) + \mathrm{dist}(\tilde{y}, \tilde{x}) + \mathrm{dist}(\tilde{x}, w_\star^{-p}\tilde{x}_\star) - C$$

for some $C > 0$ independent of w, p and q (see Figure 4.7). Next, let $p_w, q_w \in \mathbb{Z}$ such that

$$\mathrm{dist}(\tilde{x}, w_\star^{-p_w}\tilde{x}_\star) < \ell(\gamma_\star) \quad \text{and} \quad \mathrm{dist}(\tilde{y}, ww_\star^{q_w}\tilde{x}_\star) < \ell(\gamma_\star).$$

Then for any $p, q \in \mathbb{Z}$ we have

$$\mathrm{dist}(\tilde{x}, w_\star^{-p}\tilde{x}_\star) \geqslant |p - p_w|\ell(\gamma_\star) - \ell(\gamma_\star), \quad \mathrm{dist}(\tilde{y}, ww_\star^q\tilde{x}_\star) \geqslant |q - q_w|\ell(\gamma_\star) - \ell(\gamma_\star),$$

which yields

$$\mathrm{dist}(w_\star^p ww_\star^q\tilde{x}_\star, \tilde{x}_\star) \geqslant (|p - p_w| + |q - q_w|)\ell(\gamma_\star) + \mathrm{dist}(\tilde{x}, \tilde{y}) - C - 2\ell(\gamma_\star).$$

Finally, we note that

$$\mathrm{dist}(\tilde{x}, \tilde{y}) \geqslant \mathrm{dist}(ww_\star^{q_w}\tilde{x}_\star, w_\star^{-p_w}\tilde{x}_\star) - 2\ell(\gamma_\star) = \ell_\star(w_\star^{p_w}ww_\star^{q_w}) - 2\ell(\gamma_\star),$$

which completes the proof. □

Building on Lemmata 4.4.6 and 4.4.7 and Proposition 4.4.8, we prove Proposition 4.4.5.

Proof of Proposition 4.4.5. In what follows, C is a positive constant independent of L that may change at each line. First, assume that $h_1 = h_2 = h_\star$. For $j = 1, 2$ we denote by $\langle w_{\star,j} \rangle = \{w_{\star,j}^n \ : \ n \in \mathbb{Z}\}$ the infinite cyclic subgroup of $\pi_1(\Sigma_j)$ generated by $w_{\star,j}$, and we set $\pi_1(\Sigma_j)_\star = \pi_1(\Sigma_j) \setminus \langle w_{\star,j} \rangle$. Since $\ell_\star(w_{\star,j}^n) = |n|\ell(\gamma_\star)$, there is C such that for any large L it holds

$$C^{-1} \mathrm{e}^{h_\star L} \leqslant N_{\star,j}(L) \leqslant C \mathrm{e}^{h_\star L} \tag{4.4.17}$$

by (4.4.10), where $N_{\star,j}(L) = \sharp\{w \in \pi_1(\Sigma_j)_\star \ : \ \ell_\star(w) \leqslant L\}$. For $w \in \pi_1(\Sigma_j)_\star$, we set

$$\mathcal{C}_w = \{w_\star^p w w_\star^q \ : \ p, q \in \mathbb{Z}\} \subset \pi_1(\Sigma_j)_\star,$$

and we define $\mathscr{C}_j = \{\mathcal{C}_w \ : \ w \in \pi_1(\Sigma_j)_\star\}$. Note that the elements $\mathcal{C} \in \mathscr{C}_j$ are pairwise disjoint, and thus we have a partition $\bigsqcup_{\mathcal{C} \in \mathscr{C}_j} \mathcal{C}$ of $\pi_1(\Sigma_j)_\star$. We also denote

$$\ell_\star(\mathcal{C}) = \inf\{\ell_\star(w) \ : \ w \in \mathcal{C}\}, \quad \mathcal{C} \in \mathscr{C}_j, \quad j = 1, 2.$$

Then Proposition 4.4.8 yields

$$\sharp\{w \in \mathcal{C} \ : \ \ell_\star(w) \leqslant L\} \leqslant C(L - \ell_\star(\mathcal{C}) + C)^2$$

for any $\mathcal{C} \in \mathscr{C}_j$ such that $\ell_\star(\mathcal{C}) \leqslant L$. Thus

$$\begin{aligned} N_{\star,j}(L) &= \sum_{\substack{\mathcal{C} \in \mathscr{C}_j \\ \ell_\star(\mathcal{C}) \leqslant L}} \sharp\{w \in \mathcal{C} \ : \ \ell_\star(w) \leqslant L\} \\ &\leqslant C \sum_{\substack{\mathcal{C} \in \mathscr{C}_j \\ \ell_\star(\mathcal{C}) \leqslant L}} (L - \ell_\star(\mathcal{C}) + C)^2 \end{aligned}$$

Let $\beta > 0$ be a large number. Then

$$\sum_{\substack{\mathcal{C} \in \mathscr{C}_j \\ \ell_\star(\mathcal{C}) \leqslant L - \beta \log L}} (L - \ell_\star(\mathcal{C}) + C)^2 \leqslant (L + C)^2 \sharp\{\mathcal{C} \in \mathscr{C}_j \ : \ \ell_\star(\mathcal{C}) \leqslant L - \beta \log L\}. \tag{4.4.18}$$

However, using (4.4.17), we obtain

$$\sharp\{\mathcal{C} \in \mathscr{C}_j \ : \ \ell_\star(\mathcal{C}) \leqslant L - \beta \log L\} \leqslant N_{\star,j}(L - \beta \log L) \leqslant C L^{-h_\star \beta} \mathrm{e}^{h_\star L}.$$

In particular, if $h_\star \beta > 2$, and if $A_\beta(L)$ denotes the left-hand side of (4.4.18), we have the bound $A_\beta(L) \ll N_{\star,j}(L)$ as $L \to \infty$. Thus for large L it holds

$$\begin{aligned} C^{-1} N_{\star,j}(L) &\leqslant \sum_{\substack{\mathcal{C} \in \mathscr{C}_j \\ \ell_\star(\mathcal{C}) \in [L - \beta \log L, L]}} (L - \ell_\star(\mathcal{C}) + C)^2 \\ &\leqslant (\beta \log L + C)^2 \sharp\big\{\mathcal{C} \in \mathscr{C}_j \ : \ \varepsilon L \leqslant \ell(\mathcal{C}) \leqslant L\big\}, \end{aligned}$$

where $\varepsilon > 0$ is any small number. This finally yields, for any large L,

$$\sharp\{\mathcal{C} \in \mathscr{C}_j \ : \ \varepsilon L \leqslant \ell(\mathcal{C}) \leqslant L\} \geqslant C^{-1} \mathrm{e}^{h_\star L} / (\beta \log L + C)^2. \tag{4.4.19}$$

For any $\mathcal{C} \in \mathscr{C}_j$, we choose some $w_{\mathcal{C}} \in \mathcal{C}$ such that $\ell_\star(w_{\mathcal{C}}) = \ell_\star(\mathcal{C})$. Next, Lemmata 4.4.6 and 4.4.7 imply that we have a well defined and injective map

$$\mathscr{C}_1 \times \mathscr{C}_2 \to \{\gamma \in \mathcal{P} \ : \ \mathrm{i}(\gamma, \gamma_\star) = 2\}, \quad (\mathcal{C}_1, \mathcal{C}_2) \mapsto [w_{\mathcal{C}_2} w_{\mathcal{C}_1}] \equiv \gamma_{w_{\mathcal{C}_2} w_{\mathcal{C}_1}}.$$

Obviously, $\ell(\gamma_{w_2 w_1}) \leqslant \ell_\star(w_1) + \ell_\star(w_2)$ for any w_1, w_2, and thus we get for large L

$$\begin{aligned} N(2,L) &\geqslant \sharp\Big\{(\mathcal{C}_1, \mathcal{C}_2) \in \mathscr{C}_1 \times \mathscr{C}_2 \ : \ \ell_\star(\mathcal{C}_1) + \ell_\star(\mathcal{C}_2) \leqslant L \\ &\qquad\qquad \text{and } \ell_\star(\mathcal{C}_1), \ell_\star(\mathcal{C}_2) \geqslant \varepsilon L\Big\} \\ &\geqslant \sum_{\substack{\mathcal{C}_1 \in \mathscr{C}_1 \\ \varepsilon L \leqslant \ell_\star(\mathcal{C}_1) \leqslant L}} \sharp\left\{\mathcal{C}_2 \in \mathscr{C}_2 \ : \ \varepsilon L \leqslant \ell_\star(\mathcal{C}_2) \leqslant L - \ell_\star(\mathcal{C}_1)\right\} \\ &\geqslant \sum_{\substack{\mathcal{C}_1 \in \mathscr{C}_1 \\ \varepsilon L \leqslant \ell_\star(\mathcal{C}_1) \leqslant L}} \frac{C^{-1} \mathrm{e}^{h_\star(L - \ell_\star(\mathcal{C}_1))}}{(\beta \log(L - \ell_\star(\mathcal{C}_1)) + C)^2}. \end{aligned}$$

For simplicity, in what follows we will use the notations $f(\ell) = C^{-1}\mathrm{e}^{h_\star \ell}/(\beta\log(\ell) + C)^2$, and $N(\mathscr{C}_1, L) = \sharp\{\mathcal{C} \in \mathscr{C}_j \ : \ \varepsilon L \leqslant \ell(\mathcal{C}) \leqslant L\}$. Fix some large number $\mu > 0$. Note that if μ is large enough, there is $C > 0$ (depending on μ) such that for any large ℓ

$$f(\ell + \mu) - f(\ell) \geqslant C^{-1} f(\ell). \tag{4.4.20}$$

Now, there holds,

$$\begin{aligned} N(2,L) &\geqslant C^{-1} \sum_{k \in \left[\frac{\varepsilon L}{\mu}, \frac{L}{\mu}\right]} \Big(N(\mathscr{C}_1, k\mu) - N(\mathscr{C}_1, (k-1)\mu)\Big) f(L - (k-1)\mu) \\ &\geqslant C^{-1} \sum_{k \in \left[\frac{\varepsilon L}{\mu}+1, \frac{L}{\mu}-1\right]} N(\mathscr{C}_1, k\mu)\Big(f(L-(k-1)\mu) - f(L - k\mu)\Big) \\ &\qquad - N(\mathscr{C}_1, \varepsilon L + \mu) f(L - \varepsilon L), \end{aligned} \tag{4.4.21}$$

where we used an Abel transformation in the last inequality. Next, note that by (4.4.17) one has $N(\mathscr{C}_1, L) \leqslant N_{\star,1}(L) \leqslant C\mathrm{e}^{h_\star L}$. This yields

$$N(\mathscr{C}_1, \varepsilon L + \mu) f(L - \varepsilon L) = \mathcal{O}(\mathrm{e}^{h_\star L}) \tag{4.4.22}$$

as $L \to \infty$. On the other hand, (4.4.20) gives for any large L,

$$\begin{aligned} &\sum_{k \in \left[\frac{\varepsilon L}{\mu}+1, \frac{L}{\mu}-1\right]} N(\mathscr{C}_1, k\mu)\Big(f(L-(k-1)\mu) - f(L-k\mu)\Big) \\ &\qquad \geqslant \sum_{k \in \left[\frac{\varepsilon L}{\mu}+1, \frac{L}{\mu}-1\right]} N(\mathscr{C}_1, k\mu) f(L - k\mu) \\ &\qquad \geqslant C^{-1} \sum_{k \in \left[\frac{\varepsilon L}{\mu}+1, \frac{L}{\mu}-1\right]} \frac{\mathrm{e}^{h_\star k\mu}}{(\beta\log(k\mu)+C)^2} \frac{\mathrm{e}^{h_\star(L-k\mu)}}{(\beta\log(L-k\mu)+C)^2} \\ &\qquad \geqslant \frac{C^{-1} L \mathrm{e}^{h_\star L}(1-\varepsilon)}{2\mu(\log(L)+C)^4}. \end{aligned}$$

We conclude the proof of Proposition 4.4.5 in the case $h_1 = h_2$ by combining this last estimate with (4.4.21) and (4.4.22).

If $h_1 \neq h_2$, say $h_1 > h_2$ (the case $h_1 < h_2$ is identical), one is able to obtain the desired bound by considering for example the injective map $\mathscr{C}_1 \to \{\gamma \in \mathcal{P} \ : \ \mathrm{i}(\gamma, \gamma_\star) = 2\}$ given by $\mathcal{C} \mapsto [a_{\mathrm{g}} w_{\mathcal{C}}]$ and by using (4.4.19). $\square$

4.4.2.2 Upper bound

Clearly, each $\gamma \in \mathcal{P}_2$ with $\ell(\gamma) \leqslant L$ may be represented by the conjugacy class of $w_1 w_2$ for some $w_j \in \pi_1(\Sigma_j)$ with $\ell_\star(w_1) + \ell_\star(w_2) \leqslant L + C$. Therefore (7.4.2) implies

$$\begin{aligned} N(2, L) &\leqslant \sharp \{(w_1, w_2) \in \pi_1(\Sigma_1) \times \pi_1(\Sigma_2) \ : \ \ell_\star(w_1) + \ell_\star(w_2) \leqslant L + C\} \\ &\leqslant \sum_{k=0}^{L+C} C \exp(h_1 k) \exp(h_2(L - k + C)), \end{aligned}$$

which gives for large L, if $h_\star = \max(h_1, h_2)$,

$$N(2, L) \leqslant \begin{cases} CL \exp(h_\star L) & \text{if} \quad h_1 = h_2, \\ C \exp(h_\star L) & \text{if} \quad h_1 \neq h_2. \end{cases}$$

Iterating this process we obtain (with C depending on n)

$$N(2n, L) \leqslant \begin{cases} CL^{2n-1} \exp(h_\star L) & \text{if} \quad h_1 = h_2, \\ CL^{n-1} \exp(h_\star L) & \text{if} \quad h_1 \neq h_2. \end{cases}$$

4.4.3 Relative growth of closed geodesics with a small intersection angle

For $x = \gamma_\star(\tau) \in \mathrm{Im}(\gamma_\star)$, we let $v_\star(x) = \dot{\gamma}_\star(\tau)$. For any $\eta > 0$ small, we consider the number $N(n, \eta, L) = \sharp \mathcal{P}_{\eta,n}(L)$ where $\mathcal{P}_{\eta,n}(L)$ is the set of closed geodesics $\gamma : \mathbb{R}/\ell(\gamma)\mathbb{Z} \to \Sigma$ of length not greater than L, intersecting $\gamma_\star$ exactly n times, and such that there is $t \in \mathbb{R}/\ell(\gamma)\mathbb{Z}$ with $\gamma(t) \in \mathrm{Im}(\gamma_\star)$ with

$$\mathrm{angle}\big(\dot{\gamma}(t), v_\star(\gamma(t))\big) < \eta \quad \text{or} \quad \mathrm{angle}\big(\dot{\gamma}(t), -v_\star(\gamma(t))\big) < \eta.$$

The purpose of this paragraph is to prove the following estimate.

Lemma 4.4.11. *Let $n \geqslant 1$. For any $\varepsilon, L_0 > 0$, there exists $\eta_0 > 0$ such that for any $\eta \in \,]0, \eta_0[$ and any large L*

$$N(1, \eta, L) \leqslant 4N(1, L - L_0) \quad \textit{and} \quad N(n, \eta, L) \leqslant \varepsilon L^{n-1} \exp(h_\star L), \tag{4.4.23}$$

if $\gamma_\star$ is not separating and $N(2, \eta, L) \leqslant 4N(2, L - L_0)$ and

$$N(2n, \eta, L) \leqslant \begin{cases} \varepsilon L^{2n-1} \exp(h_\star L) & \textit{if} \quad h_1 = h_2, \\ \varepsilon L^{n-1} \exp(h_\star L) & \textit{if} \quad h_1 \neq h_2, \end{cases} \tag{4.4.24}$$

if $\gamma_\star$ is separating.

Proof. We first prove the lemma when $\gamma_\star$ is assumed not separating. Let $\gamma : [0, \ell(\gamma)] \to \Sigma$ be an element of $\mathcal{P}_{\eta,n}(L)$ parameterized by arc length. Let $0 \leqslant t_1 < t_2 < \cdots < t_n < \ell(\gamma)$ be such that $\gamma(t_j) \in \mathrm{Im}(\gamma_\star)$. For every $j = 1, \ldots, n$, we choose a path c_j contained in $\mathrm{Im}(\gamma_\star)$, of length not greater than $\ell(\gamma_\star)$, and linking $x_j = \gamma(t_j)$ to $x_\star$. Recall that we have a map $q_\star : \Sigma_\star \to \Sigma$ given by the identification of the boundary components of $\Sigma_\star$. Write $q_\star^{-1}(x_\star) = \{x_\star, \bar{x}_\star\}$, where we chose some $x_\star \in \Sigma_\star$ with $q_\star(x_\star) = x_\star$, as in §4.4.1. Then γ is freely homotopic to the composition

$$w_1 w_2 \cdots w_n, \quad \text{where} \quad w_j = c_{j+1}\gamma|_{[t_j, t_{j+1}]} c_j^{-1} \in \pi_1(\Sigma), \quad j = 1, \ldots, n,$$

with the convention that $t_{n+1} = \ell(\gamma)$ and $c_{n+1} = c_1$. Note also that

$$\ell_\star(w_j) \leqslant |t_{j+1} - t_j| + 2\ell(\gamma_\star).$$

In fact, the elements w_j actually define elements of the space $\pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$, that is, the space of equivalence classes of paths $c : [0,1] \to \Sigma_\star$ with $c(0), c(1) \in \{x_\star, \bar{x}_\star\}$, where two paths are equivalent if they are homotopic via an homotopy preserving the endpoints. The space $\pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$ is not a group (we may not be able to concatenate two paths) ; however, we have a natural map $\pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\}) \to \pi_1(\Sigma)$. In particular, for any $u_1, \ldots, u_n \in \pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$, the composition $u_n \cdots u_1$ is well defined in $\pi_1(\Sigma)$. For any $u \in \pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$, we will denote by $\ell_\star(u)$ the infimum of the lengths of curves in the equivalence class u.

Up to reparameterizing of γ we may assume that $t_1 = 0$, and either $\angle\big(v, v_\star\big) < \eta$ or $\angle\big(v, -v_\star\big) < \eta$, where we set $x = \gamma(0)$, $v_\star = v_\star(x)$ and $v = \dot{\gamma}(0)$. We will first assume that $\angle\big(v, v_\star\big) < \eta$. Let $L_0 > 0$ be a large number and $\varepsilon > 0$ be small. By continuity of the geodesic flow (φ_t), there is $\eta_0 > 0$ such that if $\eta < \eta_0$ one has

$$\mathrm{dist}_M(\varphi_t(v), \varphi_t(v_\star)) \leqslant \varepsilon, \quad t \in [0, L_0].$$

Let K be a positive integer such that $K \in [L_0/\ell(\gamma_\star) - 1, L_0/\ell(\gamma_\star)]$, so that

$$\mathrm{dist}_\Sigma(\pi(\varphi_{K\ell(\gamma_\star)}(v)), x) < \varepsilon.$$

Let c_K be a path in Σ of length not greater than ε linking $\pi(\varphi_{K\ell(\gamma_\star)}(v))$ and x. Then if $\varepsilon > 0$ is small enough, we have [10]

$$c_1 c_K \gamma|_{[0, K\ell(\gamma_\star)]} c_1^{-1} = a_{\mathrm{g}}^K \quad \text{in } \pi_1(\Sigma).$$

In particular, it holds $w_1 = w_1' a_{\mathrm{g}}^K$ in $\pi_1(\Sigma)$, where $w_1' = c_2 \gamma|_{[K\ell(\gamma_\star), t_2]} c_K^{-1} c_1^{-1}$. Note also that

$$\ell_\star(w_1') \leqslant |t_2 - K\ell(\gamma_\star)| + 2\ell(\gamma_\star) + \varepsilon,$$

where w_1' is seen as an element of $\pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$. Note that if we had assumed $\angle\big(v, -v_\star\big) < \eta$, we would have obtained the same factorization with a_{g}^{-K} instead of a_{g}^K. Next, denote by $A_{K,n}(L)$ the set

$$\Big\{(w_1, \ldots, w_n) \in \pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})^n \ : \ \sum_{j=1}^n \ell_\star(w_j) \leqslant L + (2n - K)\ell(\gamma_\star) + \varepsilon\Big\},$$

10. Indeed, if $\varepsilon > 0$ is small enough, we have the following property. For any $x \in \Sigma$ and $L > 0$, if we are given two paths $c, c' : [0, L] \to \Sigma$ such that $c(0) = c'(0) = c(L) = c'(L) = x$ and $\mathrm{dist}_\Sigma(c(t), c'(t)) < \varepsilon$, then c and c' define the same element in $\pi_1(\Sigma, x)$.

and consider the map

$$\Psi_{K,n,\pm} : A_{K,n}(L) \to \mathcal{P}, \quad (w_1, \ldots, w_n) \mapsto \left[w_1 \cdots w_n a_{\mathrm{g}}^{\pm K}\right].$$

Then the discussion above shows that

$$\mathcal{P}_{\eta,n}(L) \subset \mathrm{Im}(\Psi_{K,n,+}) \cup \mathrm{Im}(\Psi_{K,n,-}).$$

In particular, $N(n,\eta,L) \leqslant 2\ \sharp A_{K,n}(L)$. Next, we obtain a bound on $A_{K,n}(L)$ as follows. Let $c_\star$ be a path connecting $\bar{x}_\star$ and $x_\star$ in $\Sigma_\star$, so that the image of $c_\star^{-1}$ in $\pi_1(\Sigma)$ is b_{g} (see Figure 4.4.1). Then it is not hard to see that for any $w \in \pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$, there is $u \in \pi_1(\Sigma_\star, x_\star)$ such that w can be written in one of the forms

$$u, \quad c_\star u, \quad u c_\star^{-1}, \quad \text{or} \quad c_\star u c_\star^{-1}$$

(depending on the endpoints of w), with $\ell_\star(u) \leqslant \ell_\star(w) + 2\ell(c_\star)$. This immediately gives

$$\sharp A_{K,1}(L) \leqslant 4\ \sharp\{u \in \pi_1(\Sigma_\star)\ :\ \ell_\star(u) \leqslant L\} \leqslant C\exp(h_\star L).$$

As in §4.4.1.2 we obtain, for some $C_n > 0$ depending only n,

$$\sharp A_{K,n}(L) \leqslant C_n L^{n-1} \exp(h_\star(L - L_0))$$

where we used that $K\ell(\gamma_\star) \geqslant L_0 - \ell(\gamma_\star)$. This proves the second part of (4.4.23). For the first part, we proceed as follows. With the notations of the proof of Proposition 4.4.5, one has well defined maps

$$\Psi_{K,1,\pm,\mathrm{r}}, \Psi_{K,1,\pm,\mathrm{l}} : \{\mathcal{C} \in \mathscr{C}\ :\ \ell_\star(w) \leqslant L - K\ell(\gamma_\star)\} \to \{\gamma \in \mathcal{P}_1\ :\ \ell(\gamma) \leqslant L + 2C\}$$

given respectively by $\mathcal{C} \mapsto [a_{\mathrm{g}}^{\pm K} w b_{\mathrm{g}}]$ and $\mathcal{C} \mapsto [b_{\mathrm{g}}^{-1} w a_{\mathrm{g}}^{\pm K}]$ where w is any element of $\mathcal{C}$. Next, we remark that the above discussion implies that every $\gamma \in \mathcal{P}_{\eta,1}(L)$ can be written as

$$[a_{\mathrm{g}}^{\pm K} w b_{\mathrm{g}}] \quad \text{or} \quad [b_{\mathrm{g}}^{-1} w a_{\mathrm{g}}^{\pm K}]$$

for some $w \in \pi_1(\Sigma_\star)$ with $\ell_\star(w) \leqslant L - K\ell(\gamma_\star) + C$. Therefore the union of the images of the maps $\Psi_{K,1,\pm,\mathrm{r}}, \Psi_{K,1,\pm,\mathrm{l}}$ contains $\mathcal{P}_\eta(L + 2C)$, and thus

$$N(1,\eta,L) \leqslant 4\sharp\{\mathcal{C} \in \mathscr{C}\ :\ \ell_\star(w) \leqslant L - K\ell(\gamma_\star) + 2C\} \leqslant 4N(1, L - K\ell(\gamma_\star) + 3C),$$

where we used the first inequality of (4.4.8). This gives the first part of (4.4.23).

Next, assume that $\gamma_\star$ is separating. Then, as above, every $\gamma : [0, \ell(\gamma)] \to \Sigma$ such that $\gamma \in \mathcal{P}_{2n,\eta}(L)$ can be written as a composition $w_{1,1} w_{1,2} \cdots w_{1,n} w_{2,n}$ for some $w_{k,j} \in \pi_1(\Sigma_k)$ ($k = 1, 2$ and $j = 1, 2, \ldots, n$), with

$$\sum_{j=1}^{n} \ell_\star(w_{2,j}) + \ell_\star(w_{1,j}) \leqslant \ell(\gamma) + 4n\ell(\gamma_\star).$$

Now if η is small, we may proceed as before to obtain (up to reparameterization of γ) that $w_{1,1} = w_{\star,1}^{\pm K} w'_{1,1}$ or $w_{1,1} = w'_{1,1} w_{\star,1}^{\pm K}$ for some $w'_{1,1} \in \pi_1(\Sigma_1)$ with

$$\ell_\star(w'_{1,1}) \leqslant \ell_\star(w_{1,1}) - K\ell(\gamma_\star) + C.$$

Here K is a large number depending on η (i.e. such that $K \to \infty$ as $\eta \to 0$) and $C > 0$ is a constant independent of γ and K. Thus we get

$$N(2n, \eta, L) \leqslant C \ \sharp\Big\{(w_{1,1}, w_{2,1}, \dots, w_{1,n}, w_{2,n}) \ : \ w_{k,j} \in \pi_1(\Sigma_k), \\ \sum_{j=1}^{n} \ell_\star(w_{1,j}) + \ell_\star(w_{2,j}) \leqslant L - K\ell(\gamma_\star) + C_n\Big\}.$$

Then we obtain the second part of (4.4.24) by proceeding as in §4.4.2.2. For the first part of (4.4.24), we proceed as follows. For $w_j \in \pi_1(\Sigma_j)_\star$, we denote

$$\mathcal{C}_{w_1,w_2} = \{(w_1', w_2') \ : \ [w_1'w_2'] = [w_1 w_2]\},$$

and $\ell_\star(\mathcal{C}_{w_1,w_2}) = \inf\{\ell_\star(w_1') + \ell_\star(w_2') \ : \ (w_1', w_2') \in \mathcal{C}_{(w_1,w_2)}\}$. We also introduce the notation $\mathscr{C}_{1,2} = \{\mathcal{C}_{w_1,w_2} \ : \ w_j \in \pi_1(\Sigma_j)_\star\}$. By Lemmata 4.4.6 and 4.4.7, we have well defined maps $\Psi_{K,1,\pm,\mathrm{r}}, \Psi_{K,1,\pm,\mathrm{l}}$, mapping

$$\{\mathcal{C} \in \mathscr{C}_{1,2} \ : \ \ell_\star(\mathcal{C}_{w_1,w_2}) \leqslant L - K\ell(\gamma_\star)\} \to \{\gamma \in \mathcal{P}_2 \ : \ \ell(\gamma) \leqslant L\},$$

given respectively by $\mathcal{C} \mapsto [w_1 w_{\star,1}^{\pm K} w_2]$ and $\mathcal{C} \mapsto [w_{\star,1}^{\pm K} w_1 w_2]$. By the discussion above, the union of the images of those maps contains $\mathcal{P}_{2,\eta}(L)$. Therefore

$$N(2, \eta, L) \leqslant 4\sharp\{\mathcal{C} \in \mathscr{C}_{1,2} \ : \ \ell_\star(\mathcal{C}_{w_1,w_2}) \leqslant L - K\ell(\gamma_\star)\} \leqslant 4N(2, L - K\ell(\gamma_\star)),$$

where we used Lemmata 4.4.6 and 4.4.7 again in the last inequality. The first part of (4.4.24) follows. □

4.5 A Tauberian argument

The goal of this section is to give an asymptotic growth of the quantity

$$N_\pm(n, \chi, t) = \sum_{\substack{\gamma \in \mathcal{P} \\ \mathrm{i}(\gamma_\star, \gamma) = n \\ \ell(\gamma) \leqslant t}} I_{\star,\pm}(\gamma, \chi)$$

as $t \to +\infty$, where $\chi \in C_c^\infty(\partial \setminus \partial_0)$ and $I_{\star,\pm}(\gamma, \chi) = \prod_{z \in I_{\star,\pm}(\gamma)} \chi^2(z)$.

4.5.1 The case $\gamma_\star$ is not separating

By [DG16, Theorem 3 and §6.2], we know that the zeta function

$$\zeta_{\Sigma_\star}(s) = \prod_{\gamma \in \mathcal{P}_\star} \left(1 - \mathrm{e}^{-s\ell(\gamma)}\right)^{-1}$$

extends meromorphically to the whole complex plane, and moreover we may write

$$\zeta_{\Sigma_\star}'(s)/\zeta_{\Sigma_\star}(s) = \sum_{k=0}^{2} (-1)^k \mathrm{tr}^\flat \left(\mathrm{e}^{\pm\varepsilon s} \varphi_{\mp\varepsilon}^* R_{\pm,\delta}(s)|_{\Omega_c^k(M_\delta) \cap \ker \iota_X}\right),$$

where the flat trace is computed on M_δ. Here $\mathcal{P}_\star$ denote the set of primitive closed geodesics of $(\Sigma_\star, g)$. By [Dal99], we may apply [PP83, Proposition 9] (see also [PP90, Theorem 9.1]) to obtain that $\zeta_{\Sigma_\star}$ is holomorphic in $\{\mathrm{Re}(s) \geqslant h_\star\}$ except for a simple pole at $s = h_\star$, where $h_\star > 0$ is the topological entropy of the geodesic flow of $(\Sigma_\star, g)$ restricted to its trapped set. Write the Laurent expansion given in §4.2.6 of $R_{\pm,\delta}(s)$ near $s = h_\star$ as

$$R_{\pm,\delta}(s) = Y_{\pm,\delta}(s) + \frac{\Pi_{\pm,\delta}(h_\star)}{s - h_\star} + \sum_{j=2}^{J(h_\star)} \frac{(X \pm h_\star)^{j-1}\Pi_{\pm,\delta}(h_\star)}{(s-h_\star)^j} : \Omega_c^\bullet(M_\delta) \to \mathcal{D}'^\bullet(M_\delta).$$

By [DG16, Equation (5.8)], we have $\mathrm{tr}^\flat(\mathrm{e}^{\pm\varepsilon h_\star}\varphi_{\mp\varepsilon}^*\Pi_{\pm,\delta}(h_\star)) = \mathrm{rank}\ \Pi_{\pm,\delta}(h_\star)$ and

$$\mathrm{tr}^\flat\left(\varphi_{\mp\varepsilon}^*(X \pm h_\star)^j \Pi_{\pm,\delta}(h_\star)\right) = 0, \quad j = 1, \dots, J(h_\star) - 1.$$

We denote $\Omega^k = \Omega_c^k(M_\delta)$ and $\Omega_0^k = \Omega^k \cap \ker \iota_X$. Then by [Gui17, Propositions 2.4 and 4.4], the map $s \mapsto R_{\pm,\delta}(s)|_{\Omega_0^0}$ has no pole in $\{\mathrm{Re}(s) > 0\}$. Since $\Omega_0^2 = \Omega_0^0 \wedge \mathrm{d}\alpha$, and $R_{\pm,\delta}(s)|_{\Omega_0^2} = R_{\pm,\delta}(s)|_{\Omega_0^0} \wedge \mathrm{d}\alpha$ (because $\varphi_t^*\alpha = \alpha$), it follows that $s \mapsto R_{\pm,\delta}(s)|_{\Omega_0^2}$ has no poles in $\{\mathrm{Re}(s) > 0\}$. In particular the residue of $\zeta'_{\Sigma_\star}(s)/\zeta_{\Sigma_\star}(s)$ at $s = h_\star$ is given by $\mathrm{rank}(\Pi_{\pm,\delta}(h_\star)|_{\Omega_0^1})$, and since $\zeta_{\Sigma_\star}(s)$ has a simple pole at $s = h_\star$, this residue is equal to 1. Therefore

$$\mathrm{rank}(\Pi_{\pm,\delta}(h_\star)|_{\Omega_0^1}) = 1.$$

In particular $(X \pm h_\star)^j \Pi_{\pm,\delta} = 0$ for each $j = 1, \dots, J(h_\star) - 1$. As $R_{\pm,\delta}(s)$ commutes with ι_X, it preserves the spaces Ω_0^k. Writing $\Omega^k = \Omega_0^k \oplus \alpha \wedge \Omega_0^{k-1}$ we have for any $w = u + \alpha \wedge v$ with $\iota_X u = 0$ and $\iota_X v = 0$,

$$\Pi_{\pm,\delta}(h_\star)|_{\Omega^2}(u + \alpha \wedge v) = \Pi_{\pm,\delta}(h_\star)|_{\Omega_0^2}(u) + \alpha \wedge \Pi_{\pm,\delta}(h_\star)|_{\Omega_0^1}(v).$$

Thus $\Pi_{\pm,\delta}(h_\star)|_{\Omega^2} = \alpha \wedge \iota_X \Pi_{\pm,\delta}(h_\star)|_{\Omega_0^1}$. By Proposition 4.3.2, and the fact that $\varphi_{\pm\varepsilon}^*\Pi_{\pm,\delta}(h_\star) = \mathrm{e}^{\pm\varepsilon h_\star}\Pi_{\pm,\delta}(h_\star)$, we have near $s = h_\star$

$$(-1)^N \chi \tilde{\mathcal{S}}_\pm(s)\chi = \chi Y_\pm(s)\chi + \frac{\chi\psi^*\iota^*\iota_X\Pi_{\pm,\delta}(h_\star)\iota_*\chi}{s - h_\star}, \tag{4.5.1}$$

where $s \mapsto Y_\pm(s)$ is holomorphic in a neighborhood of $h_\star$ and N is the number operator. We denote

$$\Pi_{\pm,\partial} = \psi^*\iota^*\iota_X\Pi_{\pm,\delta}(h_\star)\iota_* : \Omega^\bullet(\partial) \to \mathcal{D}'^\bullet(\partial).$$

Then by what precedes, and since $\iota_X\Pi_{\pm,\delta}(h_\star)|_{\Omega^1} = 0$, we obtain that the rank of $\Pi_{\pm,\partial}$ is not greater than 1, and that $\Pi_{\pm,\partial} = \Pi_{\pm,\partial}|_{\Omega^1(\partial)}$. In particular, (4.5.1) reads

$$-\chi\tilde{\mathcal{S}}_\pm(s)\chi = (-1)^{N+1}\chi Y_\pm(s)\chi + \frac{\chi\Pi_{\pm,\partial}\chi}{s - h_\star}. \tag{4.5.2}$$

In what follows, we will set

$$c_\pm(\chi) = \mathrm{tr}_s^\flat(\chi\Pi_{\pm,\partial}\chi) = -\,\mathrm{tr}^\flat\left(\chi\Pi_{\pm,\partial}\chi|_{\Omega^1(\partial)}\right)$$

(see §B.3.3).

Lemma 4.5.1. *Let* $\chi \in C_c^\infty(\partial \setminus \partial_0)$ *and assume that* $c_\pm(\chi) > 0$. *Then it holds*

$$N_\pm(n,\chi,t) \sim \frac{(c_\pm(\chi)t)^n}{n!}\frac{\mathrm{e}^{h_\star t}}{h_\star t}, \quad t \to +\infty.$$

Proof. Because $\chi\Pi_{\pm,\partial}\chi$ is of rank one, it follows that [11]

$$\begin{aligned}\operatorname{tr}_{\mathrm{s}}^\flat((\chi\Pi_{\pm,\partial}\chi)^n) &= -\operatorname{tr}^\flat\left((\chi\Pi_{\pm,\partial}\chi)^n|_{\Omega^1(\partial)}\right)\\ &= -\operatorname{tr}^\flat\left(\chi\Pi_{\pm,\partial}\chi|_{\Omega^1(\partial)}\right)^n\\ &= (-1)^{n+1}c_\pm(\chi)^n\end{aligned}$$

for any $n \geqslant 1$. Writing $\operatorname{tr}_{\mathrm{s}}^\flat((-\chi\tilde{\mathcal{S}}_\pm(s)\chi)^n) = (-1)^n\operatorname{tr}_{\mathrm{s}}^\flat((\chi\tilde{\mathcal{S}}_\pm(s)\chi)^n)$, one gets, by (4.5.2),

$$-\operatorname{tr}_{\mathrm{s}}^\flat((\chi\tilde{\mathcal{S}}_\pm(s)\chi)^n) = \frac{c_\pm(\chi)^n}{(s-h_\star)^n} + \mathcal{O}((s-h_\star)^{-n+1}), \quad s \to h_\star. \tag{4.5.3}$$

Note that here, we implicitely used the fact that the flat trace of products of the form

$$(\chi Y_\pm(s)\chi)^{k_1}(\chi\Pi_{\pm,\partial}\chi)^{\ell_1}(\chi Y_\pm(s)\chi)^{k_2}(\chi\Pi_{\pm,\partial}\chi)^{\ell_2}\cdots \tag{4.5.4}$$

makes sense. Indeed, note that both $\mathrm{WF}(\chi\Pi_{\pm,\partial}\chi)$ and $\mathrm{WF}(\chi Y_\pm(s)\chi)$ are contained in the set $\mathrm{WF}(\chi\tilde{\mathcal{S}}_\pm(s)\chi)$ by (4.5.2) and Cauchy's integral formula. Thus we may reproduce the proofs of Lemma 4.3.6, Remark 4.3.7 and Proposition 4.3.8 to obtain that the composition (4.5.4) is well defined that its flat trace makes sense. Next, set $\eta_{n,\chi}(s) = -\operatorname{tr}_{\mathrm{s}}^\flat((\chi\tilde{\mathcal{S}}_\pm(s)\chi)^n)$, and

$$g_{n,\chi}(t) = \sum_{\substack{\gamma\in\mathcal{P}\\ \mathrm{i}(\gamma,\gamma_\star)=n}} \ell^\sharp(\gamma) \sum_{\substack{k\geqslant 1\\ k\ell(\gamma)\leqslant t}} I_{\star,\pm}(\gamma,\chi)^k, \quad t \geqslant 0,$$

Now if $G_{n,\chi}(s) = \int_0^{+\infty} g_{n,\chi}(t)\mathrm{e}^{-ts}\mathrm{d}t$, a simple computation leads to

$$G_{n,\chi}(s) = \frac{1}{s}\sum_{i(\gamma,\gamma_\star)=n} \ell^\sharp(\gamma)\mathrm{e}^{-s\ell(\gamma)}I_{\star,\pm}(\gamma,\chi)^{\ell(\gamma)/\ell^\sharp(\gamma)} = -\frac{\eta'_{n,\chi}(s)}{ns},$$

where the last equality comes from Proposition 4.3.8. Because one has the expansion $\eta'_{n,\chi}(s) = -nc_\pm(\chi)^n(s-h_\star)^{-(n+1)} + \mathcal{O}((s-h_\star)^{-n})$ as $s \to h_\star$ by (4.5.3) we obtain

$$G_{n,\chi}(h_\star s) = \frac{c_\pm(\chi)^n}{h_\star^{n+2}(s-1)^{n+1}} + \mathcal{O}((s-h_\star)^{-n}), \quad s \to h_\star.$$

Then applying the Tauberian theorem of Delange [Del54, Théorème III] (see Theorem A.2.1 of Appendix A) there holds

$$\frac{1}{h_\star}g_{n,\chi}(t/h_\star) \sim \frac{c_\pm(\chi)^n}{h_\star^{n+2}}\frac{\mathrm{e}^t}{n!}t^n, \quad t \to +\infty,$$

11. Indeed, the equality $\operatorname{tr}^\flat(A^n) = \operatorname{tr}^\flat(A)^n$ holds whenever A is of rank one and has a smooth kernel. By approximation this remains true for any A of rank one whenever $\operatorname{tr}^\flat(A^n)$ and $\operatorname{tr}^\flat(A)^n$ make sense.

which reads

$$g_{n,\chi}(t) \sim \frac{(c_\pm(\chi)t)^n}{n!h_\star} \exp(h_\star t). \tag{4.5.5}$$

Now note that, if $\mathcal{P}_n$ is the set of primitive closed geodesics γ with $\mathrm{i}(\gamma,\gamma_\star) = n$, there holds

$$g_{n,\chi}(t) \leqslant \sum_{\substack{\gamma\in\mathcal{P}_n \\ \ell(\gamma)\leqslant t}} \ell(\gamma)\lfloor t/\ell(\gamma)\rfloor I_{\star,\pm}(\gamma,\chi) \leqslant tN(n,\chi,t).$$

As a consequence, one gets

$$\liminf_{t\to+\infty} N_\pm(n,\chi,t)\frac{n!h_\star t}{(c_\pm(\chi)t)^n \mathrm{e}^{h_\star t}} \geqslant 1. \tag{4.5.6}$$

For the other bound, we use the a priori bound obtained in §4.4.1.2

$$N_\pm(n,\chi,t) \leqslant N(n,t) \leqslant \frac{Ct^n}{n!}\frac{\mathrm{e}^{h_\star t}}{h_\star t} \tag{4.5.7}$$

to deduce that for any $\sigma > 1$

$$\limsup_{t\to+\infty} N_\pm(n,\chi,t/\sigma)\frac{n!}{t^n}\frac{h_\star t}{\mathrm{e}^{h_\star t}} = 0. \tag{4.5.8}$$

Now we may write

$$\begin{aligned} N_\pm(n,\chi,t) &= N_\pm(n,\chi,t/\sigma) + \sum_{\substack{\gamma\in\mathcal{P} \\ \mathrm{i}(\gamma_\star,\gamma)=n \\ t/\sigma\leqslant\ell(\gamma)\leqslant t}} I_{\star,\pm}(\gamma,\chi) \\ &\leqslant N_\pm(n,\chi,t/\sigma) + \frac{\sigma}{t}\sum_{\substack{\gamma\in\mathcal{P} \\ \mathrm{i}(\gamma_\star,\gamma)=n \\ t/\sigma\leqslant\ell(\gamma)\leqslant t}} I_{\star,\pm}(\gamma,\chi)\ell(\gamma) \\ &\leqslant N_\pm(n,\chi,t/\sigma) + \frac{\sigma}{t}g_{n,\chi}(t), \end{aligned} \tag{4.5.9}$$

which gives, with (4.5.5) and (4.5.8),

$$\limsup_{t\to+\infty} N_\pm(n,\chi,t)\frac{n!}{(c_\pm(\chi)t)^n}\frac{h_\star t}{\mathrm{e}^{h_\star t}} \leqslant \sigma.$$

As $\sigma > 1$ is arbitrary, the Lemma is proven. □

Remark 4.5.2. If $c_\pm(\chi) = 0$, then with the notations of the above proof, the function $s \mapsto \eta_{1,\chi}(s)$ has no pole on the line $\{\mathrm{Re}(s) = h_\star\}$. In particular we may reproduce the arguments of the aforementioned proof, replacing $g_{n,\chi}(t)$ by $g_{n,\chi}(t) + \exp(h_\star t)$ to obtain that $s \mapsto \int_0^\infty (g_{n,\chi}(t)+\exp(h_\star t))\exp(-ts)\mathrm{d}t$ has a pole of order 1 at $s = h_\star$ and thus $g_{n,\chi}(t) + \exp(h_\star t) \sim \exp(h_\star t)$ as $t \to \infty$. This implies $g_{n,\chi}(t) \ll_{t\to\infty} \exp(h_\star t)$, yielding

$$N_\pm(1,\chi,t) \ll \exp(h_\star t)/t, \quad t \to \infty,$$

where we used the last line of (4.5.9) and (4.5.7).

4.5.2 The case $\gamma_\star$ is separating

If $\gamma_\star$ is separating, then Σ_δ consists of two surfaces $\Sigma_\delta^{(1)}$ and $\Sigma_\delta^{(2)}$. We write $M_\delta = M_\delta^{(1)} \sqcup M_\delta^{(2)}$ where $M_\delta^{(j)} = S\Sigma_\delta^{(j)}$, $j = 1, 2$, and $\partial = \partial^{(1)} \sqcup \partial^{(2)}$ with $\partial^{(j)} \subset M_\delta^{(j)}$. As before, fix $\chi \in C_c^\infty(\partial \setminus \partial_0)$. Note that, if $\tilde{\mathcal{S}}_\pm^{(j)}(s)$ denotes the restriction of $\tilde{\mathcal{S}}_\pm(s)$ to $\partial^{(j)}$, we have

$$\chi \tilde{\mathcal{S}}_\pm^{(1)}(s)\chi : \Omega^\bullet(\partial^{(1)}) \to \mathcal{D}'^\bullet(\partial^{(2)}), \quad \chi \tilde{\mathcal{S}}_\pm^{(2)}(s)\chi : \Omega^\bullet(\partial^{(2)}) \to \mathcal{D}'^\bullet(\partial^{(1)}).$$

As in §4.5.1, we have

$$-\chi \tilde{\mathcal{S}}_\pm^{(j)}(s)\chi = (-1)^{N+1}\chi Y_\pm^{(j)}(s)\chi + \frac{\chi \Pi_{\pm,\partial}^{(j)}\chi}{s - h_j}, \quad s \to h_j, \tag{4.5.10}$$

where $\operatorname{rank}(\Pi_{\pm,\partial}^{(j)}) = 1$, and $Y_\pm^{(j)}(s)$ is holomorphic near $s = h_j$ and h_j is the topological entropy of the geodesic flow of $\Sigma_\delta^{(j)}$.

4.5.2.1 The case $h_1 \neq h_2$

We may assume $h_1 > h_2$ and we define

$$c_\pm(\chi) = \operatorname{tr}_s^\flat\left(\chi \tilde{\mathcal{S}}_\pm^{(2)}(h_1)\chi^2 \Pi_{\pm,\partial}^{(1)}\chi\right) = -\operatorname{tr}^\flat\left(\chi \tilde{\mathcal{S}}_\pm^{(2)}(h_1)\chi^2 \Pi_{\pm,\partial}^{(1)}\chi|_{\Omega^1(\partial^{(1)})}\right).$$

Because $\Pi_{\pm,\partial}^{(1)} = \Pi_{\pm,\partial}^{(1)}|_{\Omega^1(\partial)}$ is of rank one, we have, as in §4.5.1,

$$\operatorname{tr}_s^\flat\left(\left(\chi \tilde{\mathcal{S}}_\pm^{(2)}(h_1)\chi^2 \Pi_{\pm,\partial}^{(1)}\chi\right)^n\right) = (-1)^{n+1} c_\pm(\chi)^n$$

for any $n \geqslant 1$. Therefore, (4.5.10) implies, by cyclicity of the flat trace (see B.3.1),

$$\begin{aligned}
-\operatorname{tr}_s^\flat\left((\chi \tilde{\mathcal{S}}_\pm(s)\chi)^{2n}\right) &= -\operatorname{tr}_s^\flat\left(\left(\chi \tilde{\mathcal{S}}_\pm^{(1)}(s)\chi^2 \tilde{\mathcal{S}}_\pm^{(2)}(s)\chi\right)^n + \left(\chi \tilde{\mathcal{S}}_\pm^{(2)}(s)\chi^2 \tilde{\mathcal{S}}_\pm^{(1)}(s)\chi\right)^n\right) \\
&= \frac{2c_\pm(\chi)^n}{(s - h_1)^n} + \mathcal{O}((s - h_1)^{-n+1}).
\end{aligned}$$

as $s \to h_1$. Now we may proceed exactly as in §4.5.1 to obtain that, if $c_\pm(\chi) > 0$,

$$N_\pm(2n, \chi, t) \sim \frac{(c_\pm(\chi)t)^n}{n!} \frac{\mathrm{e}^{h_\star t}}{h_\star t}, \quad t \to +\infty.$$

Remark 4.5.3 (Continuation of Remark 4.5.2)**.** If $h_1 \neq h_2$ and $c_\pm(\chi) = 0$, then the map $s \mapsto \operatorname{tr}_s^\flat\left((\chi \tilde{\mathcal{S}}_\pm(s)\chi)^2\right)$ has no pole on the line $\{\mathrm{Re}(s) = h_\star\}$. As in Remark 4.5.2 this yields

$$N_\pm(2, \chi, t) \ll \exp(h_\star t)/t, \quad t \to \infty.$$

4.5.2.2 The case $h_1 = h_2 = h_\star$

In that case, by denoting $c_\pm(\chi) = -\operatorname{tr}_{\mathrm{s}}^\flat(\chi \Pi^{(1)}_{\pm,\partial} \chi^2 \Pi^{(2)}_{\pm,\partial}\chi)$ we have

$$-\operatorname{tr}_{\mathrm{s}}^\flat\left((\chi \tilde{\mathcal{S}}_\pm(s)\chi)^{2n}\right) = \frac{2c_\pm(\chi)^n}{(s-h_\star)^{2n}} + \mathcal{O}((s-h_\star)^{-2n+1}), \quad s \to h_\star.$$

Again, provided that $c_\pm(\chi) \neq 0$, we may proceed exactly as in §4.5.1 to obtain

$$N_\pm(2n,\chi,t) \sim 2\frac{(c_\pm(\chi)t^2)^n}{(2n)!}\frac{e^{h_\star t}}{h_\star t}.$$

Remark 4.5.4 (Continuation of Remark 4.5.3)**.** If $h_1 = h_2$ and $c_\pm(\chi) = 0$, then the map $s \mapsto -\operatorname{tr}_{\mathrm{s}}^\flat\left((\chi\tilde{\mathcal{S}}_\pm(s)\chi)^2\right)$ may have a pole at $s = h_\star$, of order at most 1. Therefore, reproducing the arguments of §4.5.1, we see that this would imply

$$N_\pm(2,\chi,t) = \mathcal{O}(\exp(h_\star t)), \quad t \to \infty.$$

Note that here, assuming $c_\pm(\chi) = 0$ only makes us win a factor t for the bound on $N_\pm(2,\chi,t)$, whereas in Remarks 4.5.2 and 4.5.3 we could win a bit more. This is the reason for which we needed a sharper bound on $N(2,L)$ in §4.4.1.

4.6 Proof of Theorems 4.1.1 and 4.1.2

In this section we prove Theorems 4.1.1 and 4.1.2. We will apply the asymptotic growth we obtained in the last section to some appropriate sequence of functions in $C^\infty_c(\partial \setminus \partial_0)$. Let $F \in C^\infty(\mathbb{R},[0,1])$ be an even function such that $F \equiv 0$ on $[-1,1]$ and $F \equiv 1$ on $]-\infty,-2] \cup [2,+\infty[$. For any small $\eta > 0$, set

$$F_\eta(t) = \sum_{k\in\mathbb{Z}} F((t-k\pi)/\eta).$$

Then F_η is 2π-periodic and it induces a function $F_\eta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}_{\geqslant 0}$. In the coordinates from Lemma 4.2.3, we define

$$\chi_\eta(z) = F_\eta(\theta), \quad z = (\tau, 0, \theta) \in \partial.$$

Then $\chi_\eta \in C^\infty_c(\partial \setminus \partial_0)$ for any $\eta > 0$ small. Note that the function χ_η is introduced in order to forget about trajectories passing at distance not greater than η from the "glancing" set $S\gamma_\star$.

4.6.1 The case $\gamma_\star$ is not separating

Recall from §4.4 that we have the *a priori* bounds

$$C^{-1}\frac{e^{h_\star L}}{h_\star L} \leqslant N(1,L) \leqslant Ce^{h_\star L} \tag{4.6.1}$$

for L large enough. This estimate implies the following fact [12] :

$$\forall \varepsilon > 0, \quad \exists L_0 > 0, \quad \forall L_1 > 0, \quad \exists L > L_1, \quad N(1, L - L_0) \leqslant \varepsilon N(1, L).$$

In particular, we see with the first part of (4.4.23) in Lemma 4.4.11 that for any $\eta > 0$ small enough, one has

$$\liminf_{L \to +\infty} \frac{N(1, \eta, L)}{N(1, L)} \leqslant \frac{1}{2}, \tag{4.6.2}$$

where $N(1, \eta, L)$ is defined in §4.4.3.

For $\eta > 0$ small and $L > 0$, neither $c_\pm(\chi_\eta)$ nor $N_\pm(n, \chi_\eta, L)$ (see §4.5.1) depend on $\pm$, since F is an even function. We denote them simply by $c(\eta)$ and $N(n, \chi_\eta, L)$ respectively. Then we claim that $c(\eta) > 0$ if $\eta > 0$ is small enough. Indeed, if $c(\eta) = 0$ then Remark 4.5.2 implies

$$N(1, \chi_\eta, L) \ll \exp(h_\star L)/h_\star L, \quad L \to +\infty. \tag{4.6.3}$$

On the other hand we have $N(1, L) = N(1, \chi_\eta, L) + R(\eta, L)$ with

$$R(\eta, L) = N(1, L) - N(1, \chi_\eta, L) \leqslant N(1, 2\eta, L),$$

and thus, if η is small enough, (4.6.2) gives

$$\limsup_{L \to +\infty} \frac{N(1, \chi_\eta, L)}{N(1, L)} \geqslant \frac{1}{2}.$$

Since for large L it holds $C^{-1} \exp(h_\star L)/L \leqslant N(1, L)$, we obtain that (4.6.3) cannot hold, and thus $c(\eta) > 0$.

In particular we can apply Lemma 4.5.1 to get

$$\lim_{L} N(n, \chi_\eta, L) \frac{n!}{(c(\eta)L)^n} \frac{h_\star L}{\mathrm{e}^{h_\star L}} = 1.$$

As $N(n, L) \geqslant N(n, \chi_\eta, L)$ we obtain that for L large enough

$$C^{-1} \frac{L^n}{n!} \frac{\mathrm{e}^{h_\star L}}{h_\star L} \leqslant N(n, L) \leqslant C \frac{L^n}{n!} \frac{\mathrm{e}^{h_\star L}}{h_\star L}$$

(the upper bound comes from §4.4.1.2). Let $\varepsilon > 0$. Then the above estimate combined with the second part of (4.4.23) in Lemma 4.4.11 implies that for $\eta > 0$ small enough, one has

$$\limsup_{L} R(n, \eta, L) \frac{n!}{L^n} \frac{h_\star L}{\mathrm{e}^{h_\star L}} < \varepsilon,$$

12. Indeed, if it does not hold, then there is $\varepsilon > 0$ such that for any $L_0 > 0$ there is L_1 such that for any $n \geqslant 0$, it holds

$$\varepsilon < \frac{N(1, L_1 + nL_0)}{N(1, L_1 + (n+1)L_0)},$$

which gives $N(1, L_1 + (n+1)L_0)\varepsilon^n < N(1, L_1)$ for each n. Now if L_0 is large enough, we see that (4.6.1) cannot hold by making $n \to \infty$.

where $R(n,\eta,L) = N(n,L) - N(n,\chi_\eta,L)$. Thus writing

$$N(n,\chi_\eta,L) \leqslant N(n,L) \leqslant N(n,\chi_\eta,L) + R(n,\eta,L)$$

we obtain

$$c(\eta)^n \leqslant \liminf_L N(n,L)\frac{n!}{L^n}\frac{h_\star L}{\mathrm{e}^{h_\star L}} \leqslant \limsup_L N(n,L)\frac{n!}{L^n}\frac{h_\star L}{\mathrm{e}^{h_\star L}} \leqslant c(\eta)^n + \varepsilon$$

for any η small enough (depending on ε !). As $\varepsilon > 0$ is arbitrary, we finally get

$$N(n,L) \sim \frac{(c_\star L)^n}{n!}\frac{\mathrm{e}^{h_\star L}}{h_\star L}, \quad L \to +\infty$$

where $c_\star = \lim_{\eta\to 0} c(\eta) < +\infty$ (the limit exists as $\eta \mapsto c(\eta)$ is nonincreasing and bounded by above by (4.6.1)).

4.6.2 The case $\gamma_\star$ is separating

4.6.2.1 The case $h_1 \neq h_2$

In that case recall from §4.4 that we have the bound

$$\frac{C^{-1}\mathrm{e}^{h_\star L}}{\log(L)^2} \leqslant N(2,L) \leqslant C\mathrm{e}^{h_\star L}$$

for L large enough. In particular, using (4.4.24) in Lemma 4.4.11 and Remark 4.5.3 we may proceed exactly as in §4.6.1 to obtain

$$N(2n,L) \sim \frac{(c_\star L)^n}{n!}\frac{\mathrm{e}^{h_\star L}}{h_\star L}, \quad L \to +\infty$$

where $c_\star = \lim_{\eta\to 0} c_\pm(\chi_\eta)$.

4.6.2.2 The case $h_1 = h_2 = h$

In that case recall from §4.4 that we have the bound

$$\frac{C^{-1}L\mathrm{e}^{h_\star L}}{\log(L)^4} \leqslant N(2,L) \leqslant CL\mathrm{e}^{h_\star L}$$

for L large enough. In particular, using Lemma 4.4.11 and Remark 4.5.4 we may proceed exactly as in §4.6.1 to obtain

$$N(2n,L) \sim 2\frac{(c_\star L)^n}{(2n)!}\frac{\mathrm{e}^{h_\star L}}{h_\star L}, \quad L \to +\infty$$

where $c_\star = \lim_{\eta\to 0} c_\pm(\chi_\eta)$.

4.7 A Bowen-Margulis type measure

4.7.1 Description of the constant $c_\star$

In this subsection we describe the constant $c_\star$ in terms of Pollicott-Ruelle resonant states of the open system (M_δ, φ_t), assuming for simplicity that $\gamma_\star$ is not separating. By §4.2.6 we may write, since $\Pi_{\pm,\delta}(h_\star)$ is of rank one (see §4.5.1),

$$\Pi_{\pm,\delta}(h_\star)|_{\Omega^1(M_\delta)} = u_\pm \otimes (\alpha \wedge s_\mp), \quad u_\pm, \in \mathcal{D}'^1_{E^*_{\pm,\delta}}(M_\delta), \quad s_\mp \in \mathcal{D}'^1_{E^*_{\mp,\delta}}(M_\delta),$$

with $\operatorname{supp}(u_\pm, s_\pm) \subset \Gamma_{\pm,\delta}$ and $u_\pm, s_\mp \in \ker(\iota_X)$. Using the Guillemin trace formula [Gui77] and the Ruelle zeta function $\zeta_{\Sigma_\star}$, we see that the Bowen-Margulis measure μ_0 (see [Bow72]) of the open system (M_δ, φ_t), which is given by Bowen's formula

$$\mu_0(f) = \lim_{L\to+\infty} \sum_{\substack{\gamma\in\mathcal{P}_\delta \\ \ell(\gamma)\leqslant L}} \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(\gamma(\tau), \dot\gamma(\tau))\mathrm{d}\tau, \quad f \in C_c^\infty(M_\delta),$$

coincides with the distribution $f \mapsto \operatorname{tr}^\flat_{\mathrm{s}}(f\Pi_{\pm,\delta}(h)) = \int_{M_\delta} f\, u_\pm \wedge \alpha \wedge s_\mp$. Note that $\operatorname{supp}(u_\pm \wedge \alpha \wedge s_\mp) \subset K_\star$, where $K_\star \subset S\Sigma_\star$ is the trapped set. On the other hand we have by definition of $\Pi_{\pm,\partial}$,

$$c_\star = -\lim_{\eta\to 0} \operatorname{tr}^\flat_{\mathrm{s}}(\chi_\eta \Pi_{\pm,\partial}\chi_\eta) = \lim_{\eta\to 0}\int_\partial \chi_\eta \psi^*\iota^* u_\pm \wedge \iota^* s_\mp \chi_\eta.$$

4.7.2 A Bowen-Margulis type measure

In what follows we set $S_{\gamma_\star}\Sigma = \{(x,v) \in S\Sigma \ :\ x \in \gamma_\star\}$ and for any primitive geodesic $\gamma : \mathbb{R}/\ell(\gamma)\mathbb{Z} \to \Sigma$,

$$I_\star(\gamma) = \{z \in S_{\gamma_\star}\Sigma \ :\ z = (\gamma(\tau), \dot\gamma(\tau)) \text{ for some } \tau\}.$$

For any $n \geqslant 1$ we define the set $\Gamma_n \subset S_{\gamma_\star}\Sigma$ by

$$\complement\Gamma_n = \{z \in S_{\gamma_\star}\Sigma \ :\ \left(\tilde S_\pm\right)^k(z) \text{ is well defined for } k = 1, \dots, n\}.$$

Also we set $\ell_n(z) = \max(\ell_{+,n}(z), \ell_{-,n}(z))$ where

$$\ell_{\pm,n}(z) = \ell_\pm(z) + \ell_\pm(\tilde S_\pm(z)) + \cdots + \ell_\pm(\tilde S_\pm^{n-1}(z)), \quad z \in \complement\Gamma_n,$$

and $\ell_\pm(z) = \inf\{t > 0 \ :\ \varphi_{\pm t}(z) \in S_{\gamma_\star}\Sigma\}$.

We will now prove Theorem 4.1.4 which says that for any $f \in C^\infty(S_{\gamma_\star}\Sigma)$ the limit

$$\mu_n(f) = \lim_{L\to+\infty} \frac{1}{N(n,L)} \sum_{\gamma\in\mathcal{P}_n} \frac{1}{n} \sum_{z\in I_\star(\gamma)} f(z) \tag{4.7.1}$$

exists and defines a probability measure μ_n on $S_{\gamma_\star}\Sigma$ supported in Γ_n. We will also prove that, in the non separating case,

$$\mu_n(f) = -c_\star^{-n} \lim_{\eta\to 0} \operatorname{tr}^\flat_{\mathrm{s}}(f(\chi_\eta \Pi_{\pm,\partial}\chi_\eta)^n), \tag{4.7.2}$$

where $c_\star > 0$ is the constant appearing in Theorem 4.1.1. Note that here we identify f with its lift $p_\star^* f$ (which is a function on ∂), so that the above formula makes sense (recall that $p_\star : S\Sigma_\star \to S\Sigma$ is the natural projection which identifies both components of $\partial S\Sigma_\star = \partial$). Of course, a similar formula holds in the non separating case but we omit it here.

Proof of Theorem 4.1.4. Let $f \in C^\infty(S_{\gamma_\star}\Sigma)$ be a non-negative function. Then reproducing the arguments in the proof of Proposition 4.3.8, we get for $\mathrm{Re}(s)$ big enough,

$$\mathrm{tr}_s^\flat\left(f(\chi_\eta \tilde{\mathcal{S}}_\pm(s)\chi_\eta)^n\right) = \sum_{i(\gamma,\gamma_\star)=n} \left(\sum_{z\in I_\star(\gamma)} f(z)\right) \mathrm{e}^{-s\ell(\gamma)} I_\star(\gamma,\chi_\eta),$$

where χ_η is defined in §4.6 and $I_\star(\gamma,\chi_\eta) = I_{\star,\pm}(\gamma,\chi_\eta)$ (see §4.5 ; this does not depend on $\pm$ as the function F used to construct χ_η is even). Now, as f is non-negative, we may proceed exactly as in §4.5, replacing $g_{n,\chi}(t)$ by

$$g_{n,\chi_\eta,f}(t) = \sum_{\substack{\gamma\in\mathcal{P}\\ \mathrm{i}(\gamma,\gamma_\star)=n}} \left(\sum_{z\in I_\star(\gamma)} f(z)\right) \sum_{\substack{k\geqslant 1\\ k\ell(\gamma)\leqslant t}} I_\star(\gamma,\chi_\eta), \quad t \geqslant 0,$$

to obtain that

$$\lim_{L\to\infty} \frac{n!}{L^n}\frac{h_\star L}{\mathrm{e}^{h_\star L}} \sum_{\substack{\gamma\in\mathcal{P}\\ \mathrm{i}(\gamma_\star,\gamma)=n\\ \ell(\gamma)\leqslant L}} \left(\sum_{z\in I_\star(\gamma)} f(z)\right) I_\star(\gamma,\chi_\eta) = -\mathrm{Res}_{s=h_\star} \mathrm{tr}_s^\flat(f(\chi_\eta\tilde{\mathcal{S}}_\pm(s)\chi_\eta)^n). \tag{4.7.3}$$

We denote by $\nu_{n,\eta}(f)$ the left-hand side of (4.7.3). Then $\eta \mapsto \nu_{n,\eta}(f)$ is a non-negative and nonincreasing function which is bounded by above by $nc_\star^n\|f\|_\infty$ by Theorem 4.1.1. In particular the formula

$$\mu_n(f) = \lim_{\eta\to 0} \frac{1}{nc_\star^n}\nu_{n,\eta}(f), \quad f \in C^\infty(S_{\gamma_\star}\Sigma, \mathbb{R}_{\geqslant 0}),$$

defines a measure μ_n on $S_{\gamma_\star}\Sigma$, whose total mass is not greater than 1. In fact its total mass is equal to 1, since by definition of $c_\star$ one has

$$\mu_n(1) = \lim_{\eta\to 0}\frac{nc_\pm(\chi_\eta)^n}{nc_\star^n} = 1.$$

Let $\varepsilon > 0$. Then for each $f \in C^\infty(S_{\gamma_\star}\Sigma, \mathbb{R}_{\geqslant 0})$ one has by Lemma 4.4.11

$$\sum_{\substack{\gamma\in\mathcal{P}\\ \mathrm{i}(\gamma_\star,\gamma)=n\\ \ell(\gamma)\leqslant L}} \left(\sum_{z\in I_\star(\gamma)} f(z)\right)(1 - I_\star(\gamma,\chi_\eta)) \leqslant nN(n,\eta,L)\|f\|_\infty \leqslant \varepsilon nN(n,L)\|f\|_\infty$$

for large L, whenever η is small enough. In particular, setting

$$\mu_n^+(f) = \limsup_L \frac{A_f(n,L)}{nN(n,L)} \quad \text{and} \quad \mu_n^-(f) = \liminf_L \frac{A_f(n,L)}{nN(n,L)}$$

where $A_f(n,L) = \sum_{\substack{\gamma\in\mathcal{P}\\ \mathrm{i}(\gamma_\star,\gamma)=n\\ \ell(\gamma)\leqslant L}} \left(\sum_{z\in I_\star(\gamma)} f(z)\right)$, we see that for η small it holds

$$\left|\mu_n^\pm(f) - \nu_{n,\eta}(f)\right| \leqslant \varepsilon\|f\|_\infty.$$

Indeed, setting $A_f(n,\eta,L) = \sum_{\substack{\gamma\in\mathcal{P}\\ \mathrm{i}(\gamma_\star,\gamma)=n\\ \ell(\gamma)\leqslant L}} \left(\sum_{z\in I_\star(\gamma)} f(z)\right) I_\star(\gamma,\chi_\eta)$, we have

$$\limsup_L \left|\left(\frac{1}{nN(n,L)} - \frac{n!L^n}{nc_\star^n \mathrm{e}^{h_\star L}}\right) A_f(n,\eta,L)\right| = 0$$

by Theorem 4.1.1, since $A_f(n,\eta,L) \leqslant nN(n,L)$. Now we may let $\eta\to 0$ to obtain $|\mu_n^\pm(f) - \mu_n(f)| \leqslant \varepsilon\|f\|_\infty$. Since ε is arbitrary, one gets $\mu_n^\pm(f) = \mu_n(f)$. This implies that the limit (4.7.1) exists, and moreover (4.7.2) holds by (4.7.3) (provided that $\gamma_\star$ is not separating).

Next, take a general $f\in C^\infty(S_{\gamma_\star}\Sigma)$ which we no longer assume to be non-negative. We choose some smooth functions $f_{\delta,\pm}$, $\delta\in\,]0,1[$, with the property that $\|f - f_{\delta,+} + f_{\delta,-}\|_\infty \leqslant \delta$ and $\pm f_{\delta,\pm}\geqslant 0$, and we write $f_\delta = f_{\delta_+} + f_{\delta_-}$. By nonnegativeness of $\pm f_{\delta,\pm}$, the arguments above imply that $A_{f_\delta}(n,L)/(nN(n,L)) \to \mu_n(f_\delta)$ as $L\to\infty$. On the other hand $|A_f(n,L) - A_{f_\delta}(n,L)| \leqslant A_{|f-f_\delta|}(n,L) \leqslant \delta nN(n,L)$. Letting $L\to\infty$ this yields

$$\mu_n(f_\delta) - \delta \leqslant \liminf_L \frac{A_f(n,L)}{nN(n,L)} \leqslant \limsup_L \frac{A_f(n,L)}{nN(n,L)} \leqslant \mu_n(f_\delta) + \delta.$$

Since $\mu_n(f_\delta)\to\mu_n(f)$ as $\delta\to 0$, one concludes that (4.7.1) and (4.7.2) are valid for f.

Finally, if $f\in C_c^\infty(S_{\gamma_\star}\Sigma\setminus\Gamma_n)$ then there is $L>0$ such that

$$\ell_n(z)\leqslant L, \quad z\in\operatorname{supp}(f).$$

In particular for any $\gamma\in\mathcal{P}$ such that $\mathrm{i}(\gamma,\gamma_\star) = n$ and $\ell(\gamma) > L$, we have $f(z) = 0$ for any $z\in I_\star(\gamma)$. This shows that $\mu_n(f) = 0$ and the support condition for μ_n follows. □

4.8 A large deviation result

The goal of this section, which is independent of the rest of the paper, is to prove the following result, which is a consequence of a classical large deviation result by Kifer [Kif94].

Proposition 4.8.1. *There exists $I_\star > 0$ such that the following holds. For any $\varepsilon > 0$, there are $C,\delta>0$ such that for large L*

$$\frac{1}{N(L)}\sharp\left\{\gamma\in\mathcal{P}\ :\ \ell(\gamma)\leqslant L,\ \left|\frac{i(\gamma,\gamma_\star)}{\ell(\gamma)} - I_\star\right| \geqslant \varepsilon\right\} \leqslant C\exp(-\delta L). \tag{4.8.1}$$

In fact, $I_\star = 4i(\bar{m},\delta_{\gamma_\star})$ where i is the Bonahon's intersection form [Bon86], $\delta_{\gamma_\star}$ is the Dirac measure on $\gamma_\star$ in and $\bar{m}$ is the renormalized Bowen-Margulis measure on M (here we see the intersection form as a function on the space of φ-invariant measures on $S\Sigma$, as described below). Lalley [Lal96] showed a similar result for self-intersection numbers ; see also [PS06] for self intersection numbers with prescribed angles.

4.8.1 Bonahon's intersection form

Let $\mathcal{M}_\varphi(S\Sigma)$ be the set of finite positive measures on $S\Sigma$ invariant by the geodesic flow, endowed with the vague topology. For any closed geodesic γ, we denote by $\delta_\gamma \in \mathcal{M}_\varphi(S\Sigma)$ the Lebesgue measure of γ parameterized by arc length (thus of total mass $\ell(\gamma)$). Let $\mu \in \mathcal{M}_\varphi(S\Sigma)$ be the Liouville measure, that is, the measure associated to the volume form $\frac{1}{2}\alpha \wedge d\alpha$.

Proposition 4.8.2 (Bonahon [Bon88], see also Otal [Ota90])**.** *There exists a continuous function*

$$\mathrm{i} : \mathcal{M}_\varphi(S\Sigma) \times \mathcal{M}_\varphi(S\Sigma) \to \mathbb{R}_+$$

which is additive and positively homogeneous with respect to each variable, such that $\mathrm{i}(\mu, \mu) = 2\pi \mathrm{vol}(\Sigma)$ *and*

$$\mathrm{i}(\delta_\gamma, \delta_{\gamma'}) = \mathrm{i}(\gamma, \gamma'), \quad \mathrm{i}(\mu, \delta_\gamma) = 2\ell(\gamma),$$

for any closed geodesics γ, γ'*.*

Remark 4.8.3. (i) Actually, Bonahon's intersection form is a pairing on the space of *geodesic currents*. This space is naturally identified with the space of φ-invariant measure on $S\Sigma$ which are also invariant by the flip $R : (x, v) \mapsto (x, -v)$. What we mean here by $\mathrm{i}(\nu, \nu')$ for general $\nu, \nu' \in \mathcal{M}_\varphi(S\Sigma)$ is simply $\mathrm{i}(\Phi(\nu), \Phi(\nu'))$ where $\Phi : \nu \mapsto \nu + R^*\nu$ (note that $\varphi_t R = R\varphi_{-t}$ for $t \in \mathbb{R}$).

(ii) Note that the formulae for $\mathrm{i}(\mu, \mu)$ and $\mathrm{i}(\mu, \delta_\gamma)$ differ from [Bon88] ; it is due to our convention since here the Liouville measure μ corresponds to twice the Liouville current considered in [Bon88].

4.8.2 Large deviations

For any $\nu \in \mathcal{M}_\varphi(S\Sigma)$ we denote by $h(\nu)$ the measure-theoretical entropy of φ with respect to ν. Then we have the following result.

Proposition 4.8.4 (Kifer [Kif94])**.** *Let* $F \subset \mathcal{M}^1_\varphi(S\Sigma)$ *be a closed set, where* $\mathcal{M}^1_\varphi(S\Sigma)$ *is the set of* φ*-invariant probability measures on* $S\Sigma$*. Then*

$$\limsup_L \frac{1}{L} \log \frac{1}{N(L)} \sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L, \ \delta_\gamma/\ell(\gamma) \in F\} \leqslant \sup_{\nu \in F} h(\nu) - h,$$

where h *is the entropy of the geodesic flow.*

Proof of Proposition 4.8.1. We denote by $\bar{m} \in \mathcal{M}^1_\varphi(S\Sigma)$ the unique probability measure of maximal entropy, that is

$$\bar{m} = \lim_{L \to +\infty} \sum_{\substack{\gamma \in \mathcal{P} \\ \ell(\gamma) \leqslant L}} \frac{\delta_\gamma}{\ell(\gamma)},$$

where the convergence holds in the weak sense. Let $\varepsilon > 0$. Define

$$F_\varepsilon = \{\nu \in \mathcal{M}^1_\varphi(S\Sigma) \ : \ |\mathrm{i}(\nu, \delta_{\gamma_\star}) - \mathrm{i}(\bar{m}, \delta_{\gamma_\star})| \geqslant \varepsilon\}.$$

Then F_ε is closed in $\mathcal{M}^1_\varphi(S\Sigma)$, and thus compact by the Banach–Alaoglu theorem, and $\bar{m} \in \complement F_\varepsilon$ so that $\delta = h - \sup_{\nu \in F_\varepsilon} h(\nu) > 0$. In particular we obtain that for large L

$$\frac{1}{N(L)} \sharp\{\gamma \in \mathcal{P} \ : \ \delta_\gamma/\ell(\gamma) \in F_\varepsilon\} \leqslant C \exp(-\delta' L)$$

for some $0 < \delta' < \delta$ and $C > 0$. Now, by Proposition 4.8.2, $\delta_\gamma/\ell(\gamma) \in F_\varepsilon$ reads $|i(\gamma,\gamma_\star)/\ell(\gamma) - \mathrm{i}(\bar{m},\delta_{\gamma_\star})| \geqslant \varepsilon$. Let $I_\star = \mathrm{i}(\bar{m},\delta_{\gamma_\star})$. Then it is a well known fact that $\bar{m}$ have full support in $S\Sigma$, which implies $I_\star > 0$ by definition of $\mathrm{i}(\bar{m},\delta_{\gamma_\star})$ (see [Ota90]). This concludes the proof. □

Remark 4.8.5. (i) It is not hard to see that Proposition 4.8.1 implies

$$\frac{1}{N(L)} \sum_{\ell(\gamma) \leqslant L} \mathrm{i}(\gamma,\gamma_\star) \sim I_\star L$$

as $L \to +\infty$. Thus we recover [Pol85, Theorem 4].

(ii) If (Σ, g) is hyperbolic then $\bar{m}$ is the renormalized Liouville measure and we find, with Proposition 4.8.2,

$$I_\star = \frac{\ell(\gamma_\star)}{2\pi^2(\mathrm{g}-1)}.$$

(iii) Note that if $\varepsilon < I_\star$, then every closed geodesic γ which does not intersection $\gamma_\star$ satisfies $\delta_\gamma/\ell(\gamma) \in F_\varepsilon$. In particular the right hand side of (4.8.1) is bounded from below by $C \exp((h_\star - h)L)$, where we used that $N(0,L) \sim \exp(h_\star L)/h_\star L$ and $N(L) \sim \exp(hL)/hL$ as $L \to \infty$.

4.9 Extension to multi-curves

In this section, we explain how the methods used before allow to obtain asymptotic results for closed geodesics of which *several* intersection numbers are prescribed. Namely, let $r \geqslant 1$ and $\gamma_{\star,1}, \dots, \gamma_{\star,r}$ be pairwise disjoint closed geodesics of (Σ, g), and denote by $\Sigma_1, \dots, \Sigma_q$ the connected components of $\Sigma \setminus \bigcup_{i=1}^r \gamma_{\star,i}$.

4.9.1 Notations

For any $j = 1, \dots, q$, we denote by $h_j > 0$ the topological entropy of the open system $(\Sigma_j, g|_{\Sigma_j})$, and by B_j the set of indexes i such that $\gamma_{\star,i}$ is a boundary component of Σ_j. We decompose B_j as

$$B_j = S_j \sqcup O_j$$

where S_j is the set of indexes i such that $\gamma_{\star,i}$ lies in the boundary of $\Sigma_{j'}$ for some $j' \neq j$, and $O_j = B_j \setminus S_j$. In fact S_j (resp. O_j) is the set of shared (resp. unshared) boundary components of Σ_j. For any $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ we define

$$\langle \mathbf{n}, \Sigma_j \rangle = \sum_{i=1}^r n_i \left(\frac{1_{S_j}(i)}{2} + 1_{O_j}(i) \right), \quad j = 1, \dots, q.$$

This quantity represents the number of times a curve has to travel through Σ_j if it intersects n_i times $\gamma_{\star,i}$.

An *admissible* path (u,v) is the collection of two words $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$ with $u_\ell \in \{1,\ldots,r\}$ and $v_\ell \in \{1,\ldots,q\}$ for $\ell = 1,\ldots,n$, and with the following property. For any $\ell \in \mathbb{Z}/n\mathbb{Z}$ we have $u_\ell, u_{\ell+1} \in B_{v_\ell}$ and

$$v_\ell = v_{\ell+1} \implies u_{\ell+1} \in O_{v_\ell}.$$

For any admissible path $\omega = (u,v)$ we denote $\mathbf{n}(\omega) = (n_1,\ldots,n_r)$ where we set $n_i = \sharp\{\ell \,:\, u_\ell = i\}$. An admissible path ω will be called *primitive* if every non trivial cyclic permutation of ω is distinct from ω.

An element $\mathbf{n} \in \mathbb{N}^r$ will be called admissible if $\mathbf{n} = \mathbf{n}(\omega)$ for some admissible path ω. For any admissible $\mathbf{n} \in \mathbb{N}^r$ we set

$$h_{\mathbf{n}} = \max\{h_j \,:\, \langle \mathbf{n}, \Sigma_j\rangle > 0\} \quad \text{and} \quad d_{\mathbf{n}} = \sum_{h_j = h_{\mathbf{n}}} \langle \mathbf{n}, \Sigma_j \rangle.$$

The number $h_{\mathbf{n}}$ is the maximum of the entropies encountered by a closed geodesic γ satisfying $i(\gamma,\gamma_\star) = n_i$ for $i = 1,\ldots,r$, while $d_{\mathbf{n}}$ is the number of times γ will travel through a surface Σ_j with $h_j = h_{\mathbf{n}}$.

4.9.2 Statement

For any primitive geodesic $\gamma \in \mathcal{P}$ we denote

$$\mathbf{i}(\gamma, \vec{\gamma}_\star) = (i(\gamma,\gamma_{\star,1}),\ldots,i(\gamma,\gamma_{\star,r})).$$

Note that each closed geodesic $\gamma : \mathbb{R}/\ell(\gamma)\mathbb{Z} \to \Sigma$ intersecting at least one of the $\gamma_{\star,i}$'s gives rise to an admissible path $\omega(\gamma)$ (which is unique up to cyclic permutation) defined as follows. Let $(\tau_1,\ldots,\tau_n) \in (\mathbb{R}/\ell(\gamma)\mathbb{Z})^n$ be a cyclically ordered sequence such that $\gamma^{-1}\left(\bigcup_i \gamma_{\star,i}\right) = \{\tau_1,\ldots,\tau_n\}$. Then there are words $u_1\cdots u_n$ and $v_1 \cdots v_n$ such that $\gamma(\tau_\ell) \in \gamma_{\star,u_\ell}$ and $\gamma(\tau) \in \Sigma_{v_\ell}$ for any $\tau \in]\tau_\ell, \tau_{\ell+1}[$ and we set $\omega(\gamma) = (u,v)$. For two paths ω, ω', we will write $\omega \sim \omega'$ if ω is a cyclic permutation of ω'; for any admissible $\omega = (u,v)$, we will denote by $w^k = (u^k, v^k)$ the path ω concatenated k times.

Theorem 4.9.1. *Let ω be an admissible and primitive path. Then there is $c_\omega > 0$ such that for any $k \geqslant 1$*

$$\sharp\{\gamma \in \mathcal{P} \,:\, \ell(\gamma) \leqslant L,\ \omega(\gamma) \sim \omega^k\} \sim d_{\mathbf{n}(\omega)} \frac{\left(c_\omega L^{d_{\mathbf{n}(\omega)}}\right)^k}{(k d_{\mathbf{n}(\omega)})!} \frac{\mathrm{e}^{h_{\mathbf{n}(\omega)} L}}{h_{\mathbf{n}(\omega)} L} \tag{4.9.1}$$

In particular we obtain for any admissible $\mathbf{n} \in \mathbb{N}^r$

$$\sharp\{\gamma \in \mathcal{P} \,:\, \ell(\gamma) \leqslant L \,:\, \mathbf{i}(\gamma,\gamma_\star) = \mathbf{n}\} \sim C_{\mathbf{n}} L^{d_{\mathbf{n}}} \frac{\mathrm{e}^{h_{\mathbf{n}} L}}{h_{\mathbf{n}} L}$$

where $C_{\mathbf{n}} = d_{\mathbf{n}} \sum_{[\omega]:\mathbf{n}(\omega)=\mathbf{n}} c_\omega$. Here the sum runs over the equivalence classes $[\omega] = \{\omega' \,:\, \omega' \sim \omega\}$.

Note that we recover Theorems 4.1.1 and 4.1.2 : for instance, if $r = 1$, $\gamma_\star = \gamma_{\star,1}$ is separating (so that $q = 2$), $\Sigma \setminus \gamma_\star = \Sigma_1 \sqcup \Sigma_2$ and $h_1 = h_2 = h_\star > 0$, then $\gamma \in \mathcal{P}$ intersects $2k$ times $\gamma_\star$ if and only if $\omega(\gamma) \sim \omega^k$ where $\omega = (u, v)$ with $u = (1, 1)$ and $v = (1, 2)$. For this ω, we have $\mathbf{n}(\omega) = (n_1) = (2)$, $d_{\mathbf{n}(\omega)} = 2$ and $h_{\mathbf{n}(\omega)} = h_\star$, so that (4.9.1) yields

$$\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L,\ \mathrm{i}(\gamma, \gamma_\star) = 2k\} \sim 2\frac{(c_\omega L^2)^k}{(2k)!}\frac{\mathrm{e}^{h_\star L}}{h_\star L},$$

which is Theorem 4.1.2 for the case $h_1 = h_2$.

4.9.3 Proof of Theorem 4.9.1

We let $\Sigma_\star = \bigsqcup_{j=1}^q \Sigma_j$ denote the compact surface with geodesic boundary obtained by cutting Σ along $\gamma_{\star,1}, \dots, \gamma_{\star,r}$, and set

$$\partial = \{(x, v) \in S\Sigma_\star \ : \ x \in \partial\Sigma_\star\}.$$

Then the construction of §4.3 applies perfectly in this context, and we denote by

$$\mathcal{S}_\pm(s) : \Omega_c^\bullet(\partial \setminus \partial_0) \to \mathcal{D}'^\bullet(\partial)$$

the scattering operator. For any $i = 1, \dots, r$, we let $F_i \in C^\infty(\partial)$ defined by $F_i(z) = 1$ if $\pi(p(z)) \in \gamma_{\star,i}$ and $F_i(z) = 0$ if not. Here we recall that $p_\star : S\Sigma_\star \to S\Sigma$ and $\pi : S\Sigma \to \Sigma$ are the natural projections. Also we denote $\psi : \partial \simeq \partial$ the smooth map which exchanges the connected components of $(\pi \circ p_\star)^{-1}(\gamma_{\star,i})$ via the natural identification, and we set

$$\tilde{\mathcal{S}}_\pm(s) = \psi^*\mathcal{S}_\pm(s).$$

Let $\omega = (u, v)$ be a primitive admissible word of length $n \geqslant 1$ and $\chi \in C_c^\infty(\partial \setminus \partial_0)$ (recall that $\partial_0 = \cup_i p^{-1}(S\gamma_{\star,i})$ is the tangential part of ∂). Then set

$$\tilde{\mathcal{S}}_\pm(\chi, \omega, s) = F_{u_1}\chi\tilde{\mathcal{S}}_\pm^{(v_n)}(s)\chi F_{u_n} \cdots F_{u_2}\chi\tilde{\mathcal{S}}_\pm^{(v_1)}(s)\chi F_{u_1} : \Omega^\bullet(\partial_{u_1}) \to \mathcal{D}'^\bullet(\partial_{u_1}),$$

where $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ and $\tilde{S}_\pm^{(v_\ell)}$ is the scattering operator associated to the surface Σ_{v_ℓ} for $\ell = 1, \dots, n$, and $\partial_{u_1} = (\pi \circ p_\star)^{-1}(\gamma_{\star,u_1})$. As in §B.3.1, we find

$$-\operatorname{tr}_\mathrm{s}^\flat\left(\tilde{\mathcal{S}}_\pm(\chi, \omega, s)\right) = \sum_{\omega(\gamma)\sim\omega} \mathrm{e}^{-s\ell(\gamma)} \prod_{z\in I_{\star,\pm}(\gamma)} \chi^2(z),$$

where the sum runs over the (necessarily primitive) closed geodesics $\gamma : \mathbb{R}/\mathbb{Z} \to \Sigma$ with $\omega(\gamma) \sim \omega$, and where

$$I_{\star,\pm}(\gamma) = \{z \in \partial_\pm \ : \ \pi \circ p_\star(z) = \gamma(\tau) \text{ for some } \tau \in \mathbb{R}/\mathbb{Z}\}.$$

More generally, for $k \geqslant 1$ we have

$$-\operatorname{tr}_\mathrm{s}^\flat\left(\tilde{\mathcal{S}}_\pm(\chi, \omega^k, s)\right) = k\sum_{\omega(\gamma)\sim\omega^k} \frac{\ell^\sharp(\gamma)}{\ell(\gamma)}\mathrm{e}^{-s\ell(\gamma)}\left(\prod_{z\in I_{\star,\pm}(\gamma)} \chi^2(z)\right)^{\ell(\gamma)/\ell^\sharp(\gamma)}. \tag{4.9.2}$$

Note that $\max_\ell\{h_{v_\ell}\} = h_{\mathbf{n}(\omega)}$ and

$$\sharp\{\ell \in \{1,\dots,n\} \ : \ h_{v_\ell} = h_{\mathbf{n}(\omega)}\} = d_{\mathbf{n}(\omega)}.$$

Moreover, as in §4.5.1, the following holds. For any ℓ such that $h(v_\ell) = h_{\mathbf{n}(\omega)}$ we have as $s \to h_{v_{\mathbf{n}(\omega)}}$

$$F_{u_{\ell+1}}\chi\tilde{\mathcal{S}}_\pm^{(v_\ell)}(s)\chi F_{u_\ell} = -\frac{F_{u_{\ell+1}}\chi\tilde{\Pi}_{\pm,\partial_{v_\ell}}\chi F_{u_\ell}}{s - h_{\mathbf{n}(\omega)}} + \mathcal{O}_{\Omega^\bullet(\partial_{u_\ell})\to\mathcal{D}'^\bullet(\partial_{u_{\ell+1}})}(1),$$

for some operator $\tilde{\Pi}_{\pm,\partial_{v_\ell}}$ satisfying that $F_{u_{\ell+1}}\chi\tilde{\Pi}_{\pm,\partial_{v_\ell}}\chi F_{u_\ell}$ is of rank one. Thus we get, as $s \to h_{\mathbf{n}(\omega)}$,

$$\tilde{\mathcal{S}}_\pm(\chi,\omega,s) = \frac{A_\pm(\chi,\omega)}{(s - h_{\mathbf{n}(\omega)})^{d_{\mathbf{n}(\omega)}}} + \mathcal{O}_{\Omega^\bullet(\partial_{u_1})\to\mathcal{D}'^\bullet(\partial_{u_1})}\left((s-h_{\mathbf{n}(\omega)})^{1-d_{\mathbf{n}(\omega)}}\right),$$

for some operator $A_\pm(\chi,\omega) : \Omega^\bullet(\partial_{u_1}) \to \mathcal{D}'^\bullet(\partial_{u_1})$ of rank one. Note that $\tilde{\mathcal{S}}_\pm(\chi,\omega^k,s) = \tilde{\mathcal{S}}_\pm(\chi,\omega,s)^k$ for $k \geqslant 1$; thus as $s \to h_{\mathbf{n}(\omega)}$ it holds

$$-\operatorname{tr}_{\mathrm{s}}^\flat\left(\tilde{\mathcal{S}}_\pm(\chi,\omega^k,s)\right) = \frac{c_\pm(\chi,\omega)^k}{(s-h_{\mathbf{n}(\omega)})^{kd_{\mathbf{n}(\omega)}}} + \mathcal{O}\left((s-h_{\mathbf{n}(\omega)})^{1-kd_{\mathbf{n}(\omega)}}\right), \tag{4.9.3}$$

where we set $c_\pm(\chi,\omega) = -\operatorname{tr}_{\mathrm{s}}^\flat(A_\pm(\chi,\omega)) = \operatorname{tr}^\flat(A_\pm(\chi,\omega)|_{\Omega^1(\partial)})$, where we used that $\operatorname{tr}^\flat(A_\pm(\chi,\omega)^k|_{\Omega^1(\partial)}) = \operatorname{tr}^\flat(A_\pm(\chi,\omega)|_{\Omega^1(\partial)})^k$, which follows from the fact that $A_\pm(\chi,\omega)$ is of rank 1. Again, we want to apply the Tauberian Theorem A.2.1, and for this we need to know that $c_\pm(\chi,\omega) > 0$; as for the case of a single geodesic, we thus need *a priori* bounds on the growth of $\sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L,\ \omega(\gamma) \sim \omega\}$.

We claim that for some $C > 0$, we have, for L large enough,

$$\frac{C^{-1}L^{d_{\mathbf{n}(\omega)}-1}\mathrm{e}^{h_{\mathbf{n}(\omega)}L}}{(\log L + C)^{2d_{\mathbf{n}(\omega)}}} \leqslant \sharp\{\gamma \in \mathcal{P} \ : \ \ell(\gamma) \leqslant L,\ \omega(\gamma) \sim \omega\} \leqslant CL^{d_{\mathbf{n}(\omega)}-1}\mathrm{e}^{h_{\mathbf{n}(\omega)}L}. \tag{4.9.4}$$

Let us sketch the proof. Take some points $x_1,\dots,x_q \in \Sigma \setminus \bigcup_i \gamma_{\star,i}$ so that $x_j \in \Sigma_j$ for $j = 1,\dots,q$. For any $\ell = 1,\dots,n$, we consider an arbitrary smooth path $h_\ell : [0,1] \to \Sigma$ joining x_{v_ℓ} to $x_{v_{\ell+1}}$ (here $v_{n+1} = v_1$) with $h_\ell'(t) \neq 0$ and crossing $\gamma_{\star,u_\ell}$, such that

$$c(t) \in \Sigma_{v_\ell}^\circ, \quad c(t+1/2) \in \Sigma_{v_{\ell+1}}^\circ, \quad t \in]0,1/2[\,.$$

We denote $a_\ell = h_\ell|_{[0,1/2]}$, $b_\ell = h_\ell|_{[1/2,1]}$ and $y_\ell = a_\ell(1/2) = b_\ell(1/2) \in \gamma_{\star,u_\ell}$. Then we define

$$a_{\star,\ell} = a_\ell \cdot \gamma_{\star,u_\ell} \cdot a_\ell^{-1} \in \pi_1(\Sigma_{v_\ell}, x_{v_\ell}) \quad \text{and} \quad b_{\star,\ell} = b_\ell^{-1} \cdot \gamma_{\star,u_\ell} \cdot b_\ell \in \pi_1(\Sigma_{v_{\ell+1}}, x_{v_{\ell+1}}),$$

where we saw $\gamma_{\star,u_\ell}$ as an element of $\pi_1(\Sigma, y_\ell)$. If $w_\ell \in \pi_1(\Sigma_{v_\ell}, x_{v_\ell})$ for $\ell = 1,\dots,n$, we may consider the concatenation

$$w_1h_1w_2h_2\cdots w_nh_n \in \pi_1(\Sigma, x_{v_1}). \tag{4.9.5}$$

Then, proceeding as in Lemmas 4.4.3, 4.4.6 and 4.4.7, one is able to show that if w_ℓ is not a power of $a_{\star,\ell}$ or $b_{\star,\ell-1}$ for each $\ell = 1,\dots,n$, where $b_{\star,0} = b_{\star,n}$, then we have

$\omega(\gamma) \sim \omega$, where γ is the closed geodesic represented by $[w_1 h_1 w_2 h_2 \cdots w_n h_n]$; moreover, if $[w_1 h_1 w_2 h_2 \cdots w_n h_n] = [w'_1 h_1 w'_2 h_2 \cdots w'_n h_n]$ as conjugacy classes of $\pi_1(\Sigma, x_{v_1})$ for some $w'_\ell \in \pi_1(\Sigma_{v_\ell}, x_{v_\ell})$, then there are $p_\ell \in \mathbb{Z}$ such that

$$w_\ell = (b_{\star,\ell-1})^{-p_{\ell-1}} w'_\ell (a_{\star,\ell})^{p_\ell} \in \pi_1(\Sigma_{v_\ell}, x_{v_\ell}), \quad \ell = 1, \dots, n, \tag{4.9.6}$$

where $p_n = p_0$. Next, for any ℓ, we choose a universal cover $(\widetilde{\Sigma}, \tilde{x}_{v_\ell})$ of (Σ, x_{v_ℓ}). Note that, as in Lemmas 4.4.1 and 4.4.4, we have natural inclusions $\pi_1(\Sigma_{v_\ell}, x_{v_\ell}) \hookrightarrow \pi_1(\Sigma, x_{v_\ell})$; we may thus define

$$\ell_{\star,v_\ell}(w) = \operatorname{dist}(\tilde{x}_{v_\ell}, w \cdot \tilde{x}_{v_\ell}), \quad w \in \pi_1(\Sigma_{v_\ell}, x_{v_\ell}).$$

Next, we denote by

$$\pi_1(\Sigma_{v_\ell}, x_{v_\ell})_{\star,\ell} = \pi_1(\Sigma_{v_\ell}, x_{v_\ell}) \setminus \big(\langle a_{\star,\ell}\rangle \cup \langle b_{\star,\ell-1}\rangle\big)$$

the set of words in $\pi_1(\Sigma_{v_\ell}, x_{v_\ell})$ that are not powers of $a_{\star,\ell}$ or $b_{\star,\ell-1}$, and for any $w \in \pi_1(\Sigma_{v_\ell}, x_{v_\ell})_{\star,\ell}$ we set

$$\mathcal{C}_{w,\ell} = \{(b_{\star,\ell-1})^p w_\ell (a_{\star,\ell})^q \ : \ p, q \in \mathbb{Z}\}.$$

We also define $\mathscr{C}_\ell = \{\mathcal{C}_{w,\ell} \ : \ w \in \pi_1(\Sigma_{v_\ell}, x_{v_\ell})_{\star,\ell}\}$, and

$$\ell_{\star,v_\ell}(\mathcal{C}) = \inf\{\ell_{\star,v_\ell}(w) \ : \ w \ \in \mathcal{C}\}, \quad \mathcal{C} \in \mathscr{C}_\ell.$$

Then we may reproduce *exactly* the proof of Proposition 4.4.8 to obtain that for each $\ell = 1, \dots, n$, it holds for any $L > 0$

$$\sharp\{w \in \mathcal{C} \ : \ \ell_{\star,v_\ell}(w) \leqslant L\} \leqslant C(L - \ell_{\star,v_\ell}(\mathcal{C}) + C)^2, \quad \mathcal{C} \in \mathscr{C}_\ell, \tag{4.9.7}$$

for some constant $C > 0$ independent of L and $\mathcal{C}$. Next, using the orbital counting

$$\sharp\{w \in \pi_1(\Sigma_{v_\ell}, x_{v_\ell}) \ : \ \ell_{\star,v_\ell}(w) \leqslant L\} \sim A_\ell \mathrm{e}^{h_{v_\ell} L}, \tag{4.9.8}$$

we have, as in the proof of Proposition 4.4.5, see (4.4.19),

$$\sharp\{\mathcal{C} \in \mathscr{C}_\ell \ : \ \varepsilon L \leqslant \ell_{\star,v_\ell}(\mathcal{C}) \leqslant L\} \geqslant C^{-1}\mathrm{e}^{h_{v_\ell} L}/(\beta \log L + C)^2 \tag{4.9.9}$$

for any small $\varepsilon > 0$. Next, for any $\mathcal{C} \in \mathscr{C}_\ell$, we choose some $w_{\mathcal{C}} \in \mathcal{C}$ such that $\ell_{\star,v_\ell}(w_{\mathcal{C}}) = \ell_{\star,v_\ell}(\mathcal{C})$. Then by (4.9.6) we have a well defined and injective map

$$\mathscr{C}_1 \times \cdots \mathscr{C}_n \to \mathcal{P}, \quad (\mathcal{C}_1, \dots, \mathcal{C}_n) \mapsto [w_{\mathcal{C}_1} h_1 \cdots w_{\mathcal{C}_n} h_n],$$

and moreover for some $C > 0$ it holds

$$\ell([w_{\mathcal{C}_1} h_1 \cdots w_{\mathcal{C}_n} h_n]) \leqslant nC + \sum_{\ell=1}^n \ell_{\star,v_\ell}(\mathcal{C}_\ell).$$

Therefore one obtains that $\sharp\{\gamma \in \mathcal{P} \ : \ \omega(\gamma) \sim \omega, \ \ell(\gamma) \leqslant L\}$ is bounded from below by

$$\sharp\Big\{(\mathcal{C}_1, \dots, \mathcal{C}_n) \ : \ \sum_{\ell=1}^n \ell_{\star,v_\ell}(\mathcal{C}_\ell) \leqslant L - nC\Big\}. \tag{4.9.10}$$

Finally, by induction on $d_{\mathbf{n}(\omega)} = \sharp\{\ell \ : \ h_{v_\ell} = h_{\mathbf{n}(\omega)}\}$ (recall that by definition $h_{\mathbf{n}(\omega)} = \max_\ell h_{v_\ell}$), one may show, by using (4.9.9) and some Abel transformations as in the proof of Proposition 4.4.5, that (4.9.10) is bounded from below by [13]

$$\frac{CL^{d_{\mathbf{n}(\omega)}-1}\mathrm{e}^{h_{\mathbf{n}(\omega)}}}{(\log L + C)^{2d_{\mathbf{n}(\omega)}}}.$$

This yields the lower bound of (4.9.4). The upper bound is obtained as in §4.4.2.2, by noting that every $\gamma \in \mathcal{P}$ such that $\omega \sim \omega(\gamma)$ can be obtained by a concatenation of the form 4.9.5.

A suitable version of Lemma 4.4.11 is also valid in this context. Indeed, if γ is given by $[w_1 h_1 \cdots w_n h_n]$ and intersects one of the $\gamma_{\star,u_\ell}$'s with a small angle η, then proceeding as in the proof of Lemma 4.4.11, one can see that for some ℓ, we have $w_\ell = (b_{\star,\ell-1})^K w'_\ell$ or $w_\ell = w'_\ell(a_{\star,\ell})^K$ for some $w'_\ell \in \pi_1(\Sigma_{v_\ell}, x_{v_\ell})$ satisfying $\ell_{\star,v_\ell}(w'_\ell) \leqslant \ell_{\star,v_\ell}(w_\ell)+|K|\ell(\gamma_\star)+C$; here $K \in \mathbb{Z}$ can be chosen so that $|K|$ is very large, depending on η. Therefore, as in Lemma 4.4.11, we get that for any $L_0 > 0$ and $\varepsilon > 0$, there is $\eta > 0$ such that for large L, it holds

$$N(\omega, \eta, L) \leqslant C_\omega N(\omega, L - L_0) \quad \text{and} \quad N(\omega^k, \eta, L) \leqslant \varepsilon C_{k,\omega} L^{kd_{\mathbf{n}(\omega)}-1}\mathrm{e}^{h_{\mathbf{n}(\omega)}L}, \quad (4.9.11)$$

where the constants C_ω and $C_{k,\omega}$ only depend on ω and k. Here, $N(\omega, L)$ (resp. $N(\omega, \eta, L)$) is the number of geodesics γ of length not greater than L, such that $\omega(\gamma) \sim \omega$ (resp. and intersecting one of the $\gamma_{\star,u_\ell}$'s with an angle smaller than η).

Finally, combining (4.9.2), (4.9.3), (4.9.4) and (4.9.11), we may proceed exactly as in §§4.5,4.6 to obtain Theorem 4.9.1 with

$$c_\omega = \lim_{\mathrm{supp}(1-\chi) \to \partial_0} c_\pm(\chi, \omega).$$

4.10 Closed geodesics minimize intersection numbers

In this section we prove Lemma 4.2.1. We proceed by contradiction and assume that it holds $\mathrm{i}(\gamma_1, \gamma_2) < |\gamma_1 \cap \gamma_2|$. As γ_1, γ_2 are not powers of each other, the images of γ_1 and γ_2 intersect transversally (otherwise their images would coincide by unicity of the geodesic equation). Since $\mathrm{i}(\gamma_1, \gamma_2) < |\gamma_1 \cap \gamma_2|$, we may find loops $\alpha_j : \mathbb{R}/\mathbb{Z} \to \Sigma$, $j = 1, 2$, with $\alpha_j \sim \gamma_j$, and $|\alpha_1 \cap \alpha_2| < |\gamma_1 \cap \gamma_2|$, and we may moreover assume that α_1 and α_2 intersect transversally. Let $H_j : [0,1] \times \mathbb{R}/\mathbb{Z} \to \Sigma$, $j = 1, 2$, be smooth homotopies between γ_j and α_j, and define $H : [0,1] \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \Sigma \times \Sigma$ by setting

$$H(s, \tau_1, \tau_2) = (H_1(s, \tau_1), H_2(s, \tau_2)), \quad (s, \tau_1, \tau_2) \in [0,1] \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}.$$

Let $\Delta(\Sigma) = \{(x, x) \ : \ x \in \Sigma\}$ be the diagonal in Σ. Then $H(0, \cdot)$ and $H(1, \cdot)$ are transversal to $\Delta(\Sigma)$, in the sense that for every $k = 0, 1$ and $(\tau_1, \tau_2) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$

13. Indeed, we construct enough closed geodesics by considering products of the form $[w_{\mathcal{C}_1} h_1 \cdots w_{\mathcal{C}_n} h_n]$ where $\ell_{\star,v_\ell}(\mathcal{C}_\ell) \leqslant C$ if $h_{v_\ell} < h_{\mathbf{n}(\omega)}$ and such that

$$\sum_{\ell \ : \ h_{v_\ell} = h_{\mathbf{n}(\omega)}} \ell_{\star,v_\ell}(\mathcal{C}_\ell) \leqslant L.$$

with $H(k,\tau_1,\tau_2)\in\Delta(\Sigma)$, it holds

$$\mathrm{d}H(k,\tau_1,\tau_2)T_{(k,\tau_1,\tau_2)}\big(\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}\big)+T_{H(k,\tau_1,\tau_2)}\Delta(\Sigma)=T_{H(k,\tau_1,\tau_2)}(\Sigma\times\Sigma).$$

In particular by [GP10, Corollary p.73], we may assume that H is globally transversal to $\Delta(\Sigma)$, so that $H^{-1}(\Delta(\Sigma))$ is a smooth 1-dimensional submanifold of $[0,1]\times(\mathbb{R}/\mathbb{Z})^2$. Now

$$|\gamma_1\cap\gamma_2|=|H^{-1}(\Delta(\Sigma))\cap(\{0\}\times(\mathbb{R}/\mathbb{Z})^2)|\quad\text{and}\quad|\alpha_1\cap\alpha_2|=|H^{-1}(\Delta(\Sigma))\cap(\{1\}\times(\mathbb{R}/\mathbb{Z})^2)|.$$

Since $|\gamma_1\cap\gamma_2|>|\alpha_1\cap\alpha_2|$, and because $H^{-1}(\Delta(\Sigma))$ is smooth, we may find a smooth path $c:[0,1]\to[0,1]\times(\mathbb{R}/\mathbb{Z})^2$ such that $c(0)\neq c(1)$ and

$$\mathrm{Im}(c)\subset H^{-1}(\Delta(\Sigma))\quad\text{and}\quad c(0),c(1)\in\{0\}\times(\mathbb{R}/\mathbb{Z})^2.$$

Write $c=(S,T_1,T_2)$ for some smooth functions $S:[0,1]\to[0,1]$ and $T_j:[0,1]\to\mathbb{R}/\mathbb{Z}$, and for $u\in[0,1]$ define the path $c_u=(uS,T_1,T_2):[0,1]\to[0,1]\times(\mathbb{R}/\mathbb{Z})^2$. Let $x_k=H(c(k))\in\Sigma$ for $k=0,1$. Then define the paths

$$\beta_{j,u}=\pi_j\circ H\circ c_u:[0,1]\to\Sigma,\quad j=1,2,\quad u\in[0,1],$$

where $\pi_1,\pi_2:\Sigma\times\Sigma\to\Sigma$ are the projections over the first and second factor, respectively. As $c_1=c$ and $\mathrm{Im}(c)\subset H^{-1}(\Delta(\Sigma))$ we have $\beta_{1,1}=\beta_{2,1}$. In particular, the paths $\beta_{1,0}$ and $\beta_{2,0}$ are homotopic within the space of curves linking x_0 and x_1, since for each u one has $\beta_{j,u}(k)=x_k$ for $j=1,2$ and $k=0,1$. Moreover, the paths $\beta_{1,0}$ and $\beta_{2,0}$ are subpaths of γ_1 and γ_2, respectively, and in particular geodesic paths. Let $\widetilde{\Sigma}$ be a universal cover of Σ and take $\tilde{x}_0\in\widetilde{\Sigma}$ a lift of x_0. For $j=1,2$, let $\tilde{\beta}_j:[0,1]\to\widetilde{\Sigma}$ be the unique lift of $\beta_{j,0}$ starting at $\tilde{x}_0$. Then $\tilde{\beta}_1(1)=\tilde{\beta}_2(1)$ since the paths $\beta_{j,0}$, $j=1,2$, are homotopic in Σ via an homotopy preserving endpoints. In particular, we found two distinct geodesic segments of $\widetilde{\Sigma}$ joining $\tilde{x}_0$ and $\tilde{\beta}_0(1)$ (the image of the paths $\tilde{\beta}_{j,0}$, $j=1,2$, cannot coincide since $c(0)\neq c(1)$ and the intersection $\gamma_1\cap\gamma_2$ is transversal). Thus the exponential map $\exp_{\tilde{x}_0}:T_{\tilde{x}_0}\widetilde{\Sigma}\to\widetilde{\Sigma}$ at $\tilde{x}_0$ is not a diffeomorphism, and $\widetilde{\Sigma}$ cannot be negatively curved in virtue of the Cartan–Hadamard theorem (see for example [Lee97, Theorem 11.5]). This completes the proof.

4.11 An elementary fact about pullbacks of distributions

Lemma 4.11.1. *Let $K\in\mathcal{D}'(\mathbb{R}^d\times\mathbb{R}^d)$ be a compactly supported distribution. We assume that $\mathrm{WF}(K)\subset\Gamma$ where $\Gamma\subset T^*(\mathbb{R}^d\times\mathbb{R}^d)$ is a closed conical subset such that*

$$\Gamma\cap N^*\Delta=\emptyset,\quad N^*\Delta=\{(x,\xi,x,-\xi)\ :\ (x,\xi)\in T^*\mathbb{R}^d\}.$$

In particular the pullback i^*K, where $i:x\mapsto(x,x)$, is well defined. Then for $N\in\mathbb{N}_{\geqslant1}$ large enough, the following holds. Let $u\in C_c^N(\mathbb{R}^d)$ and assume that the pullback $i^*(\pi_1^*uK)$ is well defined, where $\pi_1:(x,x)\mapsto x$ is the projection on the first factor. Then

$$i^*(\pi_1^*u\cdot K)=u\cdot i^*K.$$

Proof. Let $K_\varepsilon \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $\varepsilon \in]0,1]$, be a sequence of distributions supported in a fixed compact set such that $K_\varepsilon \to K$ in $\mathcal{D}'_\Gamma(\mathbb{R}^d \times \mathbb{R}^d)$. Let $\Gamma' \subset T^*(\mathbb{R}^d \times \mathbb{R}^d)$ an open conical subset containing $N^*\Delta$. As K_ε is compactly supported we may assume that $|t-q| > \delta_0$ for any $(t,q) \in \Gamma \times \Gamma'$ such that $|t| = |q| = 1$ for some $\delta_0 > 0$. By definition of the convergence in $\mathcal{D}'_\Gamma(\mathbb{R}^d \times \mathbb{R}^d)$ (see [Hör90, Definition 8.2.2]), for every N there is $C_N > 0$ such that for any $\varepsilon > 0$ small enough,

$$\left|\widehat{K}_\varepsilon(q)\right| \leqslant C_N \langle q \rangle^{-N}, \quad q \in \Gamma'. \tag{4.11.1}$$

Let $\Gamma'' \subset \Gamma'$ another open conical subset containing $N^*\Delta$ and let $\delta > 0$ such that for any $q \in \Gamma''$ and $t \in \mathbb{R}^{2d}$ one has

$$|t-q| < \delta|q| \quad \Longrightarrow \quad t \in \Gamma'. \tag{4.11.2}$$

Then for any $q \in \Gamma''$

$$\begin{aligned}(2\pi)^{2d}\left|\widehat{K_\varepsilon \pi_1^* u}(q)\right| &\leqslant \int_{\mathbb{R}^{2d}_t} |\widehat{K}_\varepsilon(t)||\widehat{\pi_1^* u}(q-t)|\mathrm{d}t \\ &\leqslant \int_{|t-q|<\delta|q|} |\widehat{K}_\varepsilon(t)||\widehat{\pi_1^* u}(q-t)|\mathrm{d}t + \int_{|t-q|\geqslant\delta|q|} |\widehat{K}_\varepsilon(t)||\widehat{\pi_1^* u}(q-t)|\mathrm{d}t.\end{aligned}$$

Let $N_1, N_2 \in \mathbb{N}_{\geqslant 1}$. We have, with $\langle t \rangle = \sqrt{1+|t|^2}$, using (4.11.1) and (4.11.2), assuming that $u \in C_c^{N_2}(\mathbb{R}^d)$ with $N_2 \geqslant 2d+1$,

$$\begin{aligned}\int_{|t-q|<\delta|t|} |\widehat{K}_\varepsilon(t)||\widehat{\pi_1^* u}(q-t)|\mathrm{d}t &\leqslant C_{N_1,N_2} \int_{|t-q|<\delta|q|} \langle t \rangle^{-N_1} \langle q-t \rangle^{-N_2} \mathrm{d}t \\ &\leqslant C'_{N_1,N_2} \langle q \rangle^{-N_1+N_2} \int_{\mathbb{R}^d} \langle t \rangle^{-N_2} \mathrm{d}t.\end{aligned}$$

where we used Peetre's inequality. On the other hand, we have with k being the order of K, and any $N_3 \in \mathbb{N}_{\geqslant 1}$ such that $u \in C_c^{N_3}(\mathbb{R}^d)$

$$\begin{aligned}\int_{|t-q|\geqslant\delta|q|} |\widehat{K}_\varepsilon(t)||\widehat{\pi_1^* u}(q-t)|\mathrm{d}t &\leqslant C_{k,N_3} \int_{|t-q|\geqslant\delta|q|} \langle t \rangle^{k} \langle q-t \rangle^{-N_3} \\ &\leqslant C'_{k,N_3} \langle q \rangle^{-N_3+(k+2d+1)} \int_{\mathbb{R}^{2d}} \langle t \rangle^{-2d-1} \mathrm{d}t.\end{aligned}$$

Therefore, if $u \in C^N(\mathbb{R}^d)$ with $N = k+2d+1+N'$ we have

$$(2\pi)^{2d}\left|\widehat{K_\varepsilon \pi_1^* u}(q)\right| \leqslant C_N \langle q \rangle^{-N'}, \quad q \in \Gamma''. \tag{4.11.3}$$

Note that for $\varphi \in C_c^\infty(\mathbb{R}^d)$ one has

$$\langle i^*(K_\varepsilon \pi_1^* u), \varphi \rangle = \int_{\mathbb{R}^d_x} \varphi(x) \int_{\mathbb{R}^d_\xi \times \mathbb{R}^d_\eta} \widehat{K_\varepsilon \pi_1^* u}(\xi,\eta) \mathrm{e}^{ix(\xi+\eta)} \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}x.$$

Indeed (4.11.3) shows that the integral in (ξ,η) converges near $N^*\Delta$ if $N' \geqslant 2d+1$, and far from $N^*\Delta$ we can use the stationary phase method to get enough convergence in (ξ,η), so that the above integral makes sense as an oscillatory integral and coincides with $\langle i^*(K_\varepsilon \pi_1^* u), \varphi \rangle$, since this formula is obviously true if u is smooth. Moreover all the above estimates are uniform in ε, and thus letting $\varepsilon \to 0$ we obtain the desired result, since obviously $i^*(K_\varepsilon \pi_1^* u) = u(i^* K_\varepsilon)$ for each $\varepsilon \in]0,1]$. □

Deuxième partie

Séries dynamiques et topologie

Chapitre 5

Séries de Poincaré pour les surfaces à bord

Dans ce chapitre, on considère une surface à courbure négative avec un bord totalement géodésique. Nous montrons le prolongement d'une de séries de Poincaré comptant les orthogéodésiques ainsi que pour des séries de Poincaré qui comptent les arcs géodésiques reliant deux points. Nous calculons aussi leurs valeurs à l'origine ; pour la série comptant les arcs reliant deux points, cette valeur coïncide avec l'inverse de la caractéristique d'Euler de la surface. Ce chapitre contient l'article *Poincaré series for surfaces with boundary* [Cha21].

Sommaire

5.1 Introduction

Let (Σ, g) be a connected oriented negatively curved surface with totally geodesic boundary $\partial\Sigma$. We denote by $\mathcal{G}^\perp$ the set of orthogeodesics of Σ, that is, the set of geodesics $\gamma : [0, \ell] \to \Sigma$ (parameterized by arc length) such that $\gamma(0), \gamma(\ell) \in \partial\Sigma$,

$\gamma'(0) \perp T_{\gamma(0)}\partial\Sigma$ and $\gamma'(\ell) \perp T_{\gamma(\ell)}\partial\Sigma$. For large $\mathrm{Re}(s)$ the Poincaré series

$$\eta(s) = \sum_{\gamma \in \mathcal{G}^\perp} \mathrm{e}^{-s\ell(\gamma)}, \tag{5.1.1}$$

where $\ell(\gamma)$ denotes the length of the geodesic arc γ, is well defined (see §5.3.2). In this chapter we will prove the following

Theorem 5.1.1. *The Poincaré series $s \mapsto \eta(s)$ extends meromorphically to the whole complex plane and vanishes at $s = 0$.*

If $x \neq y \in \Sigma$, we may also consider the Poincaré series associated to the geodesic arcs joining x to y. Namely, we set for $\mathrm{Re}(s)$ large enough

$$\eta_{x,y}(s) = \sum_{\gamma : x \rightsquigarrow y} \mathrm{e}^{-s\ell(\gamma)},$$

where the sum runs over all geodesic arcs $\gamma : [0, \ell(\gamma)] \to \Sigma$ (parameterized by arc length) such that $\gamma(0) = x$ and $\gamma(\ell) = y$. Then we have the following result.

Theorem 5.1.2. *The Poincaré series $s \mapsto \eta_{x,y}(s)$ extends meromorphically to the whole complex plane and*

$$\eta_{x,y}(0) = \frac{1}{\chi(\Sigma)},$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

We refer to §2.2.1 for a motivation of those results. This chapter is organized as follows. In §5.2 we introduce the geometrical setting and the resolvent $Q(s)$ of the geodesic flow. In §5.3 we express $\eta(s)$ and $\eta_{x,y}(s)$ with a pairing formula involving the resolvent $Q(s)$. In §5.4 we compute $\eta(0)$. Finally, we compute $\eta_{x,y}(0)$ in §5.5.

5.2 Geometrical and dynamical preliminaries

We introduce in this section the main tools that will help us to understand η and $\eta_{x,y}$.

5.2.1 Extension to a surface with strictly convex boundary

We extend (Σ, g) into a slightly larger negatively curved surface with boundary (Σ', g'). We take $\delta > 0$ small and we set

$$\Sigma_\delta = \{x \in \Sigma' : \mathrm{dist}_{g'}(x, \Sigma) < \delta\}.$$

Then since $\partial\Sigma$ is totally geodesic and (Σ', g') is negatively curved, it follows that Σ_δ has strictly convex boundary, in the sense that the second fundamental form of $\partial\Sigma_\delta$ with respect to the outward normal vector field is negative (see Lemma 4.2.5). We denote by

$$M_\delta = S\Sigma_\delta = \{(x, v) \in T\Sigma_\delta : \|v\|_g = 1\}$$

the unit tangent bundle of the surface Σ_δ, and by $\pi : M_\delta \to \Sigma_\delta$ the natural projection.

5.2.2 Structural forms

Recall from §4.2.2 the structural forms $\alpha, \beta, \psi \in \Omega^1(M_\delta)$. Namely, α is a contact form (that is, $\alpha \wedge \mathrm{d}\alpha$ is a volume form on M_δ) whose Reeb vector field is the geodesic vector field X, in the sense that

$$\iota_X \alpha = 1, \quad \iota_X \mathrm{d}\alpha = 0,$$

where ι denote the interior product. Also $\beta = R_{\pi/2}^* \alpha$ where for $\theta \in \mathbb{R}$, $R_\theta : M_\delta \to M_\delta$ the rotation of angle θ in the fibers (which is defined thanks to the orientation of Σ_δ). The volume form vol_g of Σ_δ satisfies

$$\pi^* \mathrm{vol}_g = \alpha \wedge \beta, \tag{5.2.1}$$

and ψ the connection one-form, that is,

$$\iota_V \psi = 1, \quad \mathrm{d}\alpha = \psi \wedge \beta, \quad \mathrm{d}\beta = \alpha \wedge \psi, \quad \mathrm{d}\psi = -(\kappa \circ \pi)\alpha \wedge \beta, \tag{5.2.2}$$

where V is the vector field generating $(R_\theta)_{\theta \in \mathbb{R}}$ and κ is the Gauss curvature of Σ. Then (α, β, ψ) is a global frame of $T^* M_\delta$. Recall also that H is the unique vector field on M_δ such that (X, H, V) is the dual frame of (α, β, ψ). We have the commutation relations

$$[V, X] = H, \quad [V, H] = -X, \quad [X, H] = (\kappa \circ \pi) V. \tag{5.2.3}$$

The orientation of M_δ will be chosen so that (X, H, V) is positively oriented. Also recall that, on ∂M, we have a precise description (X, H, V), as follows (see Lemma 4.2.3).

Lemma 5.2.1. *Let $\gamma_\star$ be a connected component of $\partial \Sigma$ (which is the image of a closed geodesic) and denote by $\ell_\star > 0$ its length. Then there is a tubular neighborhood U of $\pi^{-1}(\gamma_\star)$ and coordinates (ρ, τ, θ) on U with*

$$U \simeq (-\delta, \delta)_\rho \times (\mathbb{R}/\ell_\star \mathbb{Z})_\tau \times (\mathbb{R}/2\pi\mathbb{Z})_\theta,$$

and such that

$$|\rho(z)| = \mathrm{dist}(z, \gamma_\star), \quad S_z\Sigma = \{(\tau(z), \rho(z), \theta) \ : \ \theta \in \mathbb{R}/2\pi\mathbb{Z}\}, \quad z \in U.$$

Moreover in these coordinates we have, on $\{\rho = 0\} \simeq \gamma_\star$,

$$X(z) = \cos(\theta)\partial_\tau + \sin(\theta)\partial_\rho, \quad H = -\sin(\theta)\partial_\tau + \cos(\theta)\partial_\rho, \quad V = \partial_\theta.$$

5.2.3 Extension of the geodesic vector field

We embed M_δ into a compact manifold without boundary N (for example by taking the doubling manifold); then by [DG16, Lemma 2.1], we may extend the geodesic vector field X to a vector field on N so that M_δ is convex with respect to X in the sense that for any $T \geqslant 0$,

$$x, \varphi_T(x) \in M_\delta \quad \Longrightarrow \quad \forall t \in [0, T], \quad \varphi_t(x) \in M_\delta, \tag{5.2.4}$$

where φ_t is the flow induced by X. Let $\rho_\delta \in C^\infty(M_\delta, [0,1])$ be a boundary defining function for M_δ, that is, $\rho_\delta > 0$ on $M_\delta \setminus \partial M_\delta$, $\rho_\delta = 0$ on ∂M_δ and $\mathrm{d}\rho_\delta \neq 0$ on ∂M_δ (for example we can take $\rho_\delta(x,v) = \mathrm{dist}(x, \partial\Sigma_\delta)$). Then the strict convexity of $\partial\Sigma_\delta$ implies that ∂M_δ is strictly convex in the sense that for every $x \in \partial M_\delta$ one has

$$X\rho_\delta(x) = 0 \quad \Longrightarrow \quad X^2\rho_\delta(x) < 0 \tag{5.2.5}$$

(see [DG16, §6.3]).

5.2.4 Hyperbolicity of the geodesic flow

We define

$$\Gamma_\pm = \{z \in M_\delta : \varphi_{\mp t}(z) \in M_\delta,\ t \geqslant 0\}, \quad K = \Gamma_+ \cap \Gamma_- \subset M.$$

By [Kli11, §3.9 and Theorem 3.2.17] the geodesic flow (φ_t) is hyperbolic on K, that is, for every $z \in K$ there is a decomposition

$$T_z M_\delta = E_s(z) \oplus E_u(z) \oplus \mathbb{R}X(z)$$

depending continuously on z, which is invariant by $\mathrm{d}\varphi_t$ and such that, for some $C, \nu > 0$,

$$\|\mathrm{d}\varphi_t(z)w\| \leqslant C\mathrm{e}^{-\nu t}\|w\|, \quad w \in E_s(z), \quad t \geqslant 0,$$

and

$$\|\mathrm{d}\varphi_{-t}(z)w\| \leqslant C\mathrm{e}^{-\nu t}\|w\|, \quad w \in E_u(z), \quad t \geqslant 0.$$

Moreover, by [DG16, Lemma 2.10], there are two vector subbundles $E_\pm \subset T_{\Gamma_\pm}M_\delta$ (here $T_{\Gamma_\pm}M_\delta = TM_\delta|_{\Gamma_\pm}$) with the following properties :

1. $E_+|_K = E_u$ and $E_-|K = E_s$ and $E_\pm(z)$ depends continuously on $z \in \Gamma_\pm$;
2. $\langle \alpha, E_\pm \rangle = 0$;
3. For some constants $C', \nu' > 0$ we have
$$\|\mathrm{d}\varphi_{\mp t}(z)w\| \leqslant C'\mathrm{e}^{-\nu' t}\|w\|, \quad w \in E_\pm(z), \quad z \in \Gamma_\pm, \quad t \geqslant 0;$$
4. If $z \in \Gamma_\pm$ and $w \in T_z M_\delta$ satisfy $w \notin \mathbb{R}X(z) \oplus E_\pm(z)$, then as $t \to \mp\infty$
$$\|\mathrm{d}\varphi_t(z)w\| \to \infty, \quad \frac{\mathrm{d}\varphi_t(z)w}{\|\mathrm{d}\varphi_t(z)w\|} \to E_\mp|_K.$$

Moreover, we have the following description of $E_\pm$.

Lemma 5.2.2. *There are continuous functions $r_\pm : \Gamma_\pm \to \mathbb{R}$ such that $\pm r_\pm > 0$ on $\Gamma_\pm$ and*

$$E_\pm(z) = \mathbb{R}\left(H(z) + r_\pm(z)V(z)\right), \quad z \in \Gamma_\pm. \tag{5.2.6}$$

Proof. As the contact form α is preserved by the flow φ_t, we get by property (2) above $E_\pm(z) \subset \ker\alpha(z) = \mathbb{R}(H(z) \oplus V(z))$. In a first step, we will assume that $E_\pm(z) \cap \mathbb{R}V(z) = \{0\}$ for every $z \in \Gamma_\pm$ (we shall prove it later). Since the bundles

$E_\pm$ are continuous, we deduce that there are two continuous functions $r_\pm : \Gamma_\pm \to \mathbb{R}$ such that (5.2.6) holds.

Next, we show that $\pm r_\pm > 0$. The fact that $\mathrm{d}\varphi_{\mp t}(z)E_\pm(z) \subset E_\pm(z)$ for $t \geqslant 0$ implies that the map $t \mapsto r_\pm(\varphi_{\mp t}(z))$ is smooth on $\mathbb{R}_+$ for any $z \in \Gamma_\pm$ (since $\mathbb{R}(H \oplus V)$ is preserved by $\mathrm{d}\varphi_t$). We may thus compute, on $\Gamma_\pm$,

$$[X, H + r_\pm V] = [X, H] + (Xr_\pm)V + r_\pm[X, V] = (\kappa \circ \pi + Xr_\pm)\, V - r_\pm H, \quad (5.2.7)$$

where we used the commutation relations (5.2.3). As $E_\pm$ is preserved by the flow, we must have $[X, H + r_\pm V] \in E_\pm$; thus combining (5.2.7) and (5.2.6) we obtain the following Riccati equation :

$$Xr_\pm + r_\pm^2 + \kappa \circ \pi = 0 \quad \text{on } \Gamma_\pm. \quad (5.2.8)$$

We now prove that $r_+ > 0$. Let $z \in \Gamma_+$ and set $U(t) = (H + r_+V)(\varphi_{-t}(z)) \in E_+(\varphi_{-t}(z))$ and $r_+(t) = r_+(\varphi_{-t}(z))$ for $t \geqslant 0$. By (5.2.7) and (5.2.8) we have $[-X, U(t)] = r_+(t)U(t)$ and thus

$$\mathrm{d}\varphi_{-t}(z)U(0) = \exp\left(-\int_0^t r_+(u)\mathrm{d}u\right) U(t), \quad t \geqslant 0. \quad (5.2.9)$$

On the other hand, equation (5.2.8) implies that for any $t \geqslant 0$ we have the implication

$$r_+(t) = 0 \quad \Longrightarrow \quad r_+'(t) = -Xr_+(\varphi_{-t}(z)) < 0$$

since $\kappa < 0$ everywhere. Therefore, if $r_+(t) \leqslant 0$ for some t, then $r_+(u) \leqslant 0$ for all $u \geqslant t$. This is not possible by (5.2.9) since $\mathrm{d}\varphi_{-t}(z)U(0) \to 0$ as $t \to +\infty$. We therefore proved that $r_+(t) > 0$ for all $t \geqslant 0$. Thus $r_+ > 0$ on Γ_+ and similarly, one can show that $r_-(z) < 0$ for all $z \in \Gamma_-$.

It remains to prove that $E_\pm(z) \cap \mathbb{R}V(z) = \{0\}$ for any $z \in \Gamma_\pm$. Let $z \in \Gamma_+$, and write $V(t) = V(\varphi_{-t}(z))$ and $H(t) = H(\varphi_{-t}(z))$ for $t \geqslant 0$. Then there are smooth functions $a, b : [0, \infty[\to \mathbb{R}$ such that for any $t \geqslant 0$ one has

$$\mathrm{d}\varphi_{-t}(z)V(z) = a(t)H(t) + b(t)V(t)$$

The commutation relations (5.2.3) imply that

$$a'(t) + b(t) = 0, \quad \kappa(t)a(t) + b'(t) = 0, \quad t \geqslant 0,$$

where $\kappa(t) = (\kappa \circ \pi \circ \varphi_{-t})(z)$. Thus $a''(t) + \kappa(t)a(t) = 0$; moreover we have $a(0) = 0$ and $a'(0) = -b(0) = -1$; from this it is easy to deduce that $a'(t) < 0$ for every $t \geqslant 0$. In particular there are $C_1, C_2 > 0$ such that $a(t) \leqslant -C_1$ for every $t \geqslant C_2$ and thus for some $C > 0$ we have $\|\mathrm{d}\varphi_{-t}(z)V(z)\| \geqslant C$ for any $t \geqslant 0$. As a consequence, we obtain that $V(z) \notin E_+(z)$. Similarly, one can prove that $V(z) \notin E_-(z)$ for any $z \in \Gamma_-$. This concludes the proof of the lemma. □

Remark 5.2.3. Looking carefully at the proof of Lemma 5.2.2, we see that for any z and t such that $\varphi_t(z) \in M_\delta$ we have

$$\pm \langle \varphi_t^*\beta(z), V(z)\rangle > 0 \quad \text{and} \quad \pm \langle \varphi_t^*\psi(z), H(z)\rangle > 0 \quad (5.2.10)$$

whenever $\pm t > 0$. Indeed, the first part of (5.2.10) follows from the fact that, with the notations of the proof of Lemma 5.2.2, one has $a(t) = \langle \varphi_{-t}^*\beta(z), V(z)\rangle < 0$ for $t > 0$ (since $a'(t) < 0$ and $a(0) = 0$), and reversing the time we get that $a(t) > 0$ whenever $t < 0$. The second part of (5.2.10) was not explicitly proven but the proof is very similar.

5.2.5 The resolvent

For $\mathrm{Re}(s)$ large enough, consider the operator $R(s)$ defined on $\Omega^\bullet(N)$ by

$$R(s) = \int_0^{+\infty} \mathrm{e}^{-ts} \varphi_{-t}^* \mathrm{d}t. \tag{5.2.11}$$

Here $\Omega^\bullet(N)$ denotes the space of smooth differential forms on N. Then it holds

$$(\mathcal{L}_X + s)R(s) = \mathrm{Id}_{\Omega^\bullet(N)} = R(s)(\mathcal{L}_X + s).$$

Let $\chi \in C_c^\infty(M_\delta \setminus \partial M_\delta)$ such that $\chi \equiv 1$ on $M_{\delta/2}$, and let

$$Q(s) = \chi R(s) \chi.$$

Then it follows from [DG16, Theorem 1] that the family of operators $s \mapsto \chi R(s) \chi$ extends to a family of operators

$$Q(s) : \Omega_c^\bullet(M_\delta^\circ) \to \mathcal{D}'^\bullet(M_\delta^\circ)$$

meromorphic in $s \in \mathbb{C}$, which satisfies, for $w \in \Omega_c^\bullet(M_\delta^\circ)$ supported in $\{\chi = 1\}$,

$$(\mathcal{L}_X + s)Q(s)w = w \quad \text{on} \quad \{\chi = 1\}, \tag{5.2.12}$$

for any $s \in \mathbb{C}$ which is not a pole of $s \mapsto Q(s)$. Here, M_δ° denotes the interior of M_δ and if U is a manifold, $\Omega_c^\bullet(U)$ denotes the space of compactly supported differential forms on U while $\mathcal{D}'^\bullet(U)$ denote its dual space, that is, the space of currents. In what follows, for any distribution $A \in \mathcal{D}'(T^*M_\delta \times T^*M_\delta)$, we will set

$$\mathrm{WF}'(A) = \{(z, \xi, z', \xi') \in T^*(M_\delta \times M_\delta) : (z, \xi, z', -\xi') \in \mathrm{WF}(A)\},$$

where WF is the Hörmander wavefront set, see [Hör90, §8]. The microlocal structure of $Q(s)$ is given by (see [DG16, Lemma 4.5], in what follows we identify $Q(s)$ and its Schwartz kernel)

$$\mathrm{WF}'(Q(s)) \subset \Delta(T^*M_\delta) \cup \Upsilon_+ \cup (E_+^* \times E_-^*) \tag{5.2.13}$$

where $\Delta(T^*M_\delta) = \{(\xi, \xi) : \xi \in T^*M_\delta\} \subset T^*(M_\delta \times M_\delta)$ and

$$\Upsilon_+ = \{(\Phi_t(z, \xi), (z, \xi)) : t \geqslant 0,\ \langle \xi, X(z) \rangle = 0,\ z \in M_\delta,\ \varphi_t(z) \in M_\delta\}.$$

Here Φ_t denotes the symplectic lift of φ_t on T^*M_δ, that is

$$\Phi_t(z, \xi) = (\varphi_t(z), (\mathrm{d}_z \varphi_t)^{-\top} \xi), \quad (z, \xi) \in T^*M_\delta, \quad \varphi_t(z) \in M_\delta,$$

and the subbundles $E_\pm^* \subset T_{\Gamma_\pm}^* M_\delta$ are defined by $E_\pm^*(\mathbb{R}X(z) \oplus E_\pm) = 0$. In particular, we have

$$\|\Phi_{\mp t}(z, \xi)\| \to +\infty, \quad (z, \xi) \in E_\pm^*, \quad t \to +\infty$$

and

$$E_\pm^*(z) = \mathbb{R}\left(r_\pm(z)\beta(z) - \psi(z)\right), \quad z \in \Gamma_\pm.$$

5.3 Poincaré series

In this section, we give a description of the Poincaré series $\eta(s)$ in terms of a pairing involving the operator $Q(s)$.

5.3.1 Counting measure

Let $\Lambda, \bar{\Lambda} \subset M_\delta$ be the one-dimensional submanifolds of M_δ defined by

$$\Lambda = \{(x, \nu(x)) : x \in \partial\Sigma\}, \quad \bar{\Lambda} = \{(x, -\nu(x)) : x \in \partial\Sigma\}.$$

where $\nu : \partial\Sigma \to M$ is the outward normal pointing vector to $\partial\Sigma$. Those manifolds are oriented according to the orientation of $\partial\Sigma$ which is itself oriented by ∂_τ in the coordinates of Lemma 5.2.1 ; note also that in the coordinates given by Lemma 5.2.1 we have (here $\gamma_\star$ is a connected component of $\partial\Sigma$)

$$\Lambda|_{\gamma_\star} = \{(0, \tau, \pi/2) \ : \ \tau \in \mathbb{R}/\ell_\star\mathbb{Z}\}, \quad \bar{\Lambda}|_{\gamma_\star} = \{(0, \tau, -\pi/2) \ : \ \tau \in \mathbb{R}/\ell_\star\mathbb{Z}\}; \qquad (5.3.1)$$

in particular it holds

$$T_z\Lambda = \mathbb{R}H(z), \quad T_{z'}\bar{\Lambda} = \mathbb{R}H(z'), \quad (z, z') \in \Lambda \times \bar{\Lambda}. \qquad (5.3.2)$$

For $\tau \geqslant 0$ and $z \in \Lambda$ such that $\varphi_{-\tau}(z) \in \bar{\Lambda}$, we will set $\varepsilon(\tau, z) = 1$ if

$$T_z\Lambda \oplus \mathbb{R}X(z) \oplus \mathrm{d}_{\varphi_{-\tau}(z)}\varphi_\tau \left(T_{\varphi_{-\tau}(z)}\bar{\Lambda}\right) \qquad (5.3.3)$$

has the same orientation as TM_δ, and $\varepsilon(\tau, z) = -1$ otherwise (note that the sum (5.3.3) is always direct as the component of $\mathrm{d}\varphi_\tau(z')H(z')$ on $V(\varphi_\tau(z'))$ is positive by Remark 5.2.3, since $\varphi_\tau(\bar{\Lambda}) \cap \Lambda \neq \emptyset$ implies $\tau > 0$).

Lemma 5.3.1. *For any $\tau \geqslant 0$ and $z \in \Lambda$ such that $\varphi_{-\tau}(z) \in \bar{\Lambda}$ it holds*

$$\varepsilon(\tau, z) = 1.$$

Proof. Indeed, $\bar{\Lambda}$ is oriented so that $\det_{T_{z'}\bar{\Lambda}} H(z') > 0$ for $z' \in \bar{\Lambda}$; moreover the component of $\mathrm{d}\varphi_t(z)H(z)$ on $V(\varphi_t(z))$ is positive (meaning that $\langle \psi(\varphi_t(z)), \mathrm{d}_z\varphi_t H(z)\rangle > 0$) whenever $t > 0$ (see Remark 5.2.3). Thus, since $T_z\Lambda$ is oriented so that $\det_{T_z\Lambda} H(z) < 0$ we obtain $\varepsilon(\tau, z)$ is equal to the sign of $\det_{T_zM_\delta}(-H, X, H + f(z, \tau)V)$ for some $f(z, \tau) > 0$, which is 1 as (X, H, V) is positively oriented. $\square$

In what follows, if P is an embedded, oriented, compact, k-dimensional submanifold of N, we will denote by $[P] \in \mathcal{D}'^{n-k}(N)$ the associated integration current, which is defined by

$$\int_N [P] \wedge \omega = \int_P {\iota_P}^*\omega, \quad \omega \in \Omega^k(N),$$

where $\iota_P : P \hookrightarrow N$ is the inclusion. We then have the following geometrical lemma, which is a direct adaptation of [DR20a, Lemma 4.11] in our context.

Lemma 5.3.2. *The expression*

$$\mu(t) = \sum_{\substack{\tau \geqslant 0 \\ \Lambda \cap \varphi_\tau(\bar{\Lambda}) \neq \emptyset}} \left(\sum_{z \in \Lambda \cap \varphi_\tau(\bar{\Lambda})} \varepsilon(\tau, z) \right) \delta(t - \tau),$$

makes sense and defines a distribution $\mu \in \mathcal{D}'(\mathbb{R}_{>0})$. *Moreover, it coincides with*

$$t \mapsto - \int_N [\Lambda] \wedge \left(\iota_X \varphi_{-t}^* [\bar{\Lambda}] \right).$$

Remark 5.3.3. (i) Lemma 5.3.2 can be reformulated as follows. For any $\chi \in C_c^\infty(\mathbb{R}_+)$, the product

$$A_\chi = [\Lambda] \wedge \int_{\mathbb{R}_+} \chi(t) \iota_X \varphi_{-t}^* [\bar{\Lambda}] \mathrm{d}t$$

is well defined and $\langle 1, A_\chi \rangle = -\langle \mu, \chi \rangle$ (here the first pairing takes place on N while the second one takes place on $\mathbb{R}_+$).

(ii) Lemma 5.3.2 is an elementary result coming from the theory of currents and is not specific to $(\Lambda, \bar{\Lambda}, \varphi_t)$. Indeed, this lemma will hold true for if we replace $\Lambda, \bar{\Lambda}$ and the flow φ_t by arbitrary submanifolds N_1, N_2 and another flow ψ_t, whenever the sum (5.3.3) is direct (replacing $(\Lambda, \bar{\Lambda}, \varphi_t)$ by (N_1, N_2, ψ_t)) and $\dim N_1 + \dim N_2 + 1 = \dim N$; we refer to [DR20a, Lemma 4.11] for more details.

Proof. We note that $\Lambda \cap \bar{\Lambda} = \emptyset$, and $X(z) \notin T_z \bar{\Lambda}$ for any $z \in \bar{\Lambda}$. Moreover, it holds $\dim(\Lambda) + \dim(\bar{\Lambda}) + 1 = \dim(N)$. Hence by (5.3.3) we can apply [DR20a, Lemma 4.11] to obtain the sought result (note however that here the vector field X on N may have singular points, but this is not a problem since the proof of [DR20a, Lemma 4.11] is local in nature and the singular points are far away from Λ). □

5.3.2 A pairing formula for the Poincaré series

Note that (5.2.13) implies that $Q(s)\iota_X[\bar{\Lambda}]$ is well defined. Indeed, according to [Hör90, Theorem 8.1.9] one has $\mathrm{WF}(\iota_X[\bar{\Lambda}]) \subset N^*\bar{\Lambda}$ where

$$N_z^* \bar{\Lambda} = \{\xi \in T_z^* M : \langle \xi, H(z) \rangle = 0\} = \mathbb{R}\alpha(z) \oplus \mathbb{R}\psi(z),$$

where ψ is the connection form. For $z \in \Lambda$ we have $T_z \Lambda = \mathbb{R} H(z)$ and thus

$$N_z^* \Lambda = \mathbb{R}\alpha(z) \oplus \mathbb{R}\psi(z), \tag{5.3.4}$$

(of course the same formula holds if we replace Λ by $\bar{\Lambda}$). We have $E_-^* \cap N^*\bar{\Lambda} \subset \{0\}$, since for $z \in \Gamma_- \cap \bar{\Lambda}$ we have $E_-^*(z) = \mathbb{R}(r_-(z)\beta(z) - \psi(z))$, and $r_-(z) < 0$ by Lemma 5.2.2. By (5.2.13) we can apply [Hör90, Theorem 8.2.13] to see that $Q(s)\iota_X[\bar{\Lambda}]$ is well defined, and

$$\mathrm{WF}(Q(s)\iota_X[\bar{\Lambda}]) \subset E_+^* \cup (N^*\bar{\Lambda}) \cup \{\Phi_t(z, \xi) : z \in \bar{\Lambda},\ \xi \in \mathbb{R}\psi(z),\ t \geqslant 0\}. \tag{5.3.5}$$

In particular, we have $N^*\Lambda \cap \mathrm{WF}(Q(s)\iota_X[\bar\Lambda]) = \emptyset$. Indeed, since $r_+(z) > 0$, we have as before that $N^*\Lambda \cap E_+^* \subset \{0\}$; also $N^*\Lambda \cap N^*\bar\Lambda = \emptyset$ simply because $\Lambda \cap \bar\Lambda = \emptyset$; finally, the last term in the right-hand side of (5.3.5) can only intersect $N^*\Lambda$ in a trivial way by Remark 5.2.3 and (5.3.4). Therefore, the product $[\Lambda]\wedge Q(s)\iota_X[\bar\Lambda]$ is well defined as a distribution by [Hör90, Theorem 8.2.10]. As $s \mapsto Q(s)$ is meromorphic, so is the family $s \mapsto [\Lambda] \wedge Q(s)\iota_X[\bar\Lambda]$, because the bound (5.2.13) is satisfied locally uniformly in $s \in \mathbb{C} \setminus \mathrm{Res}(\mathcal{L}_X)$ (it follows from the proof of (5.2.13) in [DG16]).

In what follows, for any closed conical subset $\Gamma \subset T^*M_\delta$, we will denote

$$\mathcal{D}'^\bullet_\Gamma(M_\delta^\circ) = \{u \in \mathcal{D}'^\bullet(M_\delta^\circ) \,:\, \mathrm{WF}(u) \subset \Gamma\}$$

endowed with its natural topology (see [Hör90, Definition 8.2.2]).

Proposition 5.3.4. *If* $\mathrm{Re}(s)$ *is large enough, the Poincaré series* $\eta(s)$ *converges, the pairing* $\langle 1, [\Lambda] \wedge Q(s)\iota_X[\bar\Lambda]\rangle$ *is well defined, and it holds*

$$\eta(s) = -\langle 1, [\Lambda] \wedge Q(s)\iota_X[\bar\Lambda]\rangle.$$

Remark 5.3.5. As we mentioned above, we already know that the pairing $\langle 1, [\Lambda] \wedge Q(s)\iota_X[\bar\Lambda]\rangle$ makes sense by using the wavefront set properties of $Q(s)$ given in [DG16]. However, we will prove below that this pairing is *a priori* well defined provided that $\mathrm{Re}(s)$ is large enough (without using the results of [DG16]) and we will see (using Lemma 5.3.2) that this implies the convergence of the series $\eta(s)$.

Corollary 5.3.6. *The function* $s \mapsto \eta(s)$ *extends meromorphically to the whole complex plane.*

Proof. We saw above that family $s \mapsto [\Lambda] \wedge Q(s)\iota_X[\bar\Lambda]$ extends meromorphically to the whole complex plane, and so does $s \mapsto \langle 1, [\Lambda] \wedge Q(s)\iota_X[\bar\Lambda]\rangle$. Thus Proposition 5.3.4 immediately implies the meromorphic continuation of η. □

Proof of Proposition 5.3.4. For $n \in \mathbb{N}_{\geqslant 1}$ we take $\chi_n \in C_c^\infty(\mathbb{R}_{>0}, [0,1])$ such that $\chi_n(t) = 1$ for any $t \in [\varepsilon, N]$ and $\chi_n(t) = 0$ for $t \geqslant n+1$, where $\varepsilon = \min(1/2, \inf_{\gamma \in \mathcal{G}^\perp} \ell(\gamma))$. Then we have

$$\langle \mu, \chi_n \mathrm{e}^{-s\cdot}\rangle \to \eta(s), \quad n \to +\infty, \tag{5.3.6}$$

for $\mathrm{Re}(s) \gg 1$ by Lemma 5.3.2, as $\varepsilon(\tau, z) = +1$ for any $z \in \Lambda$ such that $\varphi_{-\tau}(z) \in \bar\Lambda$ (note that $\eta(s)$ could *a priori* be infinite). Now consider

$$A_n(s) = \chi \int_{\mathbb{R}_+} \chi_n(t)\mathrm{e}^{-ts}\iota_X\varphi_{-t}^*[\bar\Lambda]\mathrm{d}t \in \mathcal{D}'^1(M_\delta^\circ),$$

where $\chi \in C_c^\infty(M_\delta^\circ)$ is the cutoff function introduced in §7.1.6. Note that

$$A_n(s) \to Q(s)\iota_X[\bar\Lambda] \tag{5.3.7}$$

in $\mathcal{D}'^1(M_\delta^\circ)$ when $n \to +\infty$ whenever $\mathrm{Re}(s)$ is large enough. Indeed, for any $\omega \in \Omega^\bullet(N)$ and $t \in \mathbb{R}$ we have $\|\varphi_t^* w\|_\infty \leqslant C\exp(C|t|)\|w\|_\infty$ (see for example [Bon15, Proposition A.4.1]). In particular it holds $|\langle w, \iota_X\varphi_{-t}^*[\bar\Lambda]\rangle| \leqslant C\exp(C|t|)\|w\|_\infty$, and thus, if $\mathrm{Re}(s)$ is large enough,

$$A_n(s) \to \chi\int_0^\infty \mathrm{e}^{-ts}\iota_X\varphi_{-t}^*[\bar\Lambda]\mathrm{d}t$$

as $n \to \infty$ by dominated convergence, and the integral defines a current of order 0. Now the above integral coincides with $Q(s)\iota_X[\bar{\Lambda}]$ as it follows by approximating $[\bar{\Lambda}]$ with smooth differential forms, and thus (5.3.7) holds. Using (5.3.6) and Lemma 5.3.2, we see that Proposition 5.3.4 will hold if we are able to show that the pairings $\langle [\Lambda], A_n(s)\rangle$ and $\langle 1, [\Lambda] \wedge Q(s)\iota_X[\bar{\Lambda}]\rangle$ are well defined, and that

$$\int_N [\Lambda] \wedge A_n(s) \to \langle 1, [\Lambda] \wedge Q(s)\iota_X[\bar{\Lambda}]\rangle \tag{5.3.8}$$

as $n \to \infty$. To prove (5.3.8) we will show that the convergence $A_n(s) \to Q(s)\iota_X[\bar{\Lambda}]$ actually takes place in a finer topology than that of $\mathcal{D}'^1(M_\delta^\circ)$; this is the purpose of Lemma 5.3.8 below. We will need the following result (recall that ψ is the connection form introduced in §5.2.2).

Lemma 5.3.7. *Let $\tau > 0$. Then there is $r > 0$ such that the following holds. For any $z \in M_\delta$ and $t \geqslant \tau$ such that $\varphi_t(z) \in M_\delta$, we have*

$$\left|\left\langle \varphi_t^*\psi(z), H(z)\right\rangle\right| \geqslant r\left|\left\langle \varphi_t^*\beta(z), H(z)\right\rangle\right|. \tag{5.3.9}$$

Moreover we have $\left|\left\langle \varphi_t^*\beta(z), H(z)\right\rangle\right| \geqslant 1$ for any $t \geqslant 0$.

Proof. Let $z \in M_\delta$ and $\tau > 0$. Write $\varphi_t^*\beta(z) = a(t)\beta(z) + b(t)\psi(z)$ and $\varphi_t^*\psi(z) = c(t)\beta(z) + d(t)\psi(z)$ for $t \in \mathbb{R}$. We want to show that for $t \geqslant \tau$ one has $|c(t)| \geqslant r|a(t)|$ for some $r > 0$. The structural equations (see §5.2.2) imply $\mathcal{L}_X\beta = \psi$ and $\mathcal{L}_X\psi = -\kappa\beta$. We thus obtain that a and b satisfy the following differential equation

$$y''(t) + \kappa(t)y(t) = 0 \tag{5.3.10}$$

where $\kappa(t) = \kappa(\varphi_t(z))$, with $a(0) = 1 = b'(0)$ and $a'(0) = 0 = b(0)$. Also $a'(t) = c(t)$ and $b'(t) = d(t)$. It is easy to see that (5.3.10) and the initial conditions imply $a'(t), a(t) > 0$ for $t > 0$. Thus we have $a'(t)a''(t) = -\kappa(t)a'(t)a(t) \geqslant ka'(t)a(t)$ where $k = \inf_{\Sigma_\delta}|\kappa|$. Integrating this, we get

$$c(t)^2 = a'(t)^2 \geqslant k(a(t)^2 - 1).$$

As $a'(t) > 0$ for $t > 0$ we have $a(t)^2 - 1 \geqslant a(\tau)^2 - 1$ for $t \geqslant \tau$, and thus it holds

$$c(t)^2 \geqslant Cka(t)^2, \quad t \geqslant \tau,$$

where $C = 1 - 1/a(\tau)^2 > 0$ (since $a(\tau) > a(0) = 1$). Setting $r = \sqrt{Ck}$ we obtain (5.3.9). We conclude the proof of the lemma by noting that $a(t) \geqslant a(0) = 1$. □

In what follows we set $J_n(s) = \chi \int_{\mathbb{R}_+} (\chi_{n+1}(t) - \chi_n(t))\mathrm{e}^{-ts}\iota_X\varphi_{-t}^*[\bar{\Lambda}]\mathrm{d}t \in \mathcal{D}'^1(M_\delta)$.

Lemma 5.3.8. *There exists a closed conical subset $\Gamma \subset T^*M_\delta$ not intersecting $N^*\Lambda$ such that for any continuous semi-norm q on $\mathcal{D}'^1_\Gamma(M_\delta^\circ)$ (see [Hör90, Equation (8.2.2)]), there is $C > 0$ such that*

$$q(J_n(s)) \leqslant C|s|^C\mathrm{e}^{(C - \mathrm{Re}(s))n}, \quad n \geqslant 0.$$

Proof. Let $w \in \Omega_c^2(M_\delta^\circ)$ supported in a small coordinate patch U of some point $z_0 \in \Lambda$. Now by definition of J_n it holds

$$\langle w, J_n(s)\rangle = \int_0^\infty (\chi_{n+1}(t) - \chi_n(t))\mathrm{e}^{-ts}\left(\int_{\bar\Lambda} \iota_X \varphi_t^* w\right)\mathrm{d}t.$$

Let $\xi \in T_{z_0}^* M_\delta$. We identify T^*U with $V \times \mathbb{R}^3$ for some neighborhood V of $0 \in \mathbb{R}^3$. Consider the Fourier transform $\langle w\mathrm{e}^{i\langle \xi,\cdot\rangle}, J_n\rangle$; it holds

$$\langle w\mathrm{e}^{i\langle \xi,\cdot\rangle}, J_n\rangle = \int_{t=N}^{N+2}\int_{u\in\bar\Lambda} (\chi_{n+1}(t) - \chi_n(t))\mathrm{e}^{-ts} f(t,u)\mathrm{e}^{i\langle \xi,\varphi_t(u)\rangle}\mathrm{d}u\mathrm{d}t, \tag{5.3.11}$$

where u is a choice of coordinate on $\bar\Lambda$ so that $\partial_u = H(u) \in T_u\bar\Lambda$, and f is a smooth function satisfying for any $k, \ell \geqslant 0$

$$|\partial_t^k \partial_u^\ell f(t,u)| \leqslant C_{k,\ell}\mathrm{e}^{C_{k,\ell}t}, \quad t \geqslant 0, \quad u \in \bar\Lambda,$$

for some $C_{k,\ell} > 0$ [1] ; note also that, as w is supported in the coordinate patch U, we have that $f(t,u) = 0$ whenever $\varphi_t(u) \notin U$ and thus the expression $\mathrm{e}^{i\langle\xi,\varphi_t(u)\rangle}$ is well defined on the support of f. Now we have

$$\partial_u\langle \xi, \varphi_t(u)\rangle = \langle \xi, \mathrm{d}\varphi_t(u)H(u)\rangle, \quad \partial_t\langle \xi, \varphi_t(u)\rangle = \langle \xi, X(\varphi_t(u))\rangle. \tag{5.3.12}$$

Let $\Gamma' = \{(z,\xi) \in T^*M_\delta : z \in \Lambda,\ |\langle \xi, H(z)\rangle| < \varepsilon|\xi|\}$ for $\varepsilon > 0$ small. Let $(z,\xi) \in \Gamma'$, $u \in \bar\Lambda$ and $t > 0$ such that $\varphi_t(u) = z \in U$. Then $t \geqslant \tau$ where $\tau > 0$ is a fixed number which is smaller than the length of the shortest orthogeodesic. We decompose ξ in the $(\alpha(z), \beta(z), \psi(z))$ basis as $\xi = \xi_\alpha\alpha(z) + \xi_\beta\beta(z) + \xi_\psi\psi(z)$. We have, since $\varphi_t^*\alpha(u) = \alpha(u)$,

$$\begin{aligned}\langle \xi,\ \mathrm{d}\varphi_t(u)H(u)\rangle &= \left\langle \varphi_t^*\left[\xi_\alpha\alpha(z) + \xi_\beta\beta(z) + \xi_\psi\psi(z)\right],\ H(u)\right\rangle \\ &= \langle \xi_\beta\cdot\varphi_t^*\beta(u) + \xi_\psi\cdot\varphi_t^*\psi(u),\ H(u)\rangle.\end{aligned}$$

Thus by Lemma 5.3.7 and the triangle inequality we have

$$|\langle \xi,\ \mathrm{d}\varphi_t(u)H(u)\rangle| \geqslant r|\xi_\psi| - |\xi_\beta|,$$

for some $r > 0$ depending only on τ (indeed, the above inequality is obviously true even if $r|\xi_\psi| - |\xi_\beta| \leqslant 0$). As $\xi \in \Gamma'$ we have

$$|\xi_\beta| \leqslant \frac{C\varepsilon}{1-\varepsilon}(|\xi_\psi| + |\xi_\alpha|) \tag{5.3.13}$$

for some $C > 0$. Therefore, we obtain

$$|\partial_u\langle \xi, \varphi_t(u)\rangle| \geqslant (r - c(\varepsilon))|\xi_\psi| - c(\varepsilon)|\xi_\alpha|, \quad |\partial_t\langle \xi, \varphi_t(u)\rangle| = |\xi_\alpha|,$$

1. The estimates on f follow from the fact that $\|\iota_X\varphi_t^* w\|_{C^\ell} \leqslant C_\ell \exp(C_\ell|t|)\|w\|_{C^\ell}$ (see [Bon15, Proposition A.4.1]). Thus

$$\|\partial_t^k(\iota_X\varphi_t^*w)\|_{C^\ell} = \|\iota_X(\mathcal{L}_X)^k\varphi_t^*w\|_{C^\ell} \leqslant C_{k,\ell}\exp(C_{k,\ell}|t|)\|w\|_{C^{k+\ell}}.$$

Here we denoted by $\|\cdot\|_{C^\ell}$ the C^ℓ norm on $C^\infty(N, \wedge^\bullet T^*N)$.

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Combining the estimate above with (5.3.13) we obtain that there are $c, C > 0$ such that for any $\xi \in \Gamma'$ it holds

$$C^{-1}\left\|\mathrm{d}_{t,u}\left(\mathrm{e}^{i\langle\xi,\varphi_t(u)\rangle}\right)\right\| \geqslant (r - c(\varepsilon))|\xi_\psi| + (1 - c(\varepsilon))|\xi_\alpha| \geqslant c(|\xi_\psi| + |\xi_\alpha|) \geqslant \frac{c}{2}|\xi|,$$

provided that ε is small enough. In particular we may apply the non-stationary phase method (see for example [Zwo12, Lemma 3.14]) to obtain that for any $L > 0$ we have C_L such that

$$\left|\left\langle w\mathrm{e}^{i\langle\xi,\cdot\rangle}, J_n(s)\right\rangle\right| \leqslant C_L|s|^L\mathrm{e}^{(C_L-\mathrm{Re}(s))n}\langle\xi\rangle^{-L}, \quad \xi \in \Gamma',$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ for some norm $|\cdot|$ on T^*M_δ. By setting $\Gamma = \complement\Gamma'$, we obtain the sought result. □

This last result implies that for any continuous semi norm q of $\mathcal{D}'^1_\Gamma(M_\delta^\circ)$, we have

$$q\left(A_n(s) - Q(s)\iota_X[\bar\Lambda]\right) \to 0$$

as $n \to +\infty$ if $\mathrm{Re}(s)$ is large enough (depending on q). For any finite set Q of continuous semi norms of $\mathcal{D}'_\Gamma(M_\delta^\circ)$ we define

$$\mathcal{D}'^\bullet_{\Gamma,Q}(M_\delta^\circ) = \{u \in \mathcal{D}'^\bullet(M_\delta^\circ) \ : \ q(u) < \infty, \ q \in Q\}.$$

This set is endowed with the following topology : we say that $u_n \to u$ in $\mathcal{D}'^\bullet_{\Gamma,Q}(M_\delta^\circ)$ if the convergence holds in $\mathcal{D}'^\bullet(M_\delta^\circ)$ and $\sup_n q(u_n) < \infty$ for any $q \in Q$. Then, since $[\Lambda]$ is compactly supported in M_δ° and $\mathrm{WF}'([\Lambda]) \cap \Gamma = \emptyset$, we may reproduce the proof of the Hörmander's theorem about product of distributions [Hör90, Theorem 8.2.10] to obtain that there exists a finite set Q of semi norms of $\mathcal{D}'^\bullet_\Gamma(M_\delta^\circ)$ (which depends on $[\Lambda]$) such that the product $[\Lambda] \wedge u$ is well defined for any $u \in \mathcal{D}'^\bullet_{\Gamma,Q}(M_\delta^\circ)$ and such that the map

$$\mathcal{D}'^\bullet_{\Gamma,Q}(M_\delta^\circ) \to \mathcal{D}'^\bullet(M_\delta^\circ), \quad u \mapsto [\Lambda] \wedge u, \tag{5.3.14}$$

is continuous. By Lemma 5.3.8, if $\mathrm{Re}(s)$ is large enough, the sequence $n \mapsto q(A_n(s))$ is bounded for any $q \in Q$ and letting $n \to \infty$ yields $q(Q(s)\iota_X[\bar\Lambda]) < \infty$ for every $q \in Q$. Thus the products $[\Lambda] \wedge A_n(s)$ and $[\Lambda] \wedge Q(s)\iota_X[\bar\Lambda]$ are well defined, and by continuity of the map (5.3.14), we get

$$[\Lambda] \wedge A_n(s) \to [\Lambda] \wedge Q(s)\iota_X[\bar\Lambda], \quad n \to \infty,$$

where the convergence holds in $\mathcal{D}'^\bullet(M_\delta^\circ)$. In particular, (5.3.8) holds. This completes the proof of Proposition 5.3.4. □

5.4 Value of the Poincaré series at the origin

In this section we show that $\eta(s)$ vanishes at $s = 0$.

5.4.1 Behavior of $Q(s)$ at $s = 0$

By [DG16, Theorem 2], we have the Laurent development

$$Q(s) = Y(s) + \sum_{j=1}^{J} \frac{\chi(\mathcal{L}_X)^{j-1}\Pi\chi}{s^j} \tag{5.4.1}$$

for some $J \geqslant 1$, where $s \mapsto Y(s)$ is holomorphic near $s = 0$, and $\Pi : \Omega_c^\bullet(M_\delta^\circ) \to \mathcal{D}'^\bullet(M_\delta^\circ)$ is a finite rank projector satisfying

$$\mathrm{supp}(\Pi) \subset \Gamma_+ \times \Gamma_- \quad \text{and} \quad \mathrm{WF}(\Pi) \subset E_+^* \times E_-^*. \tag{5.4.2}$$

Moreover, it holds $\mathrm{ran}(\Pi) = C^\bullet$ where

$$C^\bullet = \left\{ u \in \mathcal{D}'^\bullet_{E_+^*}(M_\delta^\circ) \ : \ \mathrm{supp}(u) \subset \Gamma_+, \ (\mathcal{L}_X)^J u = 0 \right\}.$$

Elements of $C^\bullet$ are called generalized resonant states for X (for the resonance 0). A generalized resonant state u is simply called a resonant state if $\mathcal{L}_X u = 0$. In what follows, we will set $\Omega_0^\bullet = \Omega_c^\bullet(M_\delta^\circ) \cap \ker(\iota_X)$ and $C_0^\bullet = C^\bullet \cap \ker(\iota_X)$. Since $\mathcal{L}_X \alpha = 0$ we have the decomposition

$$C^\bullet = C_0^\bullet \oplus \alpha \wedge C_0^{\bullet - 1}, \tag{5.4.3}$$

and this decomposition is preserved by Π. We now invoke a result of Hadfield (see [Had18, Propositions 3, 4, 5]) which implies that

$$C_0^0 = \{0\}, \quad C_0^2 = \{0\} \quad \text{and} \quad \mathcal{L}_X(C_0^1) = \{0\}. \tag{5.4.4}$$

In particular by (5.4.3) we have $\mathcal{L}_X \Pi = 0$ and thus (5.4.1) yields

$$Q(s) = Y(s) + \frac{\chi\Pi\chi}{s}. \tag{5.4.5}$$

Now let us decompose Π as

$$\Pi|_{\Omega_c^1(M_\delta^\circ)} = \sum_{j=1}^{r} u_j \otimes \beta_j,$$

where $(u_j, \beta_j) \in \mathcal{D}'^1_{E_+^*}(M_\delta^\circ) \times \mathcal{D}'^2_{E_-^*}(M_\delta^\circ)$ satisfy $\mathrm{supp}\, u_j \subset \Gamma_+$ and $\mathrm{supp}\, \beta_j \subset \Gamma_-$ (such a decomposition necessarily exists by (5.4.2)). Then β_j is a coresonant state for X, meaning that it is a resonant state for $-X$; applying [Had18, Propositions 3, 4, 5] for the vector field $-X$, we therefore obtain that $\beta_j = \alpha \wedge s_j$ for some coresonant state $s_j \in \mathcal{D}'^1_{E_-^*}(M_\delta^\circ)$ (indeed, we note that (5.4.4) gives $C^2 = \alpha \wedge C_0^1$, and we apply this to the vector field $-X$ instead of X). Also, it follows from [Had18, Lemma 6] that the currents u_j and s_j are closed.

Summarizing the above results, we get

$$\Pi = \sum_{j=1}^{r} u_j \otimes \alpha \wedge s_j \tag{5.4.6}$$

where $(u_j, s_j) \in \mathcal{D}'^1_{E_+^*}(M_\delta^\circ) \times \mathcal{D}'^1_{E_-^*}(M_\delta^\circ)$ satisfy

$$\operatorname{supp}(u_j) \subset \Gamma_+, \quad \operatorname{supp}(s_j) \subset \Gamma_-, \quad \mathrm{d}u_j = \mathrm{d}s_j = 0, \quad \iota_X u_j = \iota_X s_j = 0. \tag{5.4.7}$$

In particular, we have $\int_{M_\delta^\circ} \iota_X[\bar{\Lambda}] \wedge \alpha \wedge s_j = \int_{M_\delta^\circ} [\bar{\Lambda}] \wedge s_j$ and thus

$$\langle [\Lambda], \Pi \iota_X [\bar{\Lambda}] \rangle = \sum_{j=1}^r \left(\int_{M_\delta^\circ} [\Lambda] \wedge u_j \right) \left(\int_{M_\delta^\circ} [\bar{\Lambda}] \wedge s_j \right).$$

Note that those products make sense since by Lemma 5.2.2 it holds

$$E_+^* \cap N^*\Lambda \subset \{0\} \quad \text{and} \quad E_-^* \cap N^*\bar{\Lambda} \subset \{0\}.$$

Let $\eta > 0$ and set $\Gamma_+^\eta = \{z \in M_\delta : \operatorname{dist}(z, \Gamma_+) < \eta\}$. By [Had18, Lemma 6], we may find $f_j \in \mathcal{D}'(M_\delta^\circ)$ such that

$$\operatorname{supp}(f_j) \subset \Gamma_+^\eta, \quad \mathrm{WF}(f_j) \subset E_+^*, \quad \mathcal{L}_X f_j \in C_c^\infty(M_\delta^\circ),$$

and such that $v_j = u_j - \mathrm{d}f_j$ is smooth. Now since $[\Lambda]$ is compactly supported in M_δ°, we have $\displaystyle\int_{M_\delta^\circ} [\Lambda] \wedge \mathrm{d}f_j = 0$ (since $\mathrm{d}[\Lambda] = 0$ as $\partial\Lambda = \emptyset$) and thus

$$\int_M [\Lambda] \wedge u_j = \int_M [\Lambda] \wedge v_j. \tag{5.4.8}$$

Finally, take the coordinates (ρ, τ, θ) given by Lemma 5.2.1 near $\partial M = \{\rho = 0\}$ (here we assume for simplicity that ∂M is connected but the exact same proof applies if it is not). We have $\Lambda = \{(0, \tau, +\pi/2) : \tau \in \mathbb{R}/\ell_\star\mathbb{Z}\}$, and $\partial M_\delta = \{(\delta, \tau, \theta) : \tau, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$. Consider

$$S_1 = \{(\rho, \tau, \pi/2) : \rho \in [0, 2\delta/3],\ \tau \in \mathbb{R}/\ell_\star\mathbb{Z}\}$$

and

$$S_2 = \{(2\delta/3, \tau, \theta) : \tau \in \mathbb{R}/\ell_\star\mathbb{Z},\ \theta \in [\pi/2, 3\pi/2]\}.$$

Let $[S] = [S_1] + [S_2]$ and note that $\mathrm{d}[S] = [\Lambda] - [W]$ by Stokes' theorem, where

$$W = \{(2\delta/3, \tau, 3\pi/2) : \tau \in \mathbb{R}/\ell_\star\mathbb{Z}\}$$

is the incoming normal set of $\{\rho = 2\delta/3\}$, which is oriented with ∂_τ. Now if η is small enough

$$\int [\Lambda] \wedge v_j = \int [W] \wedge v_j = 0$$

since $W \cap \Gamma_+^\eta = \emptyset$ (indeed, W is at positive distance of Γ_+) and $\operatorname{supp} v_j \subset \Gamma_+^\eta$. Thus we obtained that $s \mapsto \eta(s)$ has no pole at $s = 0$, and by Proposition 5.3.4 it holds

$$\eta(0) = -\langle [\Lambda], Y(0)\iota_X[\bar{\Lambda}] \rangle.$$

5.4.2 Value at $s = 0$

In this subsection we prove Theorem 5.1.1, that is, $\eta(0) = 0$. Let us define

$$S_1 = \{\varphi_t(z) : 0 \leqslant t \leqslant \delta/2,\ z \in \Lambda\}, \quad S_2 = \{R_\theta(z) : 0 \leqslant \theta \leqslant \pi,\ z \in \Lambda'\},$$

where $R_\theta : M_\delta \to M_\delta$ is the rotation of angle θ, and $\Lambda' = \varphi_{\delta/2}(\Lambda)$. We orient Λ' with the orientation of Λ. The manifold S_1 is oriented by declaring that $(\partial_t, \partial_\tau)$ is positively oriented (here ∂_τ is any positive basis of $T\Lambda$), and S_2 is oriented by declaring that the basis $(\partial_\theta, \partial'_\tau)$, where ∂'_τ is any positive basis of $T\Lambda'$. Let $\Lambda'' = R_\pi(\Lambda')$. Note that (5.2.13) implies, by multiplication of wavefront sets (see [Hör90, Theorem 8.2.14]),

$$\mathrm{WF}(Y(0)\iota_X[\bar\Lambda]) \subset E_+^* \cup N^*\bar\Lambda \cup \bigcup_{\substack{t\geqslant 0\\ z\in\bar\Lambda}} \mathbb{R}\Phi_t(\psi(z)), \tag{5.4.9}$$

where $Y(0)$ comes from (5.4.5). In what follows, we will set $Y = Y(0)\iota_X[\bar\Lambda]$ for simplicity. Since $E_+^* \cap N^*\Lambda \subset \{0\}$, and because δ is small, we have

$$E_+^* \cap N^*\Lambda' \subset \{0\}. \tag{5.4.10}$$

Next, by analytic continuation and (5.2.11), it holds

$$\mathrm{supp}(Y) \subset \overline{\bigcup_{t\geqslant 0} \varphi_t(\bar\Lambda)}.$$

The right hand side of the above equation is disjoint from Λ'' by strict convexity of M_δ. Thus

$$\mathrm{supp}(Y) \cap \Lambda'' = \emptyset. \tag{5.4.11}$$

Now for $z \in \Lambda$ and $t \in (0, \delta/2)$, we have $N^*_{\varphi_t(z)}(S_1 \setminus \partial S_1) \subset \mathbb{R}\Phi_t(\psi(z))$ and $N^*(S_2 \setminus \partial S_2) \subset \{\xi : \langle \xi, V\rangle = 0\}$. In particular, by Lemmas 5.2.2 and 5.3.7, we have

$$\mathrm{WF}(Y) \cap N^*_{\varphi_t(z)}(S_j \setminus \partial S_j), \quad j = 1, 2. \tag{5.4.12}$$

Finally, for $z \in \Lambda$, we have for $j = 1, 2$, setting $z' = \varphi_{\delta/2}(z) \in \Lambda'$,

$$\mathrm{WF}([S_j]) \cap T^*_{z'}M_\delta \subset \mathbb{R}\alpha(z') \oplus \mathbb{R}\Phi_{\delta/2}(\psi(z)). \tag{5.4.13}$$

Combining (5.4.9), (5.4.10), (5.4.11), (5.4.12) and (5.4.13), we obtain that the intersection $\mathrm{WF}([S_j]) \cap \mathrm{WF}(Y)$ is contained in

$$\left(\bigcup_{z\in\Lambda} \mathbb{R}\Phi_{\delta/2}(\psi(z))\right) \cap \left(\bigcup_{\substack{t\geqslant 0\\ \bar z\in\bar\Lambda}} \mathbb{R}\Phi_t(\psi(\bar z))\right).$$

However, by Lemma 5.3.7, for any $\bar z \in \bar\Lambda$ and $t > 0$, we have $\Phi_t(\psi(\bar z)) \notin \mathbb{R}\psi(\varphi_t(\bar z))$. Therefore the above intersection is contained in the zero section and we get

$$\mathrm{WF}([S_j]) \cap \mathrm{WF}(Y) = \emptyset, \quad j = 1, 2, \tag{5.4.14}$$

and in particular the product $[S_j] \wedge Y$ is well defined. By Stokes' theorem, taking into account the orientations, we have

$$\mathrm{d}[S_1] = [\Lambda'] - [\Lambda], \quad \mathrm{d}[S_2] = [\Lambda''] - [\Lambda']. \tag{5.4.15}$$

Then by (8.2.9) and the facts that $\mathrm{d}[\bar{\Lambda}] = 0$, $[\mathrm{d}, Y(0)] = 0$ (on $\{\chi = 1\}$) and $[\iota_X, Y(0)] = 0$ we have, by using (5.4.1),

$$\mathrm{d}Y = \mathrm{d}\iota_X Y(0)[\bar{\Lambda}] = \mathcal{L}_X Y(0)[\bar{\Lambda}] = [\bar{\Lambda}] - \Pi([\bar{\Lambda}]) = [\bar{\Lambda}] \quad \text{on} \quad \{\chi = 1\}$$

as $\Pi([\bar{\Lambda}]) = 0$ by §5.4.1 (we showed that $\int_{M_\delta^\circ} [\Lambda] \wedge u_j = 0$ for all j but the same holds for $[\bar{\Lambda}]$ and s_j). By Stokes' theorem, since $[S_j] \wedge Y$ is compactly supported in M_δ° and $\mathrm{d}Y = [\bar{\Lambda}]$ on $\{\chi = 1\} \supset \mathrm{supp}([S_j])$ $(j = 1, 2)$,

$$\begin{aligned}\int_{M_\delta} [\Lambda] \wedge Y &= -\int_{M_\delta} \mathrm{d}[S_1] \wedge Y - \int_{M_\delta} \mathrm{d}[S_2] \wedge Y + \int_{M_\delta} [\Lambda''] \wedge Y \\ &= \int_{M_\delta} [S_1] \wedge [\bar{\Lambda}] + \int_{M_\delta} [S_2] \wedge [\bar{\Lambda}] + \int_{M_\delta} [\Lambda''] \wedge Y.\end{aligned}$$

Finally, we have $\mathrm{supp}([S_j]) \cap \mathrm{supp}([\bar{\Lambda}]) = \emptyset$ and by (5.4.11) we conclude that

$$\eta(0) = \int_{M_\delta} [\Lambda] \wedge Y = 0.$$

5.5 Poincaré series for geodesic arcs linking two points

We fix $x \neq y \in \Sigma$. We consider

$$\eta_{x,y}(s) = \sum_{\gamma : x \rightsquigarrow y} \mathrm{e}^{-s\ell(\gamma)},$$

where the sum runs over all the (oriented) geodesics joining x to y. For $a \in \Sigma$ we will set $\Lambda_a = S_a\Sigma$. Note that $T_z\Lambda_a = \mathbb{R}V(z)$ for $z \in \Lambda_a$ (this follows from the definition of V in §5.2.2), and we orient Λ_a according to V. In this context, we have the counterpart of Proposition 5.3.4, as follows.

Proposition 5.5.1. *For* $\mathrm{Re}(s)$ *large enough it holds*

$$\eta_{x,y}(s) = -\langle [\Lambda_x],\ Q(s)\iota_X[\Lambda_y] \rangle.$$

Note that the above pairing makes sense, since we have the inclusion $\mathrm{WF}([\Lambda_x]) \subset N^*\Lambda_x$ which gives $\mathrm{WF}(Q(s)\iota_X[\Lambda_y]) \cap \mathrm{WF}([\Lambda_x]) = \emptyset$ (the emptiness of the last intersection can be seen by proceeding as in §5.3.2).

Sketch of the proof. Using Remark 5.2.3, we see that for $t \geqslant 0$ and $z \in \Lambda_x$ such that $\varphi_{-t}(z) \in \Lambda_y$, one has the direct sum

$$T_zM = T_z\Lambda_x \oplus \mathbb{R}X(z) \oplus \mathrm{d}_{\varphi_{-t}(z)}\varphi_t(T_{\varphi_{-t}(z)}\Lambda_y).$$

Moreover we check that the orientation of the right-hand side has the same orientation of M, again by Remark 5.2.3. Thus we have the counterpart of Lemma 5.3.2 in this context and for any $\chi \in C_c^\infty(\mathbb{R}_+)$ it holds

$$\sum_{\gamma : x \rightsquigarrow y} \chi(\ell(\gamma)) = -\int_N [\Lambda_x] \wedge \int_{\mathbb{R}_+} \chi(t) \iota_X \varphi_{-t}^* [\Lambda_y] \mathrm{d}t.$$

Now we may proceed as in the proof of Proposition 5.3.4 to obtain the sought result, by approximating the function $t \mapsto \exp(-ts)$ with compactly supported functions of the form $t \mapsto \chi_n(t) \exp(-ts)$ and taking the limit as $n \to \infty$ (one should use appropriate versions of Lemmas 5.3.7 and 5.3.8 to justify the convergence of the pairings). □

This result implies that $s \mapsto \eta_{x,y}(s)$ extends meromorphically to the whole complex plane, since $s \mapsto Q(s)$ does. To compute its value at zero, we will need the following

Lemma 5.5.2. *There exists* $[S] \in \mathcal{D}_c'^1(M_\delta^\circ)$ *with* $\operatorname{supp}([S]) \subset M$, $\mathrm{WF}([S]) \cap \mathrm{WF}([\Lambda_y]) = \emptyset$ *and*

$$[\Lambda_x] = -\frac{1}{\chi(\Sigma)}[\bar{\Lambda}] - \mathrm{d}[S], \qquad \int_{M_\delta} [S] \wedge [\Lambda_y] = \frac{1}{\chi(\Sigma)}. \tag{5.5.1}$$

Proof. Here we adapt the arguments of [DR20a, §6.3.2]. Let $f_1 : \Sigma_\delta \to \mathbb{R}$ be a smooth function which coincides with $-\rho$ on $\{|\rho| \leqslant \delta\}$ (here ρ is the coordinate given by Lemma 5.2.1) and such that $\mathrm{d}f_1(y) \neq 0$ for any $y \in \partial\Sigma$. The set of Morse functions being open and dense [**?**, Theorem 5.6] in $C^\infty(\Sigma_\delta)$, we may find a Morse function $f_2 \in C^\infty(\Sigma_\delta)$ which is arbitrarily close to f_1 in the C^1 norm. Let $\chi_0 \in C^\infty(\Sigma_\delta, [0,1])$ such that $\chi_0 = 1$ near $\partial\Sigma$ and $\operatorname{supp} \chi_0 \subset \{|\rho| \leqslant \delta/2\}$. Note that $\|\mathrm{d}f_1\| = \|\mathrm{d}\rho\| \geqslant C$ on $\{|\rho| \leqslant \delta\}$ for some $C > 0$, where $\|\cdot\|$ is any norm on $T^*\Sigma_\delta$. In particular, if f_2 is chosen close enough to f_1 in the C^1 topology, the function $f = \chi_0 f_1 + (1-\chi_0) f_2$ is also a Morse function. Indeed, f coincides with f_2 on $\Sigma_\delta \setminus \{|\rho| \leqslant \delta\}$; moreover $f - f_1 = (1-\chi_0)(f_2 - f_1)$ so that $\|\mathrm{d}f\| \geqslant C/2$ on $\{|\rho| \leqslant \delta\}$ whenever f_2 is close enough to f_1. Next we set

$$S_f = \left\{ \left(b, \frac{\nabla^g f(b)}{\|\nabla^g f(b)\|} \right) \; : \; b \in \Sigma \setminus \operatorname{crit}(f) \right\} \subset M_\delta,$$

where $\nabla^g f \in C^\infty(\Sigma_\delta, T\Sigma_\delta)$ is the gradient of f with respect to the metric g, and $\operatorname{crit}(f) = \{\mathrm{d}f = 0\}$ is the set of critical points of f. We orient S_f according to the orientation of Σ. Then by [DR20a, Lemma 6.7] and Stokes' theorem, we obtain that the integration current $[S_f]$ extends to a current on N and we have [2]

$$\mathrm{d}[S_f] = -[\bar{\Lambda}] - \sum_{a \in \operatorname{crit}(f)} (-1)^{\operatorname{ind}_f(a)} [\Lambda_a],$$

2. Indeed, the boundary of S_f (near ∂M) is $\bar{\Lambda}$. In the coordinates of Lemma 5.2.1, $\bar{\Lambda} = \{(\tau, 0, -\pi/2)\}$ is oriented by $\partial_\tau \equiv H$; as the outward normal pointing vector at $\partial\Sigma$ is ∂_ρ and $(\partial_\rho, \partial_\tau)$ is negatively oriented, we obtain that the boundary term coming from Stokes' formula must be $-[\bar{\Lambda}]$.

where $\mathrm{ind}_f(a)$ is the index of a as a critical point of $\nabla^g f$, that is, the number of negative eigenvalues of the linearization of $\nabla^g f$ at the point a. Note that (up to taking f_2 very close to f_1), we have $y \notin \mathrm{crit}(f)$. Thus for each $a \in \mathrm{crit}(f)$ we may find a path $\gamma_a : [0,1] \to \Sigma$ joining a to x, and avoiding y. Setting

$$\theta_a = \left\{ (\gamma_a(t), v) \ : \ v \in S_{\gamma_a(t)}\Sigma, \ t \in [0,1] \right\}, \quad a \in \mathrm{crit}(f),$$

we have $\mathrm{d}[\theta_a] = [\Lambda_x] - [\Lambda_a]$. The Poincaré-Hopf formula (see [?, p.35]) yields [3]

$$\sum_{a \in \mathrm{crit}(f)} (-1)^{\mathrm{ind}_f(a)} = \chi(\Sigma).$$

In particular, by setting

$$[S] = \frac{1}{\chi(\Sigma)} \left([S_f] - \sum_a (-1)^{\mathrm{ind}_f(a)} \theta_a \right),$$

we obtain the first part of (5.5.1). For the second part, we first note that $\theta_a \cap \Lambda_y = 0$. Moreover, S_f intersects (transversally) Λ_y only at the point $(y, \nabla^g f(y)/\|\nabla^g f(y)\|)$. Looking at the orientations we get that $\int_N [S_f] \wedge [\Lambda_y] = 1$ (this follows from (5.2.1) and the fact that Λ_y is oriented according to ψ). Finally the wavefront set condition follows from the transversality of the intersection, and the lemma follows. □

Before proving Theorem 5.1.2, we state a result about regularization of currents ; this is a version of the de Rham regularization procedure (see [?, §15, Proposition 1]) which takes into account the wavefront sets.

Lemma 5.5.3. *There are operators*

$$R_\varepsilon : \mathcal{D}'^\bullet(N) \to \Omega^\bullet(N), \quad A_\varepsilon : \mathcal{D}'^\bullet(N) \to \mathcal{D}'^\bullet(N), \quad \varepsilon \in [0,1],$$

such that for any $u \in \mathcal{D}'^\bullet(N)$, the following holds.

(i) *We have the identities $R_\varepsilon - \mathrm{Id} = \mathrm{d}A_\varepsilon + A_\varepsilon \mathrm{d}$ and $[d, R_\varepsilon] = 0$;*

(ii) *The supports of $R_\varepsilon u$ and $A_\varepsilon u$ are contained in the $C\varepsilon-$neighborhood of the support of u for some $C > 0$ independent of ε ;*

(iii) *For any closed conical neighborhood Γ of $\mathrm{WF}(u)$ (i.e. $\mathrm{WF}(u) \subset \Gamma^\circ$), there is $\varepsilon_0 > 0$ such that $\mathrm{WF}(A_\varepsilon u) \subset \Gamma$ for each $\varepsilon \in [0, \varepsilon_0]$, and moreover the families $(A_\varepsilon u)_{\varepsilon \in [0,\varepsilon_0]}$ and $(R_\varepsilon u)_{\varepsilon \in [0,\varepsilon_0]}$ are bounded in $\mathcal{D}'^\bullet_\Gamma(N)$;*

(iv) *We have $R_\varepsilon u \to u$ in $\mathcal{D}'^\bullet(N)$ as $\varepsilon \to 0$.*

Proof. Let $X_1, \dots, X_n$ be vector fields on N generating TN everywhere, and denote the associated flows by $\varphi_{1,t}, \dots, \varphi_{n,t}$ for $t \in \mathbb{R}$. Let $\varepsilon > 0$ and $\chi \in C_c^\infty(\mathbb{R}^n, [0,1])$ such that $\chi = 1$ near 0 and $\int_{\mathbb{R}^n} \chi(\mathbf{t}) \mathrm{d}\mathbf{t} = 1$. For $u \in \mathcal{D}'^\bullet(N)$ we define

$$R_\varepsilon u = \int_{\mathbb{R}^n} \chi(\mathbf{t}) \varphi^*_{1,\varepsilon t_1} \cdots \varphi^*_{n,\varepsilon t_n} u \ \mathrm{d}\mathbf{t}.$$

3. Note that $\nabla^g f$ is actually inward pointing, but this is irrelevant since $\dim \Sigma = 2$.

For $z \in N$ and $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$ we will set $\Psi_{\varepsilon,z}(\mathbf{t}) = \Phi_{\varepsilon\mathbf{t}}(z)$ where

$$\Phi_{\mathbf{t}} = \varphi_{n,t_n} \circ \cdots \circ \varphi_{1,t_1}.$$

We claim that $R_\varepsilon u$ is smooth. Indeed, if u is a 0-form, then we have

$$R_\varepsilon u(z) = \langle \chi, \Psi^*_{\varepsilon,z} u \rangle, \quad z \in N,$$

where the pairing is taken in $\mathbb{R}^n$. Indeed, this formula is true for u smooth and thus it remains true for any distribution u by continuity of the pullback $\Psi^*_{\varepsilon,z} : \mathcal{D}'^{\bullet}(N) \to \mathcal{D}'^{\bullet}(\mathbb{R}^n)$ (this follows from [Hör90, Theorem 6.1.2] as $\Psi_{\varepsilon,z}$ is a submersion $\mathbb{R}^n \to N$ whenever $\varepsilon > 0$, since the vector fields X_j generate TN). In particular $R_\varepsilon u$ is smooth, because $\Psi_{\varepsilon,z}$ depends smoothly on the variable z. If $u \in \Omega^k(N)$, we write locally $u = \sum_\ell u_\ell e_\ell$ for some basis (e_ℓ) of $\wedge^k T^*N$; writing $\Phi^*_{\varepsilon\mathbf{t}} e_\ell = \sum_j \alpha_{\ell,j}(\mathbf{t}) e_j$ we get by what precedes

$$R_\varepsilon u(z) = \sum_{\ell,j} \langle \alpha_{\ell,j}\chi, \Psi^*_{\varepsilon,z} u_\ell \rangle e_j(z),$$

and thus $R_\varepsilon u$ is smooth. It is immediate to see that $R_\varepsilon u \to u$ in the distributional sense as $\varepsilon \to 0$, which is point (iv). Next, note that

$$(R_\varepsilon - \mathrm{Id})u = \int_0^\varepsilon \partial_r \left(\int_{\mathbb{R}^n} \chi(\mathbf{t}) \Phi^*_{r\mathbf{t}} u \, \mathrm{d}\mathbf{t} \right) \mathrm{d}r.$$

By Cartan's formula one has $\partial_r \Phi^*_{r\mathbf{t}} = \mathrm{d}B_{r\mathbf{t}} + B_{r\mathbf{t}}\mathrm{d}$ where $B_{\mathbf{t}} : \mathcal{D}'^{\bullet}(N) \to \mathcal{D}'^{\bullet-1}(N)$ is defined by

$$B_{\mathbf{t}} = \sum_{j=1}^{n} \varphi^*_{1,t_1} \cdots \varphi^*_{j,t_j} \iota_{X_j} \varphi^*_{j+1,t_{j+1}} \cdots \varphi^*_{n,t_n}, \quad \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n.$$

Thus by setting $A_\varepsilon = \displaystyle\int_0^\varepsilon \int_{\mathbb{R}^n} \chi(\mathbf{t}) B_{r\mathbf{t}} \, \mathrm{d}\mathbf{t}\mathrm{d}r$ we obtain (i). Property (ii) is clear and thus it remains to show that (iii) holds. Let $u \in \mathcal{D}'^{\bullet}(N)$ and let Γ be a conical neighborhood of $\mathrm{WF}(u)$. Take $(z_0, \xi_0) \in \complement\Gamma$, and a conical neighborhood Γ_0 of ξ_0 such that $\overline{\Gamma}_0 \cap \Gamma = \emptyset$. Let $\varepsilon_0 > 0$ small enough so that $\Phi^*_{\varepsilon\mathbf{t}}(\Gamma_0) \cap \Gamma = \emptyset$ for any $\varepsilon \in [0, \varepsilon_0]$ and $\mathbf{t} \in \operatorname{supp}\chi$. Let $\omega \in \Omega^{\bullet}(N)$ be supported in a coordinate chart near z_0; we have for $\xi \in \Gamma_0$

$$\int_N \omega \mathrm{e}^{i\langle \xi, \cdot \rangle} \wedge A_\varepsilon u = \int_0^\varepsilon \int_{\mathbb{R}^n} \chi(\mathbf{t}) \left(\int_N \omega \mathrm{e}^{i\langle \xi, \cdot \rangle} \wedge B_{r\mathbf{t}} u \right) \mathrm{d}\mathbf{t}\mathrm{d}r.$$

Thanks to the expression of $B_{r\mathbf{t}}$ one can see that the right hand side can be written as

$$\int_0^\varepsilon \int_{\mathbb{R}^n} \chi(\mathbf{t}) \left(\int_N f(\mathbf{t}, r) \mathrm{e}^{i\langle \Xi(\mathbf{t},r), \cdot \rangle} \wedge u \right) \mathrm{d}\mathbf{t}\mathrm{d}r \tag{5.5.2}$$

where $f(\mathbf{t}, r)$ is a smooth function depending smoothly on $(\mathbf{t}, r)$ and $\Xi : \mathbb{R}^n \times [0, \varepsilon] \to T^*N$ is a smooth function satisfying $\Xi(\mathbf{t}, r) \in \Gamma_0$ on $\operatorname{supp}\chi$ and $|\Xi(\mathbf{t}, r)| \geqslant C|\xi|$. Since Γ does not intersect $\overline{\Gamma}_0$, the integral on N in (5.5.2) decays rapidly (i.e. faster than $\langle \xi \rangle^{-k}$ for any $k \geqslant 0$) as $\xi \to \infty$ and $\xi \in \Gamma_0$, with speed decay which is locally uniform with respect to $(\mathbf{t}, r) \in \operatorname{supp}\chi \times [0, \varepsilon]$. The result follows. □

Proof of Theorem 5.1.2. By (5.4.8) and Lemma 5.5.2, we have

$$\begin{aligned}\int_{M_\delta}[\Lambda_x]\wedge u_j &= \int_{M_\delta}[\Lambda_x]\wedge v_j\\ &= -\frac{1}{\chi(\Sigma)}\int_{M_\delta}[\bar\Lambda]\wedge v_j - \int_{M_\delta}\mathrm{d}[S]\wedge v_j\\ &= 0\end{aligned}$$

since $\mathrm{supp}(v_j)\subset\Gamma_+^\eta$ with $\Gamma_+^\eta\cap\bar\Lambda=\emptyset$ and $\mathrm{d}v_j=\mathrm{d}(u_j-\mathrm{d}f_j)=0$ by (5.4.7). This shows that $\eta_{x,y}(s)$ has no pole at $s=0$ and that $\eta_{x,y}(0)=-\langle[\Lambda_x],Y(0)\iota_X[\Lambda_y]\rangle$. Now since $Y(0)\iota_X[\Lambda_y]$ is compactly supported in M_δ° we may view this pairing as a pairing on N, so that

$$\eta_{x,y}(0)=-\int_N[\Lambda_x]\wedge Y(0)\iota_X[\Lambda_y].$$

From (8.2.9) we deduce that $\mathrm{d}Y(0)\iota_X[\Lambda_y]=[\Lambda_y]+u$ for some current u supported far from $M_{\delta/2}$. Let $\varepsilon>0$ small. As $\mathrm{d}[\Lambda_x]=0$, we have by Lemma 5.5.3 that $[\Lambda_x]=R_\varepsilon[\Lambda_x]-\mathrm{d}A_\varepsilon[\Lambda_x]$, with $\mathrm{WF}(A_\varepsilon[\Lambda_x])$ close to $\mathrm{WF}([\Lambda_x])$; thus we may compute

$$\begin{aligned}\int_N[\Lambda_x]\wedge Y(0)\iota_X[\Lambda_y] &= \int_N R_\varepsilon[\Lambda_x]\wedge Y(0)\iota_X[\Lambda_y]-\int_N \mathrm{d}A_\varepsilon[\Lambda_x]\wedge([\Lambda_y]+u)\\ &= -\int_N \mathrm{d}R_\varepsilon[S]\wedge Y(0)\iota_X[\Lambda_y]-\frac{1}{\chi(\Sigma)}\int_N R_\varepsilon[\bar\Lambda]\wedge Y(0)\iota_X[\Lambda_y]\\ &\quad -\int_N \mathrm{d}A_\varepsilon[\Lambda_x]\wedge([\Lambda_y]+u),\end{aligned}$$

where we used Lemma 5.5.2 in the last equality. By point (ii) of Lemma 5.5.3, the second integral vanishes for small ε since $\bar\Lambda\cap\mathrm{supp}(Y(0)\iota_X[\Lambda_y])=\emptyset$; the third one also vanishes to zero as $\mathrm{supp}([\Lambda_x])\cap\mathrm{supp}([\Lambda_y])=\emptyset$. Finally the first one writes

$$\int_N R_\varepsilon[S]\wedge([\Lambda_y]+u),$$

and thus it converges to $1/\chi(\Sigma)$ as $\varepsilon\to 0$ thanks to the second equation of (5.5.1) and points (ii), (iii) and (iv) of Lemma 5.5.3 (since $\mathrm{supp}([S])\cap\mathrm{supp}(u)=\emptyset$). This concludes the proof of Theorem 5.1.2. □

Chapitre 6

Torsion dynamique pour les flots de contact hyperboliques

Dans ce chapitre, on introduit la *torsion dynamique* associé à une paire (ϑ, ρ), où ϑ est une forme de contact sur une variété fermée M dont le champ de Reeb induit un flot d'Anosov (φ_t) et ρ est une représentation du groupe fondamental de M. Cet objet est défini comme le produit entre la valeur renormalisée de la fonction zêta de Ruelle de (φ_t) à l'origine et la torsion du complexe de dimension finie des états résonants de Pollicott-Ruelle pour la résonance zéro. Nous montrons que la torsion dynamique est invariante par perturbations de la forme de contact, et qu'elle se comporte comme la torsion de Turaev sur l'espace des représentations. Ce chapitre reproduit l'article *Dynamical torsion for contact Anosov flows* [CD19] écrit en collaboration avec Nguyen Viet Dang.

Sommaire

6.1 Introduction

In this chapter, we prove the results regarding the dynamical torsion announced in the introduction of this thesis (see §2.2.2.2). Let M be a closed, oriented n-dimensional manifold, with n odd. Let (E, ∇) be a flat vector bundle over M. Then ∇ induces a differential

$$\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E), \quad \nabla^2 = 0,$$

where $\Omega^\bullet(M, E)$ is the space of E-valued differential forms on M. Recall that ∇ will be called *acyclic* if the associated de Rham cohomology groups $H^\bullet(\nabla) = \ker(\nabla)/\mathrm{im}(\nabla)$ are trivial.

We assume that there is a contact form $\vartheta \in \Omega^1(M)$ such that its associated Reeb vector field $X = X_\vartheta$ has the Anosov property, and we denote by

$$\mathcal{L}_X^\nabla = \iota_X \nabla + \nabla \iota_X \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$$

the Lie derivative in the X direction twisted by ∇. In §6.4, we will introduce a chirality operator associated to the contact form ϑ,

$$\Gamma_\vartheta : \Omega^\bullet(M, E) \to \Omega^{n-\bullet}(M, E), \quad \Gamma_\vartheta^2 = \mathrm{Id},$$

analogous to the usual Hodge star operator associated to a Riemannian metric, such that

$$\Gamma_\vartheta \mathcal{L}_X^\nabla = \mathcal{L}_X^\nabla \Gamma_\vartheta.$$

For $\mathrm{Re}(s)$ large, we let

$$\zeta_{X,\nabla}(s) = \prod_\gamma \det\left(1 - \rho([\gamma]) \mathrm{e}^{-s\tau(\gamma)}\right),$$

be the twisted Ruelle zeta function of the pair (X, ∇), where the product runs over all primitive periodic orbits of the flow generated by X and where $\tau(\gamma)$ is the period of γ (cf. §6.3.5). Recall from Chapter 2 that this zeta function has a meromorphic extension to the whole complex plane.

Let $C^\bullet \subset \mathcal{D}'^\bullet(M, E)$ be the finite dimensional space of Pollicott-Ruelle generalized resonant states of $\mathcal{L}_X^\nabla$ for the resonance 0, that is,

$$C^\bullet = \left\{ u \in \mathcal{D}'^\bullet(M, E),\ \mathrm{WF}(u) \subset E_u^*,\ \exists N \in \mathbb{N},\ \left(\mathcal{L}_X^\nabla\right)^N u = 0 \right\},$$

where WF is the Hörmander wavefront set, $E_u^* \subset T^*M$ is the unstable cobundle of X [1], cf. §6.3, and $\mathcal{D}'(M,E)$ denotes the space of E-valued currents. Since ∇ commutes with $\mathcal{L}_X^\nabla$, it induces a differential $\nabla : C^\bullet \to C^{\bullet+1}$. Then a result of Dang–Rivière [DR19b] implies that the complex $(C^\bullet, \nabla)$ is acyclic if we assume that ∇ is. Because Γ_ϑ commutes with $\mathcal{L}_X^\nabla$, it induces a chirality operator on $C^\bullet$. Therefore we can compute the torsion $\tau(C^\bullet, \Gamma_\vartheta)$ of the finite dimensional complex $(C^\bullet, \nabla)$ with respect to Γ_ϑ, as described in [BK07c] (see §6.2).

Then we define the *dynamical torsion* τ_ϑ as the product

$$\tau_\vartheta(\nabla)^{(-1)^q} = \pm \underbrace{\tau(C^\bullet, \Gamma_\vartheta)^{(-1)^q}}_{\text{finite dimensional torsion}} \times \underbrace{\lim_{s\to 0} s^{-m(X,\rho)}\zeta_{X,\nabla}(s)}_{\text{renormalized zeta function at } s=0} \in \mathbb{C}\setminus 0,$$

where the sign $\pm$ will be given later, $m(X,\nabla)$ is the order of $\zeta_{X,\nabla}(s)$ at $s=0$ and $q = \frac{\dim(M)-1}{2}$ is the dimension of the unstable bundle of X. Note that the order $m(X,\rho) \in \mathbb{Z}$ is a priori not stable under perturbations of (X,ρ), in fact both terms in the product may not be invariant under small changes of ϑ whereas the dynamical torsion τ_ϑ has interesting invariance properties as we will see below.

6.1.1 Main properties of the dynamical torsion

We recall here the results announced in §2.2.2.2. Denote by $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ the set of acyclic representations $\pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$ and by $\mathcal{A} \subset \mathcal{C}^\infty(M, TM)$ the space of contact forms on M whose Reeb vector field induces an Anosov flow. This is an open subset of the space of contact forms. For any $\vartheta \in \mathcal{A}$, we denote by X_ϑ its Reeb vector field. In the spirit of Ray–Singer's result on the invariance of the analytic torsion with respect to the Riemannian metric [RS71], our first result shows $\tau_\vartheta(\rho)$ is invariant by small perturbations of the contact form $\vartheta \in \mathcal{A}$. Here, for any representation ρ, the number $\tau_\vartheta(\rho)$ is by definition $\tau_\vartheta(\nabla_\rho)$, where (E_ρ, ∇_ρ) is any flat vector bundle whose holonomy is given by ρ.

Theorem 6.1.1 (Local invariance of the dynamical torsion)**.** *Let (M,ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Let $(\vartheta_\tau)_{\tau\in(-\varepsilon,\varepsilon)}$ be a smooth family in $\mathcal{A}$. Then $\partial_\tau \log \tau_{\vartheta_\tau}(\rho) = 0$ for any $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d)$.*

Remark 6.1.2. In the case where the representation ρ is not acyclic, we can still define $\tau_\vartheta(\rho)$ as an element of the determinant line $\det H^\bullet(M,\rho)$ and this element is invariant under perturbations of $\vartheta \in \mathcal{A}$, cf. Remarks 6.4.5 and 6.5.2.

Our second result aims to compare τ_ϑ with Turaev's refined version of the Reidemeister torsion $\tau_{\mathfrak{e},\mathfrak{o}}$, which depends on some choice of Euler structure $\mathfrak{e}$ and orientation $\mathfrak{o}$ (see §6.7.2 for a detailed exposition of these notions).

Theorem 6.1.3. *Let (M,ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Then $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d) \mapsto \tau_\vartheta(\rho)$ is holomorphic* [2] *and*

1. That is, E_u^* is the annihilator of $E_u \oplus \mathbb{R}X$ where $E_u \subset TM$ denotes the unstable bundle of the flow.

2. $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ is a variety over $\mathbb{C}$, see subsection 6.9.2 for the right notion of holomorphicity.

there exists an Euler structure $\mathfrak{e}$ such that for any cohomological orientation $\mathfrak{o}$ and any smooth family $(\rho_u)_{u\in(-\varepsilon,\varepsilon)}$ of $\mathrm{Rep}_{\mathrm{ac}}(M,d)$,

$$\partial_u \log \tau_\vartheta(\rho_u) = \partial_u \log \tau_{\mathfrak{e},\mathfrak{o}}(\rho_u)$$

Moreover, if $\dim M = 3$ and $b_1(M) \neq 0$, the map $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is of modulus one on the connected components of $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ containing an acyclic and unitary representation.

Finally, our third result aims to describe how $\partial_u \log \tau_\vartheta(\rho_u)$ depends on the choice of the contact Anosov vector field X_ϑ.

Theorem 6.1.4. *Let (M,ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Let $(\rho_u)_{|u|\leqslant\varepsilon}$ be a smooth family in $\mathrm{Rep}_{\mathrm{ac}}(M,d)$. Then for any $\eta \in \mathcal{A}$*

$$\partial_u \log \tau_\eta(\rho_u) = \partial_u \log \tau_\vartheta(\rho_u) + \partial_u \log \underbrace{\det \langle \rho_u, \mathrm{cs}(X_\vartheta, X_\eta)\rangle}_{\text{topological}}$$

where $\mathrm{cs}(X_\vartheta, X_\eta) \in H_1(M,\mathbb{Z})$ is the Chern-Simons class of the pair of vector fields (X_ϑ, X_η).

The Chern–Simons class $\mathrm{cs}(X_\vartheta, X_\eta) \in H_1(M,\mathbb{Z})$ measures the obstruction to find a homotopy among non singular vector fields connecting X_ϑ and X_η (see §6.7.1).

Because the dynamical torsion is constructed with the help of the dynamical zeta function $\zeta_{X,\rho}$, we deduce from the above theorem some informations about the behavior of $\zeta_{X,\rho}(s)$ near $s=0$, as follows.

Corollary 6.1.5. *Let M be a closed odd dimensional manifold. Then for every connected open subsets $\mathcal{U} \subset \mathrm{Rep}_{\mathrm{ac}}(M,d)$ and $\mathcal{V} \subset \mathcal{A}$, there exists a constant C such that for every Anosov contact form $\vartheta \in \mathcal{V}$ and every representation $\rho \in \mathcal{U}$,*

$$\zeta_{X_\vartheta,\rho}(s)^{(-1)^q} = C s^{(-1)^q m(\rho, X_\vartheta)} \frac{\tau_{\mathfrak{e}_{X_\vartheta},\mathfrak{o}}(\rho)}{\tau\left(C^\bullet(\vartheta,\rho), \Gamma_\vartheta\right)} \left(1 + \mathcal{O}(s)\right), \tag{6.1.1}$$

where X_ϑ is the Reeb vector field of ϑ, (E_ρ, ∇_ρ) is the flat vector bundle over M induced by ρ, $C^\bullet(\vartheta,\rho) \subset \mathcal{D}'^\bullet(M,E_\rho)$ is the space of generalized resonant states for the resonance 0 of $\mathcal{L}^{\nabla_\rho}_{X_\vartheta}$ and $m(X_\vartheta,\rho)$ is the vanishing order of $\zeta_{X_\vartheta,\rho}(s)$ at $s=0$.

6.1.2 Methods of proof

Let us briefly sketch the proofs of Theorems 6.1.1 and 6.1.3 which rely essentially on two variational arguments : we compute the variation of $\tau_\vartheta(\nabla)$ when we perturb the contact form ϑ and the connection ∇. As we do so, the space $C^\bullet(\vartheta,\nabla)$ of Pollicott-Ruelle resonant states of $\mathcal{L}^\nabla_{X_\vartheta}$ for the resonance 0 may radically change. Therefore, it is convenient to consider the space $C^\bullet_{[0,\lambda]}(\vartheta,\nabla)$ instead, which consists of the generalized resonant states for $\mathcal{L}^\nabla_{X_\vartheta}$ for resonances s such that $|s| \leqslant \lambda$, where $\lambda \in (0,1)$ is chosen so that $\{|s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}^\nabla_{X_\vartheta}) = \emptyset$. Then using [BK07c, Proposition 5.6] and multiplicativity of torsion, one can show that

$$\tau_\vartheta(\nabla) = \pm\tau\left(C^\bullet_{[0,\lambda]}(\vartheta,\nabla), \Gamma_\vartheta\right) \zeta^{(\lambda,\infty)}_{X_\vartheta,\rho}(0)^{(-1)^q}, \tag{6.1.2}$$

where $\zeta_{X_\vartheta,\rho}^{(\lambda,\infty)}$ is a renormalized version of $\zeta_{X_\vartheta,\rho}$ (we remove all the poles and zeros of $\zeta_{X_\vartheta,\rho}$ within $\{s \in \mathbb{C},\ |s| \leq \lambda\}$), see §6.4. Thus we can work with the space $C^\bullet_{[0,\lambda]}(\vartheta, \nabla)$, which behaves nicely under perturbations of X thanks to Bonthonneau's construction of uniform anisotropic Sobolev spaces for families of Anosov flows [Bon20], and also under perturbations of ∇.

Now consider a smooth family of contact forms $(\vartheta_t)_t$ for $|t| < \varepsilon$ such that their Reeb vector fields $(X_t)_t$ induce Anosov flows. Then Theorem 6.5.1 says that for any acyclic ∇, the map $t \mapsto \tau_{\vartheta_t}(\nabla)$ is differentiable and its derivative vanishes. This follows from a computation, using a result of [BK07c] about the variation of the torsion of a finite dimensional complex when the chirality operator is perturbed, and on a variation formula of the map $t \mapsto \zeta_{X_t,\rho}(s)$ for $\mathrm{Re}(s)$ big enough obtained in [DGRS18].

Next, consider a smooth family of flat connections $z \mapsto \nabla(z)$, where z is a complex number varying in a small neighborhood of the origin and write $\nabla(z) = \nabla + z\alpha + o(z)$ where $\alpha \in \Omega^1(M, \mathrm{End}(E))$. Then we show in §6.6, in the same spirit as before, that $z \mapsto \tau_\vartheta(\nabla(z))$ is complex differentiable and its logarithmic derivative reads

$$\partial_z|_{z=0} \log \tau_\vartheta(\nabla(z)) = -\operatorname{tr}_{\mathrm{s}}^\flat \alpha K \mathrm{e}^{-\varepsilon \mathcal{L}^\nabla_{X_\vartheta}},$$

where $\varepsilon > 0$ is small enough, $\operatorname{tr}_{\mathrm{s}}^\flat$ is the super flat trace, cf. §B.3.1, and $K : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ is a cochain contraction, that is, it satisfies $\nabla K + K \nabla = \mathrm{Id}_{\Omega^\bullet(M,E)}$. On the other hand, we can compute, using the formalism of [?],

$$\partial_z|_{z=0} \log \tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\nabla(z)) = -\operatorname{tr}_{\mathrm{s}}^\flat \alpha \widetilde{K} \mathrm{e}^{-\varepsilon \mathcal{L}^\nabla_{-\widetilde{X}}} - \int_e \operatorname{tr} \alpha,$$

where $\mathfrak{e}_\vartheta$ is an Euler structure canonically associated to ϑ, $\widetilde{K}$ is another cochain contraction, $\widetilde{X}$ is a Morse-Smale gradient vector field and $e \in C_1(M, \mathbb{Z})$ is a singular one-chain representing the Euler structure $\mathfrak{e}_\vartheta$, cf. §6.7. Now using the fact that K and $\widetilde{K}$ are cochain contractions, one can see that

$$\alpha \left(K \mathrm{e}^{-\varepsilon \mathcal{L}^\nabla_{X_\vartheta}} - \widetilde{K} \mathrm{e}^{-\varepsilon \mathcal{L}^\nabla_{\widetilde{X}}} \right) = \alpha R_\varepsilon + [\nabla, \alpha G_\varepsilon],$$

where R_ε is an operator of degree -1 whose kernel is, roughly speaking, the union of graphs of the maps $\mathrm{e}^{-\varepsilon X_u}$, where $(X_u)_u$ is a non-degenerate family of vector fields interpolating X_ϑ and $\widetilde{X}$, cf. §6.7.3, and G_ε is some operator of degree -2. Therefore we obtain by cyclicity of the flat trace

$$\partial_z|_{z=0} \log \frac{\tau_\vartheta(\nabla(z))}{\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\nabla(z))} = \operatorname{tr}_{\mathrm{s}}^\flat \alpha R_\varepsilon - \int_e \operatorname{tr} \alpha = 0, \tag{6.1.3}$$

where the last equality comes from differential topology arguments. Using the analytical structure of the representation variety, we may deduce from (6.1.3) the claim of Theorem 6.1.3. Theorem 6.1.4 then follows from the invariance of the dynamical torsion under small perturbations of the flow, the fact that $\tau_{\mathfrak{e},\mathfrak{o}}(\rho) = \tau_{\mathfrak{e}',\mathfrak{o}}(\rho)\langle \det \rho, h \rangle$ for any other Euler structure $\mathfrak{e}'$, where $h \in H_1(M, \mathbb{Z})$ satisfies $\mathfrak{e} = \mathfrak{e}' + h$ (we have that $H_1(M, \mathbb{Z})$ acts freely and transitively on the set of Euler structures, cf. §6.7), and the fact that, in our notations, $\mathfrak{e}_\eta - \mathfrak{e}_\vartheta = \mathrm{cs}(X_\vartheta, X_\eta)$ for any other contact form η.

6.1.3 Plan of the chapter.

This chapter is organized as follows. In §6.2, we give some preliminaries about torsion of finite dimensional complexes computed with respect to a chirality operator. In §6.3, we introduce Pollicott-Ruelle resonances. In §6.4, we compute the refined torsion of a space of generalized eigenvectors for nonzero resonances and we define the dynamical torsion. In §6.5, we prove that our torsion is unsensitive to small perturbations of the dynamics. In §6.6, we compute the variation of our torsion with respect to the connection. In §6.7, we introduce Euler structures which are some topological tools used to fix ambiguities of the refined torsion. In §6.8, we introduce the refined combinatorial torsion of Turaev using Morse theory and we compute its variation with respect to the connection. We finally compare it to the dynamical torsion in §6.9.

6.2 Torsion of finite dimensional complexes

We recall the definition of the refined torsion of a finite dimensional acyclic complex computed with respect to a chirality operator, following [BK07c]. Then we compute the variation of the torsion of such a complex when the differential is perturbed.

6.2.1 The determinant line of a complex

For a non zero complex vector space V, the determinant line of V is the line defined by $\det(V) = \wedge^{\dim V} V$. We declare the determinant line of the trivial vector space $\{0\}$ to be $\mathbb{C}$. If L is a 1-dimensional vector space, we will denote by L^{-1} its dual line. Any basis $(v_1, \dots, v_n)$ of V defines a nonzero element $v_1 \wedge \dots \wedge v_n \in \det(V)$. Thus elements of the determinant line of $\det(V)$ should be thought of as equivalence classes of oriented basis of V.

Let

$$(C^\bullet, \partial) : 0 \xrightarrow{\partial} C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^n \xrightarrow{\partial} 0$$

be a finite dimensional complex, i.e. $\dim C^j < \infty$ for all $j = 0, \dots, n$. We define the *determinant line* of the complex $C^\bullet$ by

$$\det(C^\bullet) = \bigotimes_{j=0}^{n} \det(C^j)^{(-1)^j}.$$

Let $H^\bullet(\partial)$ be the cohomology of $(C^\bullet, \partial)$, that is

$$H^\bullet(\partial) = \bigoplus_{j=0}^{n} H^j(\partial), \quad H^j(\partial) = \frac{\ker(\partial : C^j \to C^{j+1})}{\operatorname{ran}(\partial : C^{j-1} \to C^j)}.$$

We will say that the complex $(C^\bullet, \partial)$ is acyclic if $H^\bullet(\partial) = 0$. In that case, $\det H^\bullet(\partial)$ is canonically isomorphic to $\mathbb{C}$.

It remains to define the fusion homomorphism that we will later need to define the torsion of a finite dimensional based complex [FT00, §2.3]. For any finite dimensional vector spaces $V_1, \dots, V_r$, we have a fusion isomorphism

$$\mu_{V_1,\dots,V_r} : \det(V_1) \otimes \cdots \otimes \det(V_r) \to \det(V_1 \oplus \cdots \oplus V_r)$$

defined by

$$\mu_{V_1,\dots,V_r}\Big(v_1^1\wedge\cdots\wedge v_1^{m_1}\otimes\cdots\otimes v_r^1\wedge\cdots\wedge v_r^{m_r}\Big)=v_1^1\wedge\cdots\wedge v_1^{m_1}\wedge\cdots\wedge v_r^1\wedge\cdots\wedge v_r^{m_r},$$

where $m_j=\dim V_j$ for $j\in\{1,\dots,r\}$.

6.2.2 Torsion of finite dimensional acyclic complexes.

In the present paper, we want to think of torsion of finite dimensional acyclic complexes as a map $\varphi_{C^\bullet}$ from the determinant line of the complex to $\mathbb{C}$. We have a canonical isomorphism

$$\varphi_{C^\bullet}:\det(C^\bullet)\xrightarrow{\sim}\mathbb{C}, \tag{6.2.1}$$

defined as follows. Fix a decomposition

$$C^j=B^j\oplus A^j,\quad j=0,\dots,n,$$

with $B^j=\ker(\partial)\cap C^j$ and $B^j=\partial(A^{j-1})=\partial(C^{j-1})$ for every j. Then $\partial|_{A^j}:A^j\to B^{j+1}$ is an isomorphism for every j.

Fix non zero elements $c_j\in\det C^j$ and $a_j\in\det A^j$ for any j. Let $\partial(a_j)\in\det B^{j+1}$ denote the image of a_j under the isomorphism $\det A^j\to\det B^{j+1}$ induced by the isomorphism $\partial|_{A^j}:A^j\to B^{j+1}$. Then for each $j=0,\dots,n$, there exists a unique $\lambda_j\in\mathbb{C}$ such that

$$c_j=\lambda_j\mu_{B^j,A^j}\Big(\partial(a_{j-1})\otimes a_j\Big),$$

where μ_{B^j,A^j} is the fusion isomorphism defined in §6.2.1. Then define the isomorphism $\varphi_{C^\bullet}$ by

$$\varphi_{C^\bullet}\ :\ c_0\otimes c_1^{-1}\otimes\cdots\otimes c_n^{(-1)^n}\mapsto(-1)^{N(C^\bullet)}\prod_{j=0}^{n}\lambda_j^{(-1)^j}\in\mathbb{C},$$

where

$$N(C^\bullet)=\frac{1}{2}\sum_{j=0}^{n}\dim A^j\left(\dim A^j+(-1)^{j+1}\right).$$

One easily shows that $\varphi_{C^\bullet}$ is independent of the choices of a_j [Tur01, Lemma 1.3]. The number $\tau(C^\bullet,c)=\varphi_{C^\bullet}(c)$ is called the *refined torsion* of $(C^\bullet,\partial)$ with respect to the element c.

The torsion will depend on the choices of $c_j\in\det C^j$. Here the sign convention (that is, the choice of the prefactor $(-1)^{N(C^\bullet)}$ in the definition of $\varphi_{C^\bullet}$) follows Braverman–Kappeler [BK07c, §2] and is consistent with Nicolaescu [Nic03, §1]. This prefactor was introduced by Turaev and differs from [Tur86]. See [Nic03] for the motivation for the choice of sign.

Remark 6.2.1. If the complex $(C^\bullet,\partial)$ is not acyclic, we can still define a torsion $\tau(C^\bullet,c)$, which is this time an element of the determinant line $\det H^\bullet(\partial)$, cf. [BK07c, §2.4].

6.2.3 Torsion with respect to a chirality operator

We saw above that torsion depends on the choice of an element of the determinant line. A way to fix the value of the torsion without choosing an explicit basis is to use a chirality operator as in [BK07c]. Take $n = 2r + 1$ an odd integer and consider a complex $(C^\bullet, \partial)$ of length n. We will call a *chirality* operator an operator $\Gamma : C^\bullet \to C^\bullet$ such that $\Gamma^2 = \mathrm{Id}_{C^\bullet}$, and

$$\Gamma(C^j) = C^{n-j}, \quad j = 0, \dots, n.$$

Γ induces isomorphisms $\det(C^j) \to \det(C^{n-j})$ that we will still denote by Γ. If $\ell \in L$ is a non zero element of a complex line, we will denote by $\ell^{-1} \in L^{-1}$ the unique element such that $\ell^{-1}(\ell) = 1$. Fix non zero elements $c_j \in \det(C^j)$ for $j \in \{0, \dots, r\}$ and define

$$c_\Gamma = (-1)^{m(C^\bullet)} c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_r^{(-1)^r} \otimes (\Gamma c_r)^{(-1)^{r+1}} \otimes (\Gamma c_{r-1})^{(-1)^r} \otimes \cdots \otimes (\Gamma c_0)^{-1},$$

where

$$m(C^\bullet) = \frac{1}{2} \sum_{j=0}^{r} \dim C^j \left(\dim C^j + (-1)^{r+j} \right).$$

Definition 6.2.2. The element c_Γ is independent of the choices of c_j for $j \in \{0, \dots, r\}$; the *refined torsion of* $(C^\bullet, \partial)$ *with respect to* Γ is the element

$$\tau(C^\bullet, \Gamma) = \tau(C^\bullet, c_\Gamma).$$

We also have the following result which is [BK07c, Lemma 4.7] in the acyclic case about the multiplicativity of torsion.

Proposition 6.2.3. *Let* $(C^\bullet, \partial)$ *and* $(\tilde{C}^\bullet, \tilde{\partial})$ *be two acyclic complexes of same length endowed with two chirality operators* Γ *and* $\tilde{\Gamma}$. *Then*

$$\tau(C^\bullet \oplus \tilde{C}^\bullet, \Gamma \oplus \tilde{\Gamma}) = \tau(C^\bullet, \Gamma)\tau(\tilde{C}^\bullet, \tilde{\Gamma}).$$

6.2.4 Computation of the torsion with the contact signature operator

Let

$$B = \Gamma\partial + \partial\Gamma : C^\bullet \to C^\bullet.$$

B is called the *signature operator*. Let $B_+ = \Gamma\partial$ and $B_- = \partial\Gamma$. Denote

$$C^j_\pm = C^j \cap \ker(B_\mp), \quad j = 0, \dots, n.$$

We have that $B_\pm$ preserves $C^\bullet_\pm$. Note that $B_+(C^j_+) \subset C^{n-j-1}_+$, so that $B_+(C^j_+ \oplus C^{n-j-1}_+) \subset C^j_+ \oplus C^{n-j-1}_+$. Note that if B is invertible on $C^\bullet$, B_+ is invertible on $C^\bullet_+$. If B is invertible, we can compute the refined torsion of $(C^\bullet, \partial)$ using the following

Proposition 6.2.4. *[BK07c, Proposition 5.6] Assume that* B *is invertible. Then* $(C^\bullet, \partial)$ *is acyclic so that* $\det(H^\bullet(\partial))$ *is canonically isomorphic to* $\mathbb{C}$. *Moreover,*

$$\tau(C^\bullet, \Gamma) = (-1)^{r \dim C^r_+} \det\left(\Gamma\partial|_{C^r_+}\right)^{(-1)^r} \prod_{j=0}^{r-1} \det\left(\Gamma\partial|_{C^j_+ \oplus C^{n-j-1}_+}\right)^{(-1)^j}.$$

6.2.5 Super traces and determinants

Let $V^\bullet = \bigoplus_{j=0}^p V^j$ is a graded finite dimensional vector space and $A : V^\bullet \to V^\bullet$ be a degree preserving linear map. We define the *super trace* and the *super determinant* of A by

$$\mathrm{tr}_{\mathrm{s},V^\bullet} A = \sum_{j=0}^p (-1)^j \,\mathrm{tr}_{V^j} A, \qquad \det{}_{\mathrm{s},V^\bullet} A = \prod_{j=0}^p (\det{}_{V^j} A)^{(-1)^j}.$$

We also define the *graded trace* and the *graded determinant* of A by

$$\mathrm{tr}_{\mathrm{gr},V^\bullet} A = \sum_{j=0}^p (-1)^j j \,\mathrm{tr}_{V^j} A, \qquad \det{}_{\mathrm{gr},V^\bullet} A = \prod_{j=0}^p (\det{}_{V^j} A)^{(-1)^j j}.$$

6.2.6 Analytic families of differentials

The goal of the present subsection is to give a variation formula for the torsion of a finite dimensional complex when we vary the differential. This formula plays a crucial role in the variation formula of the dynamical torsion, when the representation is perturbed. Indeed, we split the dynamical torsion as the product of the torsion $\tau\left(C^\bullet(\vartheta, \rho), \Gamma_\vartheta\right)$ of some finite dimensional space of Ruelle resonant states and a renormalized value at $s = 0$ of the dynamical zeta function $\zeta_{X,\rho}(s)$. Then the following formula allows us to deal with the variation of $\tau\left(C^\bullet(\vartheta, \rho), \Gamma_\vartheta\right)$.

Let $(C^\bullet, \partial)$ be an acyclic finite dimensional complex of finite odd length n. If $S : C^\bullet : C^\bullet$ is a linear operator, we will say that it is of degree s if $S(C^k) \subset C^{k+s}$ for any k. If S and T are two operators on $C^\bullet$ of degrees s et t respectively then the supercommutator of S and T by

$$[S, T] = ST - (-1)^{st} TS.$$

Cyclicity of the usual trace gives $\mathrm{tr}_{\mathrm{s},C^\bullet}[S, T] = 0$ for any S, T.

Let U be a neighborhood of the origin in the complex plane and $\partial(z)$, $z \in U$, be a family of acyclic differentials on $C^\bullet$ which is complex differentiable at $z = 0$, that is,

$$\partial(z) = \partial + za + o(z) \tag{6.2.2}$$

for some operator $a : C^\bullet \to C^\bullet$ of degree 1. Note that $\partial(z) \circ \partial(z) = 0$ implies that the supercommutator

$$[\partial, a] = \partial a + a\partial = 0. \tag{6.2.3}$$

We will denote by $C^\bullet(z)$ the complex $(C^\bullet, \partial(z))$. Finally let $k : C^\bullet \to C^\bullet$ be a cochain contraction, that is a linear map of degree 1 such that

$$\partial k + k\partial = \mathrm{Id}_{C^\bullet}. \tag{6.2.4}$$

The existence of such map is ensured by the acyclicity of $(C^\bullet, \partial)$.

Lemma 6.2.5. *In the above notations, for any chirality operator Γ on $C^\bullet$, the map $z \mapsto \tau(C^\bullet(z), \Gamma)$ is complex differentiable at $z = 0$ and*

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=0} \log \tau(C^\bullet(z), \Gamma) = -\mathrm{tr}_{\mathrm{s},C^\bullet}(ak).$$

Note that this implies in particular that $\mathrm{tr}_{s,C^\bullet}(ak)$ does not depend on the chosen cochain contraction k. This is expected since if k' is another cochain contraction,

$$[\partial, akk'] = \partial akk' + akk'\partial = a(k - k')$$

by (6.2.3), and the supertrace of a supercommutator vanishes.

Proof. First note that for non zero elements $c, c' \in \det C^\bullet$, we have

$$\tau(C^\bullet(z), c) = [c : c'] \cdot \tau(C^\bullet(z), c'), \tag{6.2.5}$$

where $[c : c'] \in \mathbb{C}$ satisfies $c = [c : c'] \cdot c'$.

For every $j = 0, \dots, n$, fix a decomposition

$$C^j = A^j \oplus B^j,$$

where $B^j = \ker \partial \cap C^j$ and A^j is any complementary of B^j in C^j. Fix some basis $a_j^1, \dots, a_j^{\ell_j}$ of A^j; then $\partial a_j^1, \dots, \partial a_j^{\ell_j}$ is a basis of B^{j+1} by acyclicity of $(C^\bullet, \partial)$. Now let

$$c_j = a_j^1 \wedge \cdots \wedge a_j^{\ell_j} \wedge \partial a_{j-1}^1 \wedge \cdots \wedge \partial a_{j-1}^{\ell_{j-1}} \in \det C^j,$$

and

$$c = c_0 \otimes (c_1)^{-1} \otimes c_2 \otimes \cdots \otimes (c_n)^{(-1)^n} \in \det C^\bullet.$$

Now by definition of the refined torsion, we have for $|z|$ small enough

$$\tau(C^\bullet(z), c) = \pm \prod_{j=0}^{n} \det\big(A_j(z)\big)^{(-1)^{j+1}} \tag{6.2.6}$$

where the sign $\pm$ is independent of z and $A_j(z)$ is the matrix sending the basis

$$a_j^1, \dots, a_j^{\ell_j}, \partial a_{j-1}^1, \dots, \partial a_{j-1}^{\ell_{j-1}}$$

to the basis

$$a_j^1, \dots, a_j^{\ell_j}, \partial(z) a_{j-1}^1, \dots, \partial(z) a_{j-1}^{\ell_{j-1}}$$

(which is indeed a basis of C^j for $|z|$ small enough). Let $k : C^\bullet \to C^\bullet$ of degree -1 defined by

$$k\partial a_j^m = a_j^m, \quad k a_j^m = 0,$$

for every j and $m \in \{0, \dots, \ell_j\}$. Then $k\partial + \partial k = \mathrm{Id}_{C^\bullet}$ and

$$\det A_j(z) = \det_{\partial B^{j-1} \oplus B^j}\big(\partial(z) k \oplus \mathrm{Id}\big).$$

Now (6.2.2) and (6.2.6) imply the desired result, because $\tau(C^\bullet(z), \Gamma) = [c_\Gamma : c] \cdot \tau(C^\bullet(z), c)$ by (6.2.5). □

6.3 Geometrical and dynamical preliminaries

In this section, we introduce our geometrical and dynamical setting. We will adopt the formalism of Harvey–Polking [HP+79] about currents which will be convenient to compute flat traces and relate the variation of the Ruelle zeta function with topological objects.

6.3.1 Notations

Let M be an oriented closed connected manifold of odd dimension $n = 2r+1$. Let $(E, \nabla) \to M$ be a flat vector bundle over M of rank $d \geqslant 1$. We will take the notations of Appendix B ; in particular we will denote by

$$\Omega^k(M, E) = \mathcal{C}^\infty(M, \wedge^k \otimes E)$$

the space of E valued k-forms and by $\mathcal{D}'^k(M, \wedge^k \otimes E)$ the space of E-valued k-currents. Here we denoted the bundle $\wedge^k T^*M$ by $\wedge^k$ for simplicity. The space of differential forms is denoted by $\Omega^\bullet(M)$. We view the connection as a degree 1 operator (as an operator of the graded vector space $\Omega^\bullet(M, E)$)

$$\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E), \quad k = 0, \dots, n.$$

The flatness of the connection reads $\nabla^2 = 0$ and thus we obtain a cochain complex $\big(\Omega^\bullet(M, E), \nabla\big)$. We will assume that the connection ∇ is acyclic, that is, the complex $\big(\Omega^\bullet(M, E), \nabla\big)$ is acyclic, or equivalently, the cohomology groups

$$H^k(M, \nabla) = \frac{\Big\{u \in \Omega^k(M, E) \;:\; \nabla u = 0\Big\}}{\Big\{\nabla v \;:\; v \in \Omega^{k-1}(M, E)\Big\}}, \quad k = 0, \dots, n,$$

are trivial.

6.3.2 Anosov dynamics

Let X be a smooth vector field on M and denote by φ^t its flow. We will assume that X generates an Anosov flow, that is, there exists a splitting of the tangent space T_xM at every $x \in M$

$$T_xM = \mathbb{R}X(x) \oplus E_s(x) \oplus E_u(x),$$

where $E_u(x), E_s(x)$ are subspaces of T_xM depending continuously on x and invariant by the flot φ^t, such that for some constants $C, \nu > 0$ and some smooth metric $|\cdot|$ on TM one has

$$|(\mathrm{d}\varphi^t)_x v_s| \leqslant C\mathrm{e}^{-\nu t}|v_s|, \quad t \geq 0, \quad v_s \in E_s(x),$$
$$|(\mathrm{d}\varphi^t)_x v_u| \leqslant C\mathrm{e}^{-\nu |t|}|v_u|, \quad t \leq 0, \quad v_u \in E_u(x).$$

We will use the dual decomposition $T^*M = E_0^* \oplus E_u^* \oplus E_s^*$ where E_0^*, E_u^* and E_s^* are defined by

$$E_0^*(E_s \oplus E_u) = 0, \quad E_s^*(E_0 \oplus E_s) = 0, \quad E_u^*(E_0 \oplus E_u) = 0. \tag{6.3.1}$$

6.3.3 Pollicott-Ruelle resonances

Let ι_X denote the interior product with X and

$$\mathcal{L}_X^\nabla = \nabla \iota_X + \iota_X \nabla : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$$

be the Lie derivative along X acting on E-valued forms. Locally, the action of $\mathcal{L}_X^\nabla$ is given by the following. Take U a domain of a chart and write $\nabla = \mathrm{d} + A$ where

$A \in \Omega^1(M, \mathrm{End}(E))$. Take $w_1, \dots, w_\ell$ (resp. $e_1, \dots, e_d$) some local basis of $\wedge^k$ (resp. E) on U. Then for any $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant d$,

$$\mathcal{L}_X^\nabla(fw_i \otimes e_j) = (Xf)w_i \otimes e_j + f(\mathcal{L}_X w_i) \otimes e_j + fw_i \otimes A(X)e_j, \quad f \in \mathcal{C}^\infty(U),$$

where $\mathcal{L}_X$ is the standard Lie derivative acting on forms. In particular, $\mathcal{L}_X^\nabla$ is a differential operator of order 1 acting on sections of the bundle $\wedge^\bullet T^*M \otimes E$, whose principal part is diagonal and given by X.

Denote by Φ_k^t the induced flow on the vector bundle $\wedge^k T^*M \otimes E \to M$, that is,

$$\Phi_k^t(\beta \otimes v) = {}^T(\mathrm{d}\varphi^t)_x^{-1}\beta \otimes P_t^\nabla(x)v, \quad x \in M, \quad (\beta, v) \in \wedge^k(T_x^*M) \times E_x, \quad t \in \mathbb{R},$$

where $P_t^\nabla(x)$ is the parallel transport induced by ∇ along the curve $\{\varphi^s(x),\ s \in [0,t]\}$. This induces a map

$$\mathrm{e}^{t\mathcal{L}_X^\nabla} : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E).$$

For $\mathrm{Re}(s)$ big enough, the operator $\mathcal{L}_X^\nabla + s$ acting on $\Omega^\bullet(M, E)$ is invertible with inverse

$$(\mathcal{L}_X^\nabla + s)^{-1} = \int_0^\infty \mathrm{e}^{-t\mathcal{L}_X^\nabla}\mathrm{e}^{-st}\mathrm{d}t. \tag{6.3.2}$$

The results of [FS11] generalize to the flat bundle case as in [DR19b, §3] and the resolvent $\left(\mathcal{L}_X^\nabla + s\right)^{-1}$, viewed as a family of operators

$$\Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E),$$

admits a meromorphic continuation to $s \in \mathbb{C}$ with poles of finite multiplicites ; we will still denote by $\left(\mathcal{L}_X^\nabla + s\right)^{-1}$ this extension. Those poles are the *Pollicott-Ruelle resonances* of $\mathcal{L}_X^\nabla$, and we will denote this set by $\mathrm{Res}(\mathcal{L}_X^\nabla)$.

6.3.4 Generalized resonant states

Let $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. By [DZ16, Proposition 3.3] we have a Laurent expansion

$$\left(\mathcal{L}_X^\nabla + s\right)^{-1} = Y_{s_0}(s) + \sum_{j=1}^{J(s_0)} (-1)^{j-1}\frac{\left(\mathcal{L}_X^\nabla + s_0\right)^{j-1}\Pi_{s_0}}{(s - s_0)^j} \tag{6.3.3}$$

where $Y_{s_0}(s)$ is holomorphic near $s = s_0$, and

$$\Pi_{s_0} = \frac{1}{2\pi i}\int_{C_\varepsilon(s_0)} \left(\mathcal{L}_X^\nabla + s\right)^{-1}\mathrm{d}s : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E) \tag{6.3.4}$$

is an operator of finite rank. Here $C_\varepsilon(s_0) = \{|z - s_0| = \varepsilon\}$ with $\varepsilon > 0$ small enough is a small circle around s_0 such that $\mathrm{Res}(\mathcal{L}_X^\nabla) \cap \{|z - s_0| \leqslant \varepsilon\} = \{s_0\}$. Moreover the operators $Y_{s_0}(s)$ and Π_{s_0} extend to continuous operators

$$Y_{s_0}(s), \Pi_{s_0} : \mathcal{D}'^\bullet_{E_u^*}(M, E) \to \mathcal{D}'^\bullet_{E_u^*}(M, E). \tag{6.3.5}$$

The space

$$C^\bullet(s_0) = \mathrm{ran}(\Pi_{s_0}) \subset \mathcal{D}'^\bullet_{E_u^*}(M, E)$$

is called the space of generalized resonant states of $\mathcal{L}_X^\nabla$ associated to the resonance s_0.

6.3.5 The twisted Ruelle zeta function

Fix a base point $x_\star \in M$ and identify $\pi_1(M)$ with $\pi_1(M, x_\star)$. Let $\mathrm{Per}(X)$ be the set of periodic orbits of X. For every $\gamma \in \mathrm{Per}(X)$ we fix some base point $x_\gamma \in \mathrm{Im}(\gamma)$ and an arbitrary path c_γ joining x_γ to $x_\star$. This path defines an isomorphism $\psi_\gamma : \pi_1(M, x_\gamma) \cong \pi_1(M)$ and we can thus define every $\gamma \in \mathrm{Per}(X)$

$$\rho_\nabla([\gamma]) = \rho_\nabla(\psi_\gamma[\gamma]).$$

The *twisted Ruelle zeta function* associated to the pair (X, ∇) is defined by

$$\zeta_{X,\nabla}(s) = \prod_{\gamma \in \mathcal{G}_X} \det\left(\mathrm{Id} - \rho_\nabla([\gamma]) \mathrm{e}^{-s\tau(\gamma)}\right), \quad \mathrm{Re}(s) > C, \tag{6.3.6}$$

where $\mathcal{G}_X$ is the set of all primitive closed orbits of X (that is, the closed orbits that generate their class in $\pi_1(M)$), $\tau(\gamma)$ is the period of the orbit γ and $C > 0$ is some big constant depending on ρ and X satisfying

$$\|\rho_\nabla([\gamma])\| \leqslant \exp(C\tau(\gamma)), \quad \gamma \in \mathcal{G}_X, \tag{6.3.7}$$

for some norm $\| \cdot \|$ on $\mathrm{End}(E_{x_\star})$.

For every closed orbit γ, we have

$$|\det(I - P_\gamma)| = (-1)^q \det(I - P_\gamma), \tag{6.3.8}$$

for some $q \in \mathbb{Z}$ not depending on γ, where P_γ is the linearized Poincaré return map of γ, that is $P_\gamma = \mathrm{d}_x \varphi^{-\tau(\gamma)}|_{E_s(x) \oplus E_u(x)}$ for $x \in \mathrm{Im}(\gamma)$ (if we choose another point in $\mathrm{Im}(\gamma)$, the map will be conjugated to the first one). This condition is always true when φ^t is contact, in which case we have $q = \dim E_s$.

Giuletti-Pollicott-Liverani [GLP13] (see also the work of Dyatlov-Zworski [DZ16] for a microlocal proof) showed that $\zeta_{X,\nabla}$ has a meromorphic continuation to $\mathbb{C}$ whose poles and zeros are contained in $\mathrm{Res}(\mathcal{L}_X^\nabla)$; moreover, the order of $\zeta_{X,\nabla}$ near a resonance $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$ is given by [3]

$$m(s_0) = (-1)^{q+1} \sum_{k=0}^{n} (-1)^k k m_k(s_0), \tag{6.3.9}$$

where $m_k(s_0)$ is the rank of the spectral projector $\Pi_{s_0}|_{\Omega^k(M,E)}$.

3. Actually, it follows from [DZ16] that $m(s_0) = (-1)^q \sum_{k=0}^{n-1} (-1)^k m_k^0(s_0)$, where $m_k^0(s_0)$ is the dimension of $\Pi_{s_0}\left(\Omega^k(M,E) \cap \ker \iota_X\right)$. We can however repeat the arguments using the identity $\det(\mathrm{Id} - P_\gamma) = -\sum_{k=0}^{n} (-1)^k k \operatorname{tr} \wedge^k \mathrm{d}_x \varphi^{-\tau(\gamma)}$ instead of the identity $\det(\mathrm{Id} - P_\gamma) = \sum_{k=0}^{n-1} (-1)^k \operatorname{tr} \wedge^k P_\gamma$ (see [DZ16, §2.2]), and study the action of $\mathcal{L}_X^\nabla$ on the bundles $\wedge^k T^*M \otimes E$ rather than its action on the bundles $\left(\wedge^k T^*M \cap \ker \iota_X\right) \otimes E$, to obtain (6.3.9).

6.3.6 Topology of resonant states

Since ∇ commutes with $\mathcal{L}_X^\nabla$, it induces a differential on the complexes $C^\bullet(s_0)$ for any $s_0 \in \operatorname{Res}(\mathcal{L}_X^\nabla)$. It is shown in [DR19b] that the complexes $\big(C^\bullet(s_0), \nabla\big)$ are acyclic whenever $s_0 \neq 0$. Moreover, for $s_0 = 0$, the map

$$\Pi_{s_0=0} : \Omega^\bullet(M, \nabla) \longrightarrow C^\bullet(s_0 = 0)$$

is a quasi-isomorphism, that is, it induces isomorphisms at the level of cohomology groups. As the connexion ∇ is assumed to be acyclic, we obtain that the complex $\big(C^\bullet(s_0 = 0), \nabla\big)$ is also acyclic.

6.3.7 Perturbation of holonomy

Let $\gamma : [0,1] \to M$ be a smooth curve and $\alpha \in \Omega^1(M, \operatorname{End}(E))$. Let P_t (resp. $\tilde{P}_t$) be the parallel transport $E_{\gamma(0)} \to E_{\gamma(t)}$ of ∇ (resp. $\tilde{\nabla} = \nabla + \alpha$) along $\gamma|_{[0,t]}$. Then

$$\tilde{P}_t = P_t \exp\left(-\int_0^t P_{-\tau}\alpha(\dot{\gamma}(\tau))P_\tau \mathrm{d}\tau\right). \tag{6.3.10}$$

The above formula will be useful in some occasion. For simplicity, we will denote for any $A \in \mathcal{C}^\infty(M, \operatorname{End}(E))$

$$\int_\gamma A = \int_0^t P_{-\tau}A(\gamma(\tau))P_\tau \mathrm{d}\tau \in \operatorname{End}\left(E_{\gamma(0)}\right)$$

so that $\tilde{P}_1 = P_1 \exp\left(-\int_\gamma \alpha(X)\right)$.

Proof. For every vector field u along γ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(P_{-t}u(t)\big) = P_{-t}\nabla_{\dot{\gamma}(t)}u(t).$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(P_{-t}\tilde{P}_t\big) &= P_{-t}\nabla_{\dot{\gamma}(t)}\tilde{P}_t \\ &= P_{-t}\tilde{\nabla}_{\dot{\gamma}(t)}\tilde{P}_t - P_{-t}\alpha(\dot{\gamma}(t))\tilde{P}_t \\ &= \Big(-P_{-t}\alpha(\dot{\gamma}(t))P_t\Big)\big(P_{-t}\tilde{P}_t\big),\end{aligned}$$

which concludes. □

6.4 The dynamical torsion of a contact Anosov flow

From now on, we will assume that the flow φ^t is contact, that is, there exists a smooth one form $\vartheta \in \Omega^1(M)$ such that $\vartheta \wedge (\mathrm{d}\vartheta)^r$ is a volume form on M, $\iota_X\vartheta = 1$ and $\iota_X \mathrm{d}\vartheta = 0$. The purpose of this section is to define the dynamical torsion of the pair (ϑ, ∇). We first introduce a chirality operator Γ_ϑ acting on $\Omega^\bullet(M, E)$ which is defined thanks to the contact structure. Then the dynamical torsion is a renormalized version of the twisted Ruelle zeta function corrected by the torsion of the finite dimensional space of the generalized resonant states for resonance $s_0 = 0$ computed with respect to Γ_ϑ. This construction was inspired by the work of Braverman-Kappeler on the refined analytic torsion [BK07c].

6.4.1 The chirality operator associated to a contact structure

Let $V_X \to M$ denote the bundle $T^*M \cap \ker \iota_X$. Note that for $k \in \{0, \dots, n\}$, we have the decomposition

$$\wedge^k T^*M = \wedge^{k-1} V_X \wedge \vartheta \oplus \wedge^k V_X. \tag{6.4.1}$$

Indeed, if $\alpha \in \wedge^k T^*M$ we may write

$$\alpha = \underbrace{(-1)^{k+1}\iota_X\alpha \wedge \vartheta}_{\in \wedge^{k-1}V_X \wedge \vartheta} + \underbrace{\alpha - (-1)^{k+1}\iota_X \alpha \wedge \vartheta}_{\in \wedge^k V_X}.$$

Let us introduce the Lefschetz map

$$\begin{array}{rccc} \mathscr{L}: & \wedge^\bullet V_X & \to & \wedge^{\bullet+2} V_X \\ & u & \mapsto & u \wedge \mathrm{d}\vartheta. \end{array}$$

Since $\mathrm{d}\vartheta$ is a symplectic form on V_X, the maps $\mathscr{L}^{r-k}$ induce bundle isomorphisms

$$\mathscr{L}^{r-k} : \wedge^k V_X \xrightarrow{\sim} \wedge^{2r-k} V_X, \quad k = 0, \dots, r, \tag{6.4.2}$$

see for example [LM87, Theorem 16.3]. Using the above Lefschetz isomorphisms, we are now ready to introduce our chirality operator.

Definition 6.4.1. The chirality operator associated to the contact form ϑ is the operator $\Gamma_\vartheta : \wedge^\bullet T^*M \to \wedge^{n-\bullet} T^*M$ defined by $\Gamma_\vartheta^2 = 1$ and

$$\Gamma_\vartheta(f \wedge \vartheta + g) = \mathscr{L}^{r-k} g \wedge \vartheta + \mathscr{L}^{r-k+1} f, \quad f \in \wedge^{k-1} V_X, \quad g \in \wedge^k V_X, \quad k \in \{0, \dots, r\}, \tag{6.4.3}$$

where we used the decomposition (6.4.1).

Note that in particular one has for $k \in \{r+1, \dots, n\}$,

$$\Gamma_\vartheta(f \wedge \vartheta + g) = \left(\mathscr{L}^{k-r}\right)^{-1} g \wedge \vartheta + \left(\mathscr{L}^{k-1-r}\right)^{-1} f.$$

6.4.2 The refined torsion of a space of generalized eigenvectors

The operator Γ_ϑ acts also on $\Omega^\bullet(M, E)$ by acting trivially on E-coefficients. Since $\mathcal{L}_X \vartheta = 0$, Γ_ϑ and $\mathcal{L}_X^\nabla$ commute so that Γ_ϑ induces a chirality operator

$$\Gamma_\vartheta : C^\bullet(s_0) \to C^{n-\bullet}(s_0)$$

for every $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. Recall from §6.3.6 that the complexes $\left(C^\bullet(s_0), \nabla\right)$ are acyclic. The following formula motivates the upcoming definition of the dynamical torsion.

Proposition 6.4.2. *Let $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \setminus \{0, 1\}$. We have*

$$\tau(C^\bullet(s_0), \Gamma_\vartheta)^{-1} = (-1)^{Q_{s_0}} \det\nolimits_{\mathrm{gr}, C^\bullet(s_0)} \mathcal{L}_X^\nabla$$

where

$$Q_{s_0} = \sum_{k=0}^{r} (-1)^k (r + 1 - k) \dim C^k(s_0)$$

and $\tau(C^\bullet(s_0), \Gamma_\vartheta) \in \mathbb{C} \setminus 0$ is the refined torsion of the acyclic complex $\left(C^\bullet(s_0), \nabla\right)$ with respect to the chirality Γ_ϑ, cf Definition 6.2.2.

Let us first admit the above proposition ; the proof will be given in §§6.4.5,6.4.6.

6.4.3 Spectral cuts

If $\mathcal{I} \subset [0,1)$ is an interval, we set

$$\Pi_{\mathcal{I}} = \sum_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} \Pi_{s_0} \quad \text{and} \quad C^\bullet_{\mathcal{I}} = \bigoplus_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} C^\bullet(s_0).$$

Note that $\mathcal{L}_X^\nabla + s$ acts on $C^\bullet(s_0)$ for every $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$ as $-s_0 \,\mathrm{Id} + J$ where J is nilpotent. We thus have for $s \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$

$$\det_{\mathrm{gr}, C^\bullet_{\mathcal{I}}} \left(\mathcal{L}_X^\nabla + s\right)^{(-1)^{q+1}} = \prod_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} (s - s_0)^{m(s_0)}, \tag{6.4.4}$$

where $\det_{\mathrm{gr}}$ is the graded determinant, cf. §6.2.5.

Let $\lambda \in [0,1)$ such that $\mathrm{Res}(\mathcal{L}_X^\nabla) \cap \{s \in \mathbb{C} : |s| = \lambda\} = \emptyset$. Now define the meromorphic function

$$\zeta_{X,\nabla}^{(\lambda,\infty)}(s) = \zeta_{X,\nabla}(s) \det_{\mathrm{gr}, C^\bullet_{[0,\lambda]}} \left(\mathcal{L}_X^\nabla + s\right)^{(-1)^q}. \tag{6.4.5}$$

Then (6.3.9) and (6.4.4) show that $\zeta_{X,\nabla}^{(\lambda,\infty)}$ has no pole nor zero in $\{|s| \leqslant \lambda\}$, so that the number $\zeta_{X,\nabla}^{(\lambda,\infty)}(0)$ is well defined.

6.4.4 Definition of the dynamical torsion

Let $0 < \mu < \lambda < 1$ such that for every $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$, one has $|s_0| \neq \lambda, \mu$. Using Proposition 6.2.3 and Proposition 6.4.2 we obtain, with notations of §6.4.3,

$$\tau\big(C^\bullet_{[0,\lambda]}, \Gamma_\vartheta\big) = (-1)^{Q_{(\mu,\lambda]}} \left(\det_{\mathrm{gr}, C^\bullet_{(\mu,\lambda]}} \mathcal{L}_X^\nabla\right)^{-1} \tau\big(C^\bullet_{[0,\mu]}, \Gamma_\vartheta\big),$$

where for an interval $\mathcal{I}$ we set

$$Q_{\mathcal{I}} = \sum_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} Q_{s_0}.$$

This allows us to give the following

Definition 6.4.3 (Dynamical torsion)**.** The number

$$\tau_\vartheta(\nabla) = (-1)^{Q_{[0,\lambda]}} \zeta_{X,\nabla}^{(\lambda,\infty)}(0)^{(-1)^q} \cdot \tau\big(C^\bullet_{[0,\lambda]}, \Gamma_\vartheta\big) \in \mathbb{C} \setminus 0 \tag{6.4.6}$$

is independent of the spectral cut $\lambda \in (0,1)$. We will call this number the *dynamical torsion* of the pair (ϑ, ∇).

Remark 6.4.4. If $c_{X,\nabla} s^{m(0)}$ is the leading term of the Laurent expansion of $\zeta_{X,\nabla}(s)$ at $s = 0$, then taking λ small enough actually shows that

$$\tau_\vartheta(\nabla) = (-1)^{Q_0} c_{X,\nabla}^{(-1)^q} \cdot \tau\big(C^\bullet(0), \Gamma_\vartheta\big). \tag{6.4.7}$$

In particular, if $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$,

$$\tau_\vartheta(\nabla) = \zeta_{X,\nabla}(0)^{(-1)^q}. \tag{6.4.8}$$

Note that we could have taken (6.4.7) as a definition of the dynamical torsion ; however (6.4.6) is more convenient to study the regularity of the $\tau_\vartheta(\nabla)$ with respect to ϑ and ∇.

Remark 6.4.5. This definition actually makes sense even if ∇ is not acyclic. Indeed, in that case, formula (6.4.6) defines an element of the determinant line $\det H^\bullet\big(C^\bullet_{[0,\lambda]}\nabla\big)$, cf. Remark 6.2.1. Under the identification $H^\bullet(M,\nabla) = H^\bullet\big(C^\bullet_{[0,\lambda]}\nabla\big)$ given by the quasi-isomorphism $\Pi_{[0,\lambda]} : \Omega^\bullet(M,E) \to C^\bullet_{[0,\lambda]}$ (cf §6.3.6), we thus get an element of $\det H^\bullet(M,\nabla)$.

The rest of this section is devoted to the proof of Proposition 6.4.2.

6.4.5 Invertibility of the contact signature operator

To prove Proposition 6.4.2 we shall use §6.2.4 and introduce the *contact signature operator*

$$B_\vartheta = \Gamma_\vartheta \nabla + \nabla \Gamma_\vartheta : \mathcal{D}'^\bullet(M,E) \to \mathcal{D}'^\bullet(M,E),$$

where Γ_ϑ acts trivially on E. We fix in what follows some $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \setminus \{0,1\}$ and we denote $C^\bullet(s_0)$ by $C^\bullet$ for simplicity. We also set $C_0^\bullet = C^\bullet \cap \ker(\iota_X)$.

The following result will put us in position to apply Proposition 6.2.4.

Lemma 6.4.6. *The operator B_ϑ is invertible $C^\bullet \to C^\bullet$.*

Proof. We set

$$C^\bullet_{\mathrm{even}} = \bigoplus_{k \text{ even}} C^k, \quad C^\bullet_{\mathrm{odd}} = \bigoplus_{k \text{ odd}} C^k.$$

Then B_ϑ preserves the decomposition $C^\bullet = C^\bullet_{\mathrm{even}} \oplus C^\bullet_{\mathrm{odd}}$. Note that because $\Gamma_\vartheta^2 = 1$, we have $B_\vartheta|_{C^\bullet_{\mathrm{even}}} = \Gamma_\vartheta B_\vartheta|_{C^\bullet_{\mathrm{odd}}} \Gamma_\vartheta$. It thus suffices to show that B_ϑ is injective on $C^\bullet_{\mathrm{even}}$. Let $\beta \in C^\bullet_{\mathrm{even}}$ such that $B_\vartheta \beta = 0$. Write

$$\beta = \sum_{k=0}^{r} \beta_{2k} \in C^\bullet_{\mathrm{even}},$$

with

$$\beta_{2k} = f_{2k-1} \wedge \vartheta + g_{2k}, \quad f_{2k-1} \in C_0^{2k-1}, \quad g_{2k} \in C_0^{2k}, \quad k = 0, \dots, r.$$

Then $B_\vartheta \beta = 0$ writes, since $\Gamma_\vartheta \nabla(C^k) \subset C^{n-k-1}$ and $\nabla \Gamma_\vartheta(C^k) \subset C^{n-k+1}$,

$$\Gamma_\vartheta \nabla \beta_{2k} + \nabla \Gamma_\vartheta \beta_{2(k+1)} = 0, \quad k = 0, \dots, r. \tag{6.4.9}$$

Because ∇ does not leave the decomposition (6.4.1) stable, we need to introduce an operator $\Psi : C_0^\bullet \to C_0^{\bullet+1}$ which mimics the action of ∇. We define

$$\Psi \mu = \nabla \mu - (-1)^k \mathcal{L}_X^\nabla \mu \wedge \vartheta, \quad \mu \in C_0^k. \tag{6.4.10}$$

Because $\mathcal{L}_X \mathrm{d}\vartheta = 0$, the map Ψ satisfies the simple relation

$$\Psi\left(\mu \wedge \mathrm{d}\vartheta^j\right) = (\Psi\mu) \wedge \mathrm{d}\vartheta^j, \quad \mu \in C_0^\bullet, \quad j \in \mathbb{N}, \tag{6.4.11}$$

that is, Ψ commutes with $\mathscr{L}$. Also, observe that

$$\Psi^2 \mu = -\mathcal{L}_X^\nabla \mu \wedge \mathrm{d}\vartheta, \quad \mu \in C_0^\bullet. \tag{6.4.12}$$

Indeed, using the fact that $\mathcal{L}_X^\nabla$ and ∇ commute,

$$\begin{aligned}\Psi^2\mu &= \nabla\left(\nabla\mu - (-1)^k \mathcal{L}_X^\nabla \mu \wedge \vartheta\right) - (-1)^{k+1}\left(\mathcal{L}_X^\nabla\left(\nabla\mu - (-1)^k\mathcal{L}_X^\nabla\mu\wedge\vartheta\right)\right)\wedge\vartheta \\ &= \nabla^2\mu + (-1)^{k+1}\nabla\left(\mathcal{L}_X^\nabla\mu\wedge\vartheta\right) + (-1)^k\mathcal{L}_X^\nabla\nabla\mu\wedge\vartheta - \mathcal{L}_X^{\nabla\,2}\mu\wedge\vartheta\wedge\vartheta \\ &= (-1)^{k+1}(-1)^k\mathcal{L}_X^\nabla\mu\wedge\mathrm{d}\vartheta.\end{aligned}$$

Assume first that $k \leqslant r/2 - 1$. Then $2k+2 \leqslant r$; we can thus write, with (6.4.10) in mind,

$$\begin{aligned}\Gamma_\vartheta\nabla\beta_{2k} &= \Gamma_\vartheta\Big(\nabla f_{2k-1}\wedge\vartheta - f_{2k-1}\wedge\mathrm{d}\vartheta + \nabla g_{2k}\Big) \\ &= \Gamma_\vartheta\Big(\Psi f_{2k-1}\wedge\vartheta - \mathcal{L}_X^\nabla f_{2k-1}\wedge\vartheta\wedge\vartheta - f_{2k-1}\wedge\mathrm{d}\vartheta + \Psi g_{2k} + \mathcal{L}_X^\nabla g_{2k}\wedge\vartheta\Big) \\ &= \Big(\Psi f_{2k-1} + \mathcal{L}_X^\nabla g_{2k}\Big)\wedge\mathrm{d}\vartheta^{r-2k} + \Big(\Psi g_{2k} - f_{2k-1}\wedge\mathrm{d}\vartheta\Big)\wedge\mathrm{d}\vartheta^{r-2k-1}\wedge\vartheta.\end{aligned}$$

Similarly we find by (6.4.11)

$$\begin{aligned}\nabla\Gamma_\vartheta\beta_{2k+2} &= \nabla\Big(f_{2k+1}\wedge\mathrm{d}\vartheta^{r-2k-1} + g_{2k+2}\wedge\mathrm{d}\vartheta^{r-2k-2}\wedge\vartheta\Big) \\ &= \Big(\Psi f_{2k+1} - \mathcal{L}_X^\nabla f_{2k+1}\wedge\vartheta\Big)\wedge\mathrm{d}\vartheta^{r-2k-1} \\ &\qquad + \Big(\Psi g_{2k+2}\wedge\vartheta + g_{2k+2}\wedge\mathrm{d}\vartheta\Big)\wedge\mathrm{d}\vartheta^{r-2k-2}.\end{aligned} \tag{6.4.13}$$

Thus (6.4.9) writes, with the decompostion (6.4.1) in mind,

$$\Big(\Psi f_{2k+1} + g_{2k+2}\Big)\wedge\mathrm{d}\vartheta^{r-2k-1} + \Big(\Psi f_{2k-1} + \mathcal{L}_X^\nabla g_{2k}\Big)\wedge\mathrm{d}\vartheta^{r-2k} = 0 \tag{6.4.14}$$

and

$$\Big(-\mathcal{L}_X^\nabla f_{2k+1}\wedge\mathrm{d}\vartheta + \Psi g_{2k+2}\Big)\wedge\mathrm{d}\vartheta^{r-2k-2} + \Big(\Psi g_{2k} - f_{2k-1}\wedge\mathrm{d}\vartheta\Big)\wedge\mathrm{d}\vartheta^{r-2k-1} = 0. \tag{6.4.15}$$

Then applying Ψ to (6.4.15) gives, with (6.4.12) and (6.4.11),

$$\Big(-\Psi\mathcal{L}_X^\nabla f_{2k+1} - \mathcal{L}_X^\nabla g_{2k+2}\Big)\wedge\mathrm{d}\vartheta^{r-2k-1} - \mathcal{L}_X^\nabla g_{2k}\wedge\mathrm{d}\vartheta^{r-2k} - \Psi f_{2k-1}\wedge\mathrm{d}\vartheta^{r-2k} = 0.$$

Note that Ψ commutes with $\mathcal{L}_X^\nabla$ and thus with $\mathcal{L}_X^{\nabla\,-1}$ (which exists since $s_0 \neq 0$). Then applying $\mathcal{L}_X^{\nabla\,-1}$ to the above relation we get

$$\Big(-\Psi f_{2k+1} - g_{2k+2}\Big)\wedge\mathrm{d}\vartheta^{r-2k-1} - g_{2k}\wedge\mathrm{d}\vartheta^{r-2k} - \mathcal{L}_X^{\nabla\,-1}\left(\Psi f_{2k-1}\wedge\mathrm{d}\vartheta^{r-2k}\right) = 0.$$

Injecting this in (6.4.14), we obtain

$$\Big(\big(\mathcal{L}_X^{\nabla} - \mathrm{Id}\big)\, g_{2k} + \big(\mathrm{Id} - \mathcal{L}_X^{\nabla^{-1}}\big)\Psi f_{2k-1}\Big) \wedge \mathrm{d}\vartheta^{r-2k} = 0.$$

Since $\mathscr{L}^{r-2k}$ is injective on C_0^{2k} and $\mathcal{L}_X^{\nabla} - \mathrm{Id}$ is invertible (since $s_0 \neq 1$), this yields

$$\mathcal{L}_X^{\nabla} g_{2k} + \Psi f_{2k-1} = 0. \tag{6.4.16}$$

Applying $\mathcal{L}_X^{\nabla^{-1}}\Psi$ to the above equation we get

$$\Psi g_{2k} - f_{2k-1} \wedge \mathrm{d}\vartheta = 0 \tag{6.4.17}$$

by (6.4.11) ; thus (6.4.15) gives

$$\Big(\Psi g_{2k+2} - \mathcal{L}_X^{\nabla} f_{2k+1} \wedge \mathrm{d}\vartheta\Big) \wedge \mathrm{d}\vartheta^{r-2k-2} = 0.$$

Now repeating this process with k replaced by $k-1$ we obtain

$$\big(\Psi g_{2k} - \mathcal{L}_X^{\nabla} f_{2k-1} \wedge \mathrm{d}\vartheta\big) \wedge \mathrm{d}\vartheta^{r-2k} = 0.$$

By (6.4.17) this implies that

$$\big(\mathrm{Id} - \mathcal{L}_X^{\nabla}\big) f_{2k-1} \wedge \mathrm{d}\vartheta^{r-2k+1} = 0,$$

which leads to $f_{2k-1} = 0$ since $\mathscr{L}^{r-(2k-1)}$ is injective on C_0^{2k-1} and $\mathcal{L}_X^{\nabla} - \mathrm{Id}$ is invertible on $C^\bullet$; thus $g_{2k} = 0$ by (6.4.16), since $\mathcal{L}_X^{\nabla}$ is invertible. We therefore obtained

$$\beta_{2k} = 0, \quad k \leqslant r/2 - 1.$$

Next, assume $k \geqslant (r+1)/2$. Set $\tilde{k} = r - k$ and

$$\tilde{\beta}_{2\tilde{k}+1} = \Gamma_\vartheta \beta_{2k} \in C_0^{2\tilde{k}+1}, \quad \tilde{\beta}_{2\tilde{k}-1} = \Gamma_\vartheta \beta_{2k+2} \in C_0^{2\tilde{k}-1}.$$

Then (6.4.9) writes

$$\Gamma_\vartheta \nabla \tilde{\beta}_{2\tilde{k}-1} + \nabla \Gamma_\vartheta \tilde{\beta}_{2\tilde{k}+1} = 0.$$

Since $2\tilde{k} + 1 \leqslant r$ and we can do exactly as before to get $\tilde{\beta}_{2\tilde{k}-1} = 0$ which leads to $\beta_{2k+2} = 0$. Therefore we obtained

$$\beta_{2k} = 0, \quad k \geqslant (r+1)/2 + 1.$$

Therefore it remains to show that $\beta_{2p} = 0$ and $\beta_{2(p+1)} = 0$, where $p = \lfloor r/2 \rfloor$. We will assume that $r = 2p+1$ is odd and put $p' = p+1$ (the case r even is similar). Then (6.4.9) implies, since $\beta_{2k} = 0$ for every $k \neq p, p'$,

$$\nabla \Gamma_\vartheta \beta_{2p'} + \Gamma_\vartheta \nabla \beta_{2p} = 0, \quad \Gamma_\vartheta \nabla \beta_{2p'} = 0, \quad \nabla \Gamma_\vartheta \beta_{2p} = 0. \tag{6.4.18}$$

We can compute, keeping (6.4.10) in mind,

$$\begin{aligned}\nabla \Gamma_\vartheta \beta_{2p'} &= \nabla\left(\mathscr{L}^{-1} g_{2p'} \wedge \vartheta + f_{2p'-1}\right)\\ &= \Psi \mathscr{L}^{-1} g_{2p'} \wedge \vartheta + \mathcal{L}_X^{\nabla} \mathscr{L}^{-1} g_{2p'} \wedge \vartheta \wedge \vartheta\\ &\qquad + \mathscr{L}^{-1} g_{2p'} \wedge \mathrm{d}\vartheta + \Psi f_{2p'-1} - \mathcal{L}_X^{\nabla} f_{2p'-1} \wedge \vartheta,\end{aligned}$$

and

$$\Gamma_\vartheta \nabla \beta_{2p} = \Gamma_\vartheta\Big(\Psi g_{2p} + \mathcal{L}_X^\nabla g_{2p} \wedge \vartheta + \Psi f_{2p-1} \wedge \vartheta - \mathcal{L}_X^\nabla f_{2p-1} \wedge \vartheta \wedge \vartheta - f_{2p-1} \wedge \mathrm{d}\vartheta\Big)$$
$$= \Psi g_{2p} \wedge \vartheta - f_{2p-1} \wedge \mathrm{d}\vartheta \wedge \vartheta + \mathcal{L}_X^\nabla g_{2p} \wedge \mathrm{d}\vartheta + \Psi f_{2p-1} \wedge \mathrm{d}\vartheta.$$

Therefore the first equation of (6.4.18) implies, since $\mathscr{L}^{-1} g_{2p'} \wedge \mathrm{d}\vartheta = g_{2p'}$,

$$\Psi \mathscr{L}^{-1} g_{2p'} - \mathcal{L}_X^\nabla f_{2p'-1} - f_{2p-1} \wedge \mathrm{d}\vartheta + \Psi g_{2p} = 0 \tag{6.4.19}$$

and

$$g_{2p'} + \Psi f_{2p'-1} + \Psi f_{2p-1} \wedge \mathrm{d}\vartheta + \mathcal{L}_X^\nabla g_{2p} \wedge \mathrm{d}\vartheta = 0. \tag{6.4.20}$$

Applying ${\mathcal{L}_X^\nabla}^{-1}\Psi$ to (6.4.19) leads to

$$-g_{2p'} - \Psi f_{2p'-1} - \Psi {\mathcal{L}_X^\nabla}^{-1} f_{2p-1} \wedge \mathrm{d}\vartheta + -g_{2p} \wedge \mathrm{d}\vartheta = 0.$$

Therefore,

$$\left(\left(\mathrm{Id} - {\mathcal{L}_X^\nabla}^{-1}\right) \Psi f_{2p-1} + \left(\mathcal{L}_X^\nabla - \mathrm{Id}\right) g_{2p}\right) \wedge \mathrm{d}\vartheta. \tag{6.4.21}$$

As before this gives $\Psi f_{2p-1} + \mathcal{L}_X^\nabla g_{2p} = 0$ and thus with (6.4.20) one gets

$$\mathcal{L}_X^\nabla g_{2p} + \Psi f_{2p-1} = 0, \quad g_{2p'} + \Psi f_{2p'-1} = 0. \tag{6.4.22}$$

Next compute

$$\nabla \Gamma_\vartheta \beta_{2p} = g_{2p} \wedge \mathrm{d}\vartheta^2 + \Psi f_{2p-1} \wedge \mathrm{d}\vartheta^2 + \Psi g_{2p} \wedge \vartheta \wedge \mathrm{d}\vartheta - \mathcal{L}_X^\nabla f_{2p+1} \wedge \vartheta \wedge \mathrm{d}\vartheta^2$$

Therefore the third part of (6.4.18) gives (we take the $\wedge\vartheta$ component of the above equation)

$$-\mathcal{L}_X^\nabla f_{2p-1} \wedge \mathrm{d}\vartheta^2 + \Psi g_{2p} \wedge \mathrm{d}\vartheta = 0.$$

Applying ${\mathcal{L}_X^\nabla}^{-1}\Psi$ to (6.4.22) we get $\Psi g_{2p} = f_{2p-1} \wedge \mathrm{d}\vartheta$; we therefore obtain that $f_{2p-1} = 0$ by injectivity of $\mathscr{L}^2$ on C_0^{r-2}. Thus $g_{2p} = 0$ by (6.4.21).

Finally compute

$$\nabla \beta_{2p'} = \Psi f_{2p'-1} \wedge \vartheta + \Psi g_{2p'} + \mathcal{L}_X^\nabla g_{2p'} \wedge \vartheta = 0.$$

Therefore the second part of (6.4.18) implies (since $\Gamma_\vartheta \nabla \beta_{2p'} = 0$ is equivalent to $\nabla \beta_{2p'} = 0$)

$$\Psi f_{2p'-1} + \mathcal{L}_X^\nabla g_{2p'} = 0.$$

Therefore by (6.4.22) we get $\left(\mathcal{L}_X^\nabla - \mathrm{Id}\right) g_{2p'} = 0$, and thus $g_{2p'} = 0$. Using (6.4.19) we conclude that $\mathcal{L}_X^\nabla f_{2p'-1} = 0$ which leads to $f_{2p'-1} = 0$. □

6.4.6 Proof of Proposition 6.4.2

We start from Proposition 6.2.4 which gives us, in view of Lemma 6.4.6,

$$\tau(C^\bullet, \Gamma_\vartheta) = (-1)^{r \dim C_+^r} \det\left(\Gamma_\vartheta \nabla|_{C_+^r}\right)^{(-1)^r} \prod_{j=0}^{r-1} \det\left(\Gamma_\vartheta \nabla|_{C_+^j \oplus C_+^{n-j-1}}\right)^{(-1)^j}. \tag{6.4.23}$$

where we set as in §6.2.4

$$C_+^\bullet = C^\bullet \cap \ker(\nabla\Gamma_\vartheta), \quad C_-^\bullet = C^\bullet \cap \ker(\Gamma_\vartheta\nabla).$$

We first note that for $k \in \{0, \dots, r\}$ and $\beta \in \Omega^k(M, E)$, one has

$$\begin{aligned}\nabla\Gamma_\vartheta\beta = \mathscr{L}^{r-k}\Big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta + \mathscr{L}\big(\iota_X\nabla\iota_X\beta - \iota_X\beta\big)\Big)\wedge\vartheta \\ + (-1)^k\mathscr{L}^{r-k+1}\Big(\beta - \nabla\iota_X\beta + (-1)^k\iota_X\left(\beta - \nabla\iota_X\beta\right)\wedge\vartheta\Big), \\ \Gamma_\vartheta\nabla\beta = \mathscr{L}^{r-k-1}\Big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta\Big)\wedge\vartheta + (-1)^k\mathscr{L}^{r-k}\big(\iota_X\nabla\beta\big),\end{aligned} \tag{6.4.24}$$

where $\mathscr{L}^{j-r} = (\mathscr{L}^{r-j}|_{\wedge^j V_X})^{-1}$ for $0 \leqslant j \leqslant r$. Indeed, using the decomposition (6.4.1),

$$\begin{aligned}\Gamma_\vartheta\beta &= (-1)^{k+1}\iota_X\beta\wedge d\vartheta^{r-k+1} + \Big(\beta + (-1)^k\iota_X\beta\wedge\vartheta\Big)\wedge d\vartheta^{r-k}\wedge\vartheta \\ &= (-1)^{k+1}\iota_X\beta\wedge d\vartheta^{r-k+1} + \beta\wedge d\vartheta^{r-k}\wedge\vartheta,\end{aligned}$$

which leads to

$$\begin{aligned}\nabla\Gamma_\vartheta\beta &= (-1)^{k+1}\nabla\iota_X\beta\wedge d\vartheta^{r-k+1} + \nabla\beta\wedge d\vartheta^{r-k}\wedge\vartheta + (-1)^k\beta\wedge d\vartheta^{r-k+1} \\ &= (-1)^{k+1}\Big((-1)^{k+1}\iota_X\nabla\iota_X\beta\wedge\vartheta\wedge d\vartheta^{r-k+1}\Big) \\ &\quad + (-1)^{k+1}\Big(\nabla\iota_X\beta + (-1)^k\iota_X\nabla\iota_X\beta\wedge\vartheta\Big)\wedge d\vartheta^{r-k+1} \\ &\quad + \Big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta\Big)\wedge d\vartheta^{r-k}\wedge\vartheta \\ &\quad + (-1)^k\Big(\beta + (-1)^k\iota_X\beta\wedge\vartheta\Big)\wedge d\vartheta^{r-k+1} \\ &\quad - \iota_X\beta\wedge d\vartheta^{r-k+1}\wedge\vartheta,\end{aligned}$$

which is exactly the first part of (6.4.24). The second part follows directly from the decomposition (6.4.1).

Let us introduce, for $k \in \{0, \dots, r\}$, the operator $J_k : C^k \to C^k$ defined by

$$J_k\beta = f\wedge\vartheta - (-1)^k\Psi f \tag{6.4.25}$$

for any $\beta = f\wedge\vartheta + g \in C^k$ with $f \in C_0^{k-1}$ and $g \in C_0^k$, and where Ψ is defined in (6.4.10). Then we claim that J_k takes it values in C_+^k. Indeed, we have for any $f \in C_0^{k-1}$ and $g \in C_0^k$,

$$\begin{aligned}\nabla\Gamma_\vartheta(f\wedge\vartheta + g) &= \nabla\Big(g\wedge d\vartheta^{r-k}\wedge\vartheta + f\wedge d\vartheta^{r-k+1}\Big) \\ &= \Psi g\wedge d\vartheta^{r-k}\wedge\vartheta + (-1)^k g\wedge d\vartheta^{r-k+1} \\ &\qquad + \Psi f\wedge d\vartheta^{r-k+1} + (-1)^{k+1}\mathcal{L}_X^\nabla f\wedge d\vartheta^{r-k+1}\wedge\vartheta,\end{aligned}$$

which implies that $\beta = f\wedge\vartheta + g$ lies in C_+^k if and only if

$$\Big(\Psi g + (-1)^{k+1}\mathcal{L}_X^\nabla f\wedge d\vartheta\Big)\wedge d\vartheta^{r-k} = 0 \quad \text{and} \quad \Big(\Psi f + (-1)^k g\Big)\wedge d\vartheta^{r-k+1} = 0. \tag{6.4.26}$$

But now note that if $\beta = f\wedge\vartheta + g = J_k\beta' = f'\wedge\vartheta - (-1)^k\Psi f'$ for some $\beta' = f'\wedge\vartheta + g'$ then $f = f'$ and $g = -(-1)^k\Psi f$, and thus β satisfies the second part of (6.4.26). We also obtain $\Psi g = -(-1)^k\Psi^2 f = -(-1)^k\mathcal{L}_X^\nabla f\wedge \mathrm{d}\vartheta$ by (6.4.12), so the first part of (6.4.26) is also satisfied.

Therefore $J_k : C^k \to C_+^k$; moreover it is obvious that J_k is a projector. Therefore we can consider the restricted projection $J_k|_{C_+^k} : C_+^k \to C_+^k$, we will still denote by J_k.

The next lemma will be helpful to compute the determinants lying in the product (6.4.23).

Lemma 6.4.7. *Take $k \in \{0, \cdots, r-1\}$. Then for any $\beta = f\wedge\vartheta + g \in C_+^k$ with $f \in C_0^{k-1}$ and $g \in C_0^k$, one has*

$$(\Gamma_\vartheta\nabla)^2\beta = \mathcal{L}_X^\nabla\left(\mathcal{L}_X^\nabla - \mathrm{Id}\right)\beta - \left(\mathcal{L}_X^\nabla - \mathrm{Id}\right)J_k\beta.$$

Proof. Since $k < r$ we can write, thanks to (6.4.24),

$$\Gamma_\vartheta\nabla\beta = \nabla\beta\wedge\vartheta\wedge\mathrm{d}\vartheta^{r-k-1} + (-1)^k\iota_X\nabla\beta\wedge\mathrm{d}\vartheta^{r-k}.$$

Therefore

$$\begin{aligned}\nabla\Gamma_\vartheta\nabla\beta &= -(-1)^k\nabla\beta\wedge\mathrm{d}\vartheta^{r-k} + (-1)^k\nabla\iota_X\nabla\beta\wedge\mathrm{d}\vartheta^{r-k}\\ &= (-1)^k\left(\mathcal{L}_X^\nabla - \mathrm{Id}\right)\nabla\beta\wedge\mathrm{d}\vartheta^{r-k}\\ &= \Big(\iota_X\nabla\iota_X\nabla\beta - \iota_X\nabla\beta\Big)\wedge\vartheta\wedge\mathrm{d}\vartheta^{r-k}\\ &\quad + (-1)^k(\mathcal{L}_X^\nabla - \mathrm{Id})\Big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta\Big)\wedge\mathrm{d}\vartheta^{r-k},\end{aligned}$$

where we used $\nabla\iota_X\nabla\beta = \mathcal{L}_X^\nabla\nabla\beta$ and $\iota_X\nabla\iota_X\nabla\beta = \mathcal{L}_X^\nabla\iota_X\nabla\beta$. Since $\beta \in C_+^k$ one has with (6.4.24)

$$\Big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta\Big)\wedge\mathrm{d}\vartheta^{r-k} = \Big(\iota_X\beta - \iota_X\nabla\iota_X\beta\Big)\wedge\mathrm{d}\vartheta^{r-k+1}.$$

This leads to

$$\begin{aligned}\nabla\Gamma_\vartheta\nabla\beta = &\Big(\iota_X\nabla\iota_X\nabla\beta - \iota_X\nabla\beta\Big)\wedge\vartheta\wedge\mathrm{d}\vartheta^{r-k}\\ &+ (-1)^k\left(\mathcal{L}_X^\nabla - \mathrm{Id}\right)\Big(\iota_X\beta - \iota_X\nabla\iota_X\beta\Big)\wedge\mathrm{d}\vartheta^{r-k+1}.\end{aligned}$$

Since $\iota_X\nabla\iota_X\nabla\beta - \iota_X\nabla\beta = \left(\mathcal{L}_X^\nabla - \mathrm{Id}\right)\iota_X\nabla\beta$ and $\iota_X\beta - \iota_X\nabla\iota_X\beta = \left(\mathrm{Id} - \mathcal{L}_X^\nabla\right)\iota_X\beta$, we obtain

$$\nabla\Gamma_\vartheta\nabla\beta = \left(\mathcal{L}_X^\nabla - \mathrm{Id}\right)\iota_X\nabla\beta\wedge\vartheta\wedge\mathrm{d}\vartheta^{r-k} + (-1)^k\left(\mathcal{L}_X^\nabla - \mathrm{Id}\right)\left(\mathrm{Id} - \mathcal{L}_X^\nabla\right)\iota_X\beta\wedge\mathrm{d}\vartheta^{r-k+1},$$

and thus by definition of Γ_ϑ

$$\Gamma_\vartheta\nabla\Gamma_\vartheta\nabla\beta = -(-1)^k\left(\mathrm{Id} - \mathcal{L}_X^\nabla\right)^2\iota_X\beta\wedge\vartheta + \left(\mathcal{L}_X^\nabla - \mathrm{Id}\right)\iota_X\nabla\beta. \tag{6.4.27}$$

Now, writing $\beta = f\wedge\vartheta + g$ where $\iota_X f = 0$ and $\iota_X g = 0$, we have

$$\begin{aligned}\nabla\beta &= \nabla f\wedge\vartheta - (-1)^k f\wedge\mathrm{d}\vartheta + \nabla g,\\ \iota_X\nabla\beta &= \mathcal{L}_X^\nabla f\wedge\vartheta + (-1)^k\nabla f + \mathcal{L}_X^\nabla g,\\ \iota_X\beta\wedge\vartheta &= -(-1)^k f\wedge\vartheta.\end{aligned}\tag{6.4.28}$$

Injecting those relations in (6.4.27) we get

$$\Gamma_\vartheta\nabla\Gamma_\vartheta\nabla\beta = \mathcal{L}_X^\nabla\big(\mathcal{L}_X^\nabla - \mathrm{Id}\big)(f\wedge\vartheta + g) \\ - \big(\mathcal{L}_X^\nabla - \mathrm{Id}\big)\Big(f\wedge\vartheta - (-1)^k\big(\nabla f + (-1)^k\mathcal{L}_X^\nabla f\wedge\vartheta\big)\Big),$$

which concludes in view of (6.4.10) and (6.4.25). □

We now deal with the case $k = r$.

Lemma 6.4.8. *One has, for $\beta \in C_+^r$,*

$$\Gamma_\vartheta\nabla\beta = (-1)^r\Big(\big(\mathcal{L}_X^\nabla - \mathrm{Id}\big)\beta + (\mathrm{Id} - J_r)\beta\Big).$$

Proof. We have

$$\Gamma_\vartheta\nabla\beta = \mathscr{L}^{-1}\big(\nabla\beta - (-1)^r\iota_X\nabla\beta\wedge\vartheta\big) + (-1)^r\iota_X\nabla\beta.$$

Since $\beta \in C_+^r$ we have with (6.4.24) that $\nabla\beta - (-1)^r\iota_X\nabla\beta\wedge\vartheta = (\iota_X\beta - \iota_X\nabla\iota_X\beta)\wedge d\vartheta$. Therefore,

$$\Gamma_\vartheta\nabla\beta = (\iota_X\beta - \iota_X\nabla\iota_X\beta)\wedge\vartheta + (-1)^r\iota_X\nabla\beta.$$

We now conclude as in the previous lemma, using (6.4.28). □

We are now in position to finish the proof of Proposition 6.4.2. We will set, for $0 \leqslant k \leqslant n$,

$$m_k = \dim C^k, \quad m_k^0 = \dim C_0^k, \quad m_k^\pm = \dim C_\pm^k.$$

First take $k \in \{0, \cdots, r-1\}$. First take $k \in \{0, \cdots, r-1\}$. Because B_ϑ is invertible on $C^\bullet$, $\Gamma_\vartheta\nabla$ induces an isomorphism $C_+^k \to C_+^{n-k-1}$. Take any basis γ of C_+^k. Then $\Gamma_\vartheta\nabla\gamma$ is a basis of C_+^{n-k-1} and the matrix of $\Gamma_\vartheta\nabla|_{C_+^k\oplus C_+^{n-k+1}}$ in the basis $\gamma \oplus \Gamma_\vartheta\nabla\gamma$ is

$$\begin{pmatrix} 0 & \big[(\Gamma_\vartheta\nabla)^2\big]_\gamma \\ \mathrm{Id} & 0 \end{pmatrix}, \tag{6.4.29}$$

where $\big[(\Gamma_\vartheta\nabla)^2\big]_\gamma$ is the matrix of $(\Gamma_\vartheta\nabla)^2|_{C_+^k}$ in the basis γ. Define

$$\tilde{J}_k = \mathrm{Id} - J_k : C_+^k \to C_+^k.$$

Then $\tilde{J}_k$ is a projector (since J_k is) and Lemma 6.4.7 implies that J_k (and thus $\tilde{J}_k$) commutes with $\mathcal{L}_X^\nabla$. Moreover one has

$$(\Gamma_\vartheta\nabla)^2|_{\ker\tilde{J}_k} = \big(\mathcal{L}_X^\nabla - \mathrm{Id}\big)^2, \quad (\Gamma_\vartheta\nabla)^2|_{\mathrm{ran}\tilde{J}_k} = \mathcal{L}_X^\nabla\big(\mathcal{L}_X^\nabla - \mathrm{Id}\big).$$

As a consequence,

$$\det\Big((\Gamma_\vartheta\nabla)^2|_{C_+^k}\Big) = \big[s_0(1+s_0)\big]^{m_k^+ - m_{k-1}^0}(1+s_0)^{2m_{k-1}^0} \\ = {s_0}^{m_k^+ - m_{k-1}^0}(1+s_0)^{m_k^+ + m_{k-1}^0},$$

because on $C^\bullet$ (and in particular on C^k_+), one has $\mathcal{L}^\nabla_X = -s_0 \operatorname{Id} + \nu$ where ν is nilpotent, and one has $\dim\ker \tilde{J}_k = \dim \operatorname{ran} J_k = m^0_{k-1}$. Indeed, by (6.4.25) we can view J_k as a map $C^{k-1}_0 \to C^k_+$, which is obviously injective. We finally obtain with (6.4.29)

$$\det\left(\Gamma_\vartheta \nabla|_{C^k_+ \oplus C^{n-k+1}_+}\right) = (-1)^{m^+_k} {s_0}^{m^+_k - m^0_{k-1}} (1+s_0)^{m^+_k + m^0_{k-1}}. \tag{6.4.30}$$

We now deal with the case $k = r$. Lemma 6.4.8 gives

$$\Gamma_\vartheta \nabla|_{\ker \tilde{J}_r} = (-1)^r \left(\mathcal{L}^\nabla_X - \operatorname{Id}\right), \quad \Gamma_\vartheta \nabla|_{\operatorname{ran} \tilde{J}_r} = (-1)^r \mathcal{L}^\nabla_X.$$

As before, we obtain

$$\det\left(\Gamma_\vartheta \nabla|_{C^r_+}\right) = (-1)^{r m^+_r} (-1)^{m^+_r} {s_0}^{m^+_r - m^0_{r-1}} (1+s_0)^{m^0_{r-1}}. \tag{6.4.31}$$

Combining (6.4.23) with (6.4.30) and (6.4.31) we finally obtain

$$\tau(C^\bullet, \Gamma_\vartheta) = (-1)^J {s_0}^K (1+s_0)^L \tag{6.4.32}$$

where

$$J = \sum_{k=0}^{r} (-1)^k m^+_k, \quad K = \sum_{k=0}^{r} (-1)^k (m^+_k - m^0_{k-1}), \quad L = \sum_{k=0}^{r-1} (-1)^k (m^+_k - m^0_k).$$

Note that for $0 \leqslant k \leqslant r-1$ one has by acyclicity and because Γ_ϑ induces isomorphisms $C^k_+ \simeq C^{n-k}_-$ (since B_ϑ is invertible),

$$m^+_k = m^-_{n-k} = \dim\ker\left(\nabla|_{C^{n-k}}\right) = \dim\operatorname{ran}\left(\nabla|_{C^{n-k-1}}\right) = m_{n-k-1} - m^-_{n-k-1}$$

Therefore we get $m^+_k = m_{k+1} - m^+_{k+1}$, which reads

$$m^+_k + m^+_{k+1} = m_{k+1}, \quad 0 \leqslant k \leqslant r-1. \tag{6.4.33}$$

This leads to $m^+_k + m^+_{k+1} = m^0_k + m^0_{k+1}$. As a consequence, since $m^+_0 = m_0 = m^0_0$, we get

$$m^+_r - m^0_r = -(m^+_{r-1} - m^0_{r-1}) = \cdots = (-1)^r (m^+_0 - m^0_0) = 0.$$

This implies

$$m^0_k = m^+_k, \quad 0 \leqslant k \leqslant r, \tag{6.4.34}$$

which leads to $L = 0$. Moreover, since $m^0_k = m^0_{2r-k}$, we get

$$K = \sum_{k=0}^{r} (-1)^k (m^0_k - m^0_{k-1}) = \sum_{k=0}^{2r} (-1)^k m^0_k = -\sum_{k=0}^{n} (-1)^k k m_k = (-1)^q m(s_0),$$

where we used (6.3.9) in the last equality. Finally, again because $m^0_k = m^0_{2r-k}$,

$$2J = (-1)^r m^0_r + \sum_{k=0}^{2r} (-1)^k m^0_k = (-1)^r m^0_r - \sum_{k=0}^{n} (-1)^k k m_k.$$

We have

$$(-1)^r m_r^0 = \sum_{k=0}^{r} (-1)^k m_k,$$
$$\sum_{k=0}^{n} (-1)^k k m_k = \sum_{k=0}^{r} (-1)^k (2k-n) m_k,$$

where the first equality comes from (6.4.33) and (6.4.34) and the second from the fact that $m_k = m_{n-k}$. We thus obtained

$$J = \sum_{k=0}^{r} (-1)^k (r+1-k) m_k = Q_{s_0},$$

and finally by (6.4.32)

$$\tau(C^\bullet, \Gamma_\vartheta) = (-1)^{Q_{s_0}} (-s_0)^{(-1)^q m(s_0)}$$

But now recall from (6.4.4) that $\det_{\mathrm{gr}, C^\bullet} \left(\mathcal{L}_X^\nabla\right)^{(-1)^{q+1}} = (-s_0)^{m(s_0)}$. This completes the proof.

6.5 Invariance of the dynamical torsion under small perturbations of the contact form

In this section, we are interested in the behaviour of the dynamical torsion when we deform the contact form. Namely, we prove here the

Theorem 6.5.1. *Assume that $(\vartheta_t)_{t \in (-\delta, \delta)}$ is a smooth family of contact forms such that their Reeb vector fields X_t generate a contact Anosov flow for each t. Let (E, ∇) be an acyclic flat vector bundle. Then the map $t \mapsto \tau_{\vartheta_t}(\nabla)$ is real differentiable and we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \tau_{\vartheta_t}(\nabla) = 0.$$

Remark 6.5.2. In view of Remark 6.4.5, if ∇ is not assumed acyclic, then it is not hard to see that the proof (given below) of Theorem 6.5.1 is still valid and we have that $\partial_t \tau_{\vartheta_t}(\nabla) = 0$ in $\det H^\bullet(M, \nabla)$.

We will thus consider a family of contact forms and set $\vartheta = \vartheta_0$ and $X = X_0$. We also fix an acyclic flat vector bundle (E, ∇).

6.5.1 Anisotropic spaces for a family of vector fields

To study the dynamical torsion when the dynamics is perturbed, we construct with the help of [Bon20] some anisotropic Sobolev spaces on which each X_t has nice spectral properties. We refer to Section §6.11 where we briefly recall the construction of these spaces.

By §6.11.4, the set

$$\left\{(t, s),\ s \notin \mathrm{Res}(\mathcal{L}_{X_t}^\nabla)\right\}$$

is open in $(-\delta, \delta) \times \mathbb{C}$. Fix $\lambda \in (0,1)$ such that

$$\operatorname{Res}(\mathcal{L}_X^\nabla) \cap \{|s| \leqslant \lambda\} \subset \{0\}. \tag{6.5.1}$$

Then for t close enough to 0, we have $\operatorname{Res}(\mathcal{L}_{X_t}^\nabla) \cap \{|s| = \lambda\} = \emptyset$ so that the spectral projectors

$$\Pi_t = \frac{1}{2i\pi} \int_{|s|=\lambda} (\mathcal{L}_{X_t}^\nabla + s)^{-1} \mathrm{d}s : \Omega^\bullet(M,E) \to \mathcal{D}'^\bullet(M,E) \tag{6.5.2}$$

are well defined. The next proposition is a brief summary of the results from §6.11. We will denote for any $C, \rho > 0$,

$$\Omega(c,\rho) = \{\operatorname{Re}(s) > c\} \cup \{|s| \leqslant \rho\} \subset \mathbb{C}. \tag{6.5.3}$$

Proposition 6.5.3. *There is $c, \varepsilon_0 > 0$ such that for any $\rho > 0$ there exists anisotropic Sobolev spaces*

$$\Omega^\bullet(M,E) \subset \mathcal{H}_1^\bullet \subset \mathcal{H}^\bullet \subset \mathcal{D}'^\bullet(M,E),$$

each inclusion being continuous with dense image, such that the following holds.

1. *For each $t \in [-\varepsilon_0, \varepsilon_0]$, the family $s \mapsto \mathcal{L}_{X_t}^\nabla + s$ is a holomorphic family of (unbounded) Fredholm operators $\mathcal{H}_1^\bullet \to \mathcal{H}_1^\bullet$ and $\mathcal{H}^\bullet \to \mathcal{H}^\bullet$ of index 0 in the region $\Omega(c,\rho)$. Moreover*

$$\mathcal{L}_{X_t}^\nabla \in \mathcal{C}^1\Big([-\varepsilon_0, \varepsilon_0], \mathcal{L}(\mathcal{H}_1^\bullet, \mathcal{H}^\bullet)\Big).$$

2. *For every relatively compact open region $\mathcal{Z} \subset \operatorname{int} \Omega(c,\rho)$ such that $\operatorname{Res}(\mathcal{L}_X^\nabla) \cap \overline{\mathcal{Z}} = \emptyset$, there exists $t_\mathcal{Z} > 0$ such that*

$$\big(\mathcal{L}_{X_t}^\nabla + s\big)^{-1} \in \mathcal{C}^0\Big([-t_\mathcal{Z}, t_\mathcal{Z}]_t, \operatorname{Hol}\big(\mathcal{Z}_s, \mathcal{L}(\mathcal{H}_1^\bullet, \mathcal{H}^\bullet)\big)\Big).$$

3. $\Pi_t \in \mathcal{C}^1\Big([-\varepsilon_0, \varepsilon_0]_t, \mathcal{L}(\mathcal{H}^\bullet, \mathcal{H}_1^\bullet)\Big)$.

We will thus fix such Hilbert spaces for some $\rho > c+1$. We denote $C_t^\bullet = \operatorname{ran} \Pi_t \subset \mathcal{H}^\bullet$, $\Pi = \Pi_{t=0}$ and $C^\bullet = \operatorname{ran} \Pi$.

6.5.2 Variation of the torsion part

Let $\Gamma_t : C_t^\bullet \to C_t^{n-\bullet}$ be the chirality operator associated to X_t, c.f. §6.4.1. The next lemma allows us to compute the variation of the finite dimensional torsion part of the dynamical torsion.

Lemma 6.5.4. *We have that $t \mapsto \tau(C_t^\bullet, \Gamma_t)$ is real differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}t} \tau(C_t^\bullet, \Gamma_t) = -\operatorname{tr}_{\mathrm{s}, C_t^\bullet}\big(\Pi_t \vartheta_t \iota_{\dot{X}_t}\big) \tau(C_t^\bullet, \Gamma_t),$$

where $\dot{X}_t = \dfrac{\mathrm{d}}{\mathrm{d}t} X_t$.

Proof. By Proposition 6.5.3, the operator $\Pi_t|_{C^\bullet} : C^\bullet \to C_t^\bullet$ is invertible for t close enough to 0 and we will denote by Q_t its inverse. Then for t close enough to 0, one has

$$\tau(C_t^\bullet, \Gamma_t) = \tau(C^\bullet, \tilde{\Gamma}_t),$$

where $\tilde{\Gamma}_t = Q_t \Gamma_t \Pi_t|_{C^\bullet} : C^\bullet \to C^\bullet$ because ∇ and Π_t commute and the image of a $\tilde{\Gamma}_t$ invariant basis of $C^\bullet$ by the projector Π_t is a Γ_t invariant basis of $C_t^\bullet$.

Therefore [BK07c, Proposition 4.9]

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau(C_t^\bullet, \Gamma_t) = \frac{1}{2}\mathrm{tr}_{\mathrm{s},C^\bullet}\big(\dot{\tilde{\Gamma}}_t \tilde{\Gamma}_t\big)\tau(C_t^\bullet, \Gamma_t),$$

where $\dot{\tilde{\Gamma}}_t = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\Gamma}_t : C^\bullet \to C^\bullet$. Since Γ_t and Π_t commute, and by the two first points of Proposition 6.5.3, we can apply (6.10.2) to get

$$\tilde{\Gamma}_t = \Pi \Gamma_t \Pi|_{C^\bullet} + t\Pi\dot{\Gamma}\Pi + o_{C^\bullet \to C^\bullet}(t).$$

This leads to

$$\dot{\tilde{\Gamma}}\tilde{\Gamma} = \Pi\dot{\Gamma}\Gamma|_{C^\bullet},$$

where we removed the subscripts t to signify that we take all the t-dependent objects at $t = 0$. Therefore,

$$\frac{1}{2}\mathrm{tr}_{\mathrm{s},C^\bullet}\left(\dot{\tilde{\Gamma}}\tilde{\Gamma}\right) = \frac{1}{2}\mathrm{tr}_{\mathrm{s},C^\bullet}\left(\Pi\dot{\Gamma}\Gamma\right),$$

Now notice that $\Gamma_t^2 = 1$ implies $\Gamma\dot{\Gamma} + \dot{\Gamma}\Gamma = 0$. Therefore, for every $k \in \{0, \dots, r\}$,

$$\mathrm{tr}_{C^{n-k}}\, \Gamma\dot{\Gamma} = \mathrm{tr}_{C^k}\, \Gamma\Gamma\dot{\Gamma}\Gamma = \mathrm{tr}_{C^k}\, \dot{\Gamma}\Gamma = -\,\mathrm{tr}_{C^k}\, \Gamma\dot{\Gamma}.$$

Therefore we only need to compute $\mathrm{tr}_{C^k}\left(\Gamma\dot{\Gamma}\right)$ for $k \in \{0, \dots, r\}$ to get the full super trace $\mathrm{tr}_{\mathrm{s},C^\bullet}\left(\dot{\Gamma}\Gamma\right)$. Since n is odd we have

$$\frac{1}{2}\mathrm{tr}_{\mathrm{s},C^\bullet}\left(\dot{\tilde{\Gamma}}\tilde{\Gamma}\right) = \frac{1}{2}\,\mathrm{tr}_{C^\bullet}\left((-1)^{N+1}\Pi\Gamma\dot{\Gamma}\right) = \sum_{k=0}^{r}(-1)^{k+1}\,\mathrm{tr}_{C^k}\left(\Pi\Gamma\dot{\Gamma}\right).$$

Let $k \in \{0, \dots, r\}$ and $\alpha \in \Omega^k(M)$. Using the decomposition

$$\alpha = (-1)^{k-1}\iota_{X_t}\alpha \wedge \vartheta_t + \big(\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t\big),$$

we get by definition of Γ_t

$$\Gamma_t\alpha = (-1)^{k-1}\iota_{X_t}\alpha \wedge (\mathrm{d}\vartheta_t)^{r-k+1} + \big(\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t\big) \wedge (\mathrm{d}\vartheta_t)^{r-k} \wedge \vartheta_t.$$

Therefore,

$$\begin{aligned}\dot{\Gamma}_t\alpha = {} & (-1)^{k-1}\iota_{\dot{X}_t}\alpha \wedge (\mathrm{d}\vartheta_t)^{r-k+1} \\ & + (r-k+1)(-1)^{k-1}\iota_{X_t}\alpha \wedge \mathrm{d}\dot{\vartheta}_t \wedge (\mathrm{d}\vartheta_t)^{r-k} \\ & + (-1)^k\Big(\iota_{\dot{X}_t}\alpha \wedge \vartheta_t + \iota_{X_t}\alpha \wedge \dot{\vartheta}_t\Big) \wedge (\mathrm{d}\vartheta_t)^{r-k} \wedge \vartheta_t \\ & + \Big(\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t\Big) \wedge (\mathrm{d}\vartheta_t)^{r-k} \wedge \dot{\vartheta}_t \\ & + (r-k)\Big(\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t\Big) \wedge \mathrm{d}\dot{\vartheta}_t \wedge (\mathrm{d}\vartheta_t)^{r-k-1} \wedge \vartheta_t\end{aligned}$$

Now we use the decompositions

$$\begin{aligned} \mathrm{d}\dot{\vartheta}_t &= -\iota_{X_t}\mathrm{d}\dot{\vartheta}_t \wedge \vartheta_t + \left(\mathrm{d}\dot{\vartheta}_t + \iota_{X_t}\mathrm{d}\dot{\vartheta}_t \wedge \vartheta_t\right), \\ \dot{\vartheta}_t &= \dot{\vartheta}_t(X_t)\vartheta + \left(\dot{\vartheta}_t - \dot{\vartheta}_t(X_t)\vartheta\right), \\ \iota_{\dot{X}_t}\alpha &= (-1)^k \iota_{X_t}\iota_{\dot{X}_t}\alpha \wedge \vartheta_t + \left(\iota_{\dot{X}_t}\alpha + (-1)^{k+1}\iota_{X_t}\iota_{\dot{X}_t}\alpha \wedge \vartheta_t\right) \end{aligned}$$

to get, again by definition,

$$\begin{aligned} \Gamma\dot{\Gamma}\alpha = {} & (-1)^{k-1}\left(\iota_{\dot{X}}\alpha + (-1)^{k+1}\iota_X\iota_{\dot{X}}\alpha \wedge \vartheta\right) \wedge \vartheta \\ & + (-1)^{k-1}\left(\mathscr{L}^{r-k}\right)^{-1}\left((-1)^k\iota_X\iota_{\dot{X}}\alpha \wedge (\mathrm{d}\vartheta)^{r-k+1}\right) \\ & + (r-k+1)\left(\mathscr{L}^{r-k+1}\right)^{-1}\left((-1)^{k-1}\iota_X\alpha \wedge \left(\mathrm{d}\dot{\vartheta} + \iota_X\mathrm{d}\dot{\vartheta} \wedge \vartheta\right) \wedge (\mathrm{d}\vartheta)^{r-k}\right) \wedge \vartheta \\ & - (r-k+1)\left((-1)^{k-1}\iota_X\alpha\right) \wedge \iota_X\mathrm{d}\dot{\vartheta} \\ & + (-1)^k\iota_X\alpha \wedge \left(\dot{\vartheta} - \dot{\vartheta}(X)\vartheta\right) \\ & + \left(\mathscr{L}^{r-k+1}\right)^{-1}\left(\left(\alpha + (-1)^k\iota_X\alpha \wedge \vartheta\right) \wedge (\mathrm{d}\vartheta)^{r-k} \wedge \left(\dot{\vartheta} - \dot{\vartheta}(X)\vartheta\right)\right) \wedge \vartheta \\ & + \left(\alpha + (-1)^k\iota_X\alpha \wedge \vartheta\right)\dot{\vartheta}(X) \\ & + (r-k)\left(\mathscr{L}^{r-k}\right)^{-1}\left(\left(\alpha + (-1)^k\iota_X\alpha \wedge \vartheta\right) \wedge \left(\mathrm{d}\dot{\vartheta} + \iota_X\mathrm{d}\dot{\vartheta} \wedge \vartheta\right) \wedge (\mathrm{d}\vartheta)^{r-k-1}\right), \end{aligned} \tag{6.5.4}$$

where again we removed the subscripts t to signify that we take everything at $t = 0$. Now let $A_k : C_0^k \to C_0^k$ (note that here C_0^k is $C^k \cap \ker \iota_X$, cf §6.4.1, and not C_t^k at $t = 0$) defined by

$$A_k u = (r-k)\left(\mathscr{L}^{r-k}\right)^{-1}\left(u \wedge \left(\mathrm{d}\dot{\vartheta} + \iota_X\mathrm{d}\dot{\vartheta}\right) \wedge (\mathrm{d}\vartheta)^{r-k-1}\right).$$

Note that the maps defined by the second, the fourth, the fifth and the sixth terms of the right hand side of (6.5.4) are anti-diagonal, that is they have the form $\begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}$ in the decomposition $C^\bullet = C_0^{\bullet-1} \wedge \vartheta \oplus C_0^\bullet$. Therefore, since $A_r = 0$ (we also set $A_{-1} = 0$),

$$\begin{aligned} \sum_{k=0}^{r}(-1)^{k+1}\operatorname{tr}_{C^k}\left(\Pi\Gamma\dot{\Gamma}\right) &= \sum_{k=0}^{r}(-1)^{k+1}\left(\operatorname{tr}_{C^k}\Pi\vartheta\iota_{\dot{X}} + \operatorname{tr}_{C_0^k}\Pi\dot{\vartheta}(X)\right) \\ &\quad + \sum_{k=0}^{r}(-1)^{k+1}\left(\operatorname{tr}_{C_0^{k-1}}\Pi A_{k-1} + \operatorname{tr}_{C_0^k}\Pi A_k\right) \\ &= \sum_{k=0}^{r}(-1)^{k+1}\left(\operatorname{tr}_{C^k}\Pi\vartheta\iota_{\dot{X}} + \operatorname{tr}_{C_0^k}\Pi\dot{\vartheta}(X)\right). \end{aligned} \tag{6.5.5}$$

But now note that if $\alpha = f \wedge \vartheta + g \in C_0^{k-1} \wedge \vartheta \oplus C_0^k$ then

$$\vartheta \wedge \iota_{\dot{X}}\alpha = \vartheta(\dot{X})(f \wedge \vartheta) + \vartheta \wedge \iota_{\dot{X}}g.$$

This shows that for every $k \in \{0, \ldots, n\}$ one has

$$\operatorname{tr}_{C^k}\Pi\vartheta\iota_{\dot{X}} = \operatorname{tr}_{C_0^{k-1}}\Pi\vartheta(\dot{X}). \tag{6.5.6}$$

Injecting this relation in (6.5.5) we obtain, with $\vartheta(\dot X) = -\dot\vartheta(X)$ and the formula $\dot\vartheta(X)|_{C_0^{2r-k}}\mathscr{L}^{r-k} = \mathscr{L}^{r-k}\dot\vartheta(X)|_{C_0^k}$,

$$\sum_{k=0}^{r}(-1)^{k+1}\operatorname{tr}_{C^k}\left(\Pi\Gamma\dot\Gamma\right) = \sum_{k=0}^{r}(-1)^{k+1}\left(\operatorname{tr}_{C_0^{k-1}}\Pi\vartheta(\dot X) - \operatorname{tr}_{C_0^k}\Pi\vartheta(\dot X)\right)$$
$$= \sum_{k=0}^{2r}(-1)^k\operatorname{tr}_{C_0^k}\Pi\vartheta(\dot X).$$

However by (6.5.6) we have

$$\sum_{k=0}^{2r}(-1)^k\operatorname{tr}_{C_0^k}\Pi\vartheta(\dot X) = \operatorname{tr}_{C^\bullet}\left((-1)^{N+1}\Pi\vartheta\iota_{\dot X}\right),$$

which concludes the proof. □

6.5.3 Variation of the rest

Let us now interest ourselves in the variation of $t \mapsto \zeta^{(\lambda,\infty)}_{X_t,\nabla}(0)$, cf. §6.4.3. For t close enough to 0, let $P_t : TM \to TM$ be defined by

$$\begin{array}{cccccccccc} P_t : & \ker\vartheta & \oplus & \mathbb{R}X & \to & \ker\vartheta & \oplus & \mathbb{R}X_t, \\ & v & + & \mu X & \mapsto & v & + & \mu X_t. \end{array}$$

For simplicity, we will still denote $\wedge^k({}^TP_t) : \wedge^k T^*M \to \wedge^k T^*M$ by P_t. Then formula (5.4) of [DGRS18] gives that for $\operatorname{Re}(s)$ big enough, $t \mapsto \zeta_{X_t,\nabla}(s)$ is differentiable and we have for every $\varepsilon > 0$ small enough

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\log\zeta_{X,\nabla}(s) = (-1)^q s\ \operatorname{tr}_{\mathrm{s}}^\flat\left(\dot P(\mathcal{L}_X^\nabla + s)^{-1}\mathrm{e}^{-\varepsilon(\mathcal{L}_X^\nabla + s)}\right),$$

where $\dot P = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}P_t$. One can show that for every $k \in \{0,\dots,n\}$ and $\beta \in \wedge^k T^*M$ one has

$$\dot P\beta = \vartheta\wedge\iota_{\dot X}\beta. \tag{6.5.7}$$

Therefore (we differentiated at $t = 0$ but we can do the same for small t)

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\zeta_{X_t,\nabla}(s) = (-1)^q s\ \operatorname{tr}_{\mathrm{s}}^\flat\left(\vartheta_t\iota_{\dot X_t}(\mathcal{L}_{X_t}^\nabla + s)^{-1}\mathrm{e}^{-\varepsilon(\mathcal{L}_{X_t}^\nabla + s)}\right). \tag{6.5.8}$$

Now let us compute the variation of the $[0,\lambda]$ part of $\zeta^{(\lambda,\infty)}(s)$.

Lemma 6.5.5. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\det\nolimits_{\mathrm{gr},C_t^\bullet}\left(\mathcal{L}_{X_t}^\nabla + s\right)^{(-1)^{q+1}} = (-1)^{q+1}\operatorname{tr}_{\mathrm{s},C_t^\bullet}\left(\vartheta_t\iota_{\dot X_t}\mathcal{L}_{X_t}^\nabla(\mathcal{L}_{X_t}^\nabla + s)^{-1}\right).$$

Proof. We are in a position to apply Lemma 6.10.2 which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\det\nolimits_{\mathrm{gr},C_t^\bullet}\left(\mathcal{L}_{X_t}^\nabla + s\right)^{(-1)^{q+1}} = (-1)^{q+1}\operatorname{tr}_{\mathrm{gr},C_t^\bullet}\left(\Pi_t\dot{\mathcal{L}}_{X_t}^\nabla(\mathcal{L}_{X_t}^\nabla + s)^{-1}\right).$$

Denote $A_t = P_t^{-1}\dot{P}_t$. Then one can verify that

$$\iota_{X_t} = P_t^{-1}\iota_X P_t,$$

which leads to

$$\mathcal{L}^{\nabla}_{\dot{X}_t} = -\nabla A_t \iota_{X_t} + \nabla \iota_{X_t} A_t - A_t \iota_{X_t}\nabla + \iota_{X_t} A_t \nabla.$$

Using

$$\begin{aligned}(-1)^N N\nabla &= \nabla(-1)^{N+1}(N+1),\\ (-1)^N N\iota_{X_t} &= \iota_{X_t}(-1)^{N-1}(N-1),\end{aligned}$$

and the cyclicity of the trace, we get since $(\mathcal{L}^{\nabla}_{X_t}+s)^{-1}$ commute with ι_{X_t} and ∇,

$$\begin{aligned}&\operatorname{tr}_{C_t^\bullet}\Big((-1)^{N+q+1}N\Pi_t\mathcal{L}^{\nabla}_{\dot{X}_t}(\mathcal{L}^{\nabla}_{X_t}+s)^{-1}\Big)\\ &\quad= (-1)^{q+1}\operatorname{tr}_{C_t^\bullet}\Big(\Pi_t A_t\Big((-1)^N(N+1)\iota_{X_t}\nabla + (-1)^N N\nabla\iota_{X_t}\\ &\qquad\qquad -(-1)^N N\iota_{X_t}\nabla - (-1)^N(N-1)\nabla\iota_{X_t}\Big)(\mathcal{L}^{\nabla}_{X_t}+s)^{-1}\Big)\\ &\quad= (-1)^{q+1}\operatorname{tr}_{C_t^\bullet}\Big((-1)^N\Pi_t A_t\mathcal{L}^{\nabla}_{X_t}(\mathcal{L}^{\nabla}_{X_t}+s)^{-1}\Big).\end{aligned}$$

Therefore we conclude the proof by using (6.5.7) again because $P_{t=0} = \mathrm{Id}$. □

6.5.4 Proof of Theorem 6.5.1

Combining this lemma and (6.5.8) we obtain that for $\mathrm{Re}(s)$ big enough and t small enough

$$\frac{\zeta^{(\lambda,\infty)}_{X_t,\nabla}(s)}{\zeta^{(\lambda,\infty)}_{X_0,\nabla}(s)} = \exp\Bigg(-s\int_0^t \operatorname{tr}^\flat_{\mathrm{s}}\left(\vartheta_\tau\iota_{\dot{X}_\tau}(\mathcal{L}^{\nabla}_{X_\tau}+s)^{-1}\mathrm{e}^{-\varepsilon(\mathcal{L}_{X_\tau}+s)}\right)\mathrm{d}\tau \\ -\int_0^t \operatorname{tr}_{\mathrm{s},C_\tau^\bullet}\left(\Pi_\tau\vartheta_\tau\iota_{\dot{X}_\tau}\mathcal{L}^{\nabla}_{X_\tau}(\mathcal{L}^{\nabla}_{X_\tau}+s)^{-1}\right)\mathrm{d}\tau\Bigg)^{(-1)^{q+1}}. \tag{6.5.9}$$

Note that for every $s \notin \mathrm{Res}(\mathcal{L}^{\nabla}_{X_t})$ we have

$$\mathcal{L}^{\nabla}_{X_t}(\mathcal{L}^{\nabla}_{X_t}+s)^{-1} = \mathrm{Id} - s(\mathcal{L}^{\nabla}_{X_t}+s)^{-1},$$

so that

$$\operatorname{tr}_{\mathrm{s},C_t^\bullet}\Pi_t\vartheta_t\iota_{\dot{X}_t}\mathcal{L}^{\nabla}_{X_t}(\mathcal{L}^{\nabla}_{X_t}+s)^{-1} = \operatorname{tr}_{\mathrm{s},C_t^\bullet}\Pi_t\vartheta_t\iota_{\dot{X}_t} - s\operatorname{tr}_{\mathrm{s},C_t^\bullet}\Pi_t\vartheta_t\iota_{\dot{X}_t}(\mathcal{L}^{\nabla}_{X_t}+s)^{-1}. \tag{6.5.10}$$

We now fix $s_0 \in \mathbb{C}$ with $\mathrm{Re}(s_0)$ big enough so that (6.5.9) is valid and a smooth path $c : [0,1] \to \mathbb{C}$ with $c(0) = 0$, $c(1) = s_0$ and

$$c(u) \notin \mathrm{Res}(\mathcal{L}^{\nabla}_X), \quad u \in (0,1].$$

Let $\delta, t_0 > 0$ small enough so that

$$\operatorname{dist}\Big(\{|s| = \lambda\} \cup (V_\delta \cap \{|s| \geqslant \lambda\}),\ \operatorname{Res}(\mathcal{L}^\nabla_{X_t})\Big) \geqslant 2\delta, \quad |t| \leqslant t_0, \tag{6.5.11}$$

where V_δ is the open δ-neighborhood of $\operatorname{Im} c$. We moreover ask that

$$(\operatorname{Res}(\mathcal{L}^\nabla_{X_t}) \cap \{|s| \leqslant \lambda\}) \subset \{|s| \leqslant \delta\} \ \text{ and } \ (V_\delta \cap \{|s| \geqslant \lambda\}) \cap \operatorname{Res}(\mathcal{L}^\nabla_{X_t}) = \emptyset.$$

For $t \in [-t_0, t_0]$ and $s \notin \operatorname{Res}(\mathcal{L}^\nabla_{X_t})$ we define

$$Y_t(s) = \left(\mathcal{L}^\nabla_{X_t} + s\right)^{-1} (\operatorname{Id} - \Pi_t). \tag{6.5.12}$$

Then by (8.2.10), we have that $s \mapsto Y_t(s)$ is holomorphic on a neighborhood of $\{|s| \leqslant \lambda\}$ for each fixed t. This implies

$$Y_t(s) = \sum_{n=0}^{\infty} Y_{t,n} s^n, \quad |s| < \lambda, \quad |t| \leqslant t_0, \tag{6.5.13}$$

with

$$Y_{t,n} = \frac{1}{2i\pi} \int_{|s|=\lambda} Y_t(s) s^{-n-1} \mathrm{d}s. \tag{6.5.14}$$

Therefore, for every $|t| \leqslant t_0$ one has $\|Y_{t,n}\|_{\mathcal{H} \to \mathcal{H}} \leqslant 2\delta \wedge^{-n-1}$ by (6.5.11) and Proposition 6.5.3.

Let $\mathcal{Q}_t(s)$ denote the Schwartz Kernel of the operator

$$Q_t(s) = \left(\mathcal{L}^\nabla_{X_t} + s\right)^{-1} \mathrm{e}^{-\varepsilon\left(\mathcal{L}^\nabla_{X_t} + s\right)}.$$

Then [DGRS18, Proposition 6.3] gives that the map

$$[-t_0, t_0] \times \{|s| = \lambda\} \ni (t, s) \mapsto \mathcal{Q}_t(s) \in \mathcal{D}'^n_\Gamma(M \times M, E^\vee \boxtimes E)$$

is bounded for some closed conic subset $\Gamma \subset T^*(M \times M)$ not intersecting the conormal of the diagonal. Moreover by §6.11.7, we have that $[-t_0, t_0] \ni t \mapsto \Pi_t$ is bounded in $\mathcal{D}'^n_{W_s \times W_u}(M \times M, E^\vee \boxtimes E)$, and so is the map $[-t_0, t_0] \times \{|s| = \lambda\} \mapsto \left(\mathcal{L}^\nabla_{X_t} + s\right)^{-1} \Pi_t$. As a consequence (6.5.12), (6.5.13) and (6.5.14) imply that the map

$$[-t_0, t_0] \times \{|s| \leqslant 3\delta/2\} \ni (t, s) \mapsto \mathcal{Y}_t(s) \in \mathcal{D}'^n_\Gamma(M \times M, E^\vee \boxtimes E), \tag{6.5.15}$$

is bounded, where $\mathcal{Y}_t(s)$ is the Schwartz kernel of the operator $Y_t(s)\mathrm{e}^{-\varepsilon\left(\mathcal{L}^\nabla_{X_t} + s\right)}$. We also know that this map is continuous when it is seen as a map valued in $\mathcal{D}'^n$ thanks to the last point of Proposition 6.5.3 ; therefore this map is continuous when valued in $\mathcal{D}'^n_\Gamma(M \times M, E^\vee \boxtimes E$, cf. [Hör90, §8.4]. Therefore we obtain with §B.3.1 that

$$\operatorname{tr}^\flat_{\mathrm{s}} \vartheta \iota_{\dot{X}_t} Y_t(s) \in \mathcal{C}^0\Big([-t_0, t_0], \operatorname{Hol}\big(\{|s| \leqslant 3\delta/2\}\big)\Big). \tag{6.5.16}$$

But now apply [DGRS18, Theorem 4] to obtain that

$$\operatorname{tr}^\flat_{\mathrm{s}} \vartheta \iota_{\dot{X}_t} Q_t(s) \in \mathcal{C}^0\Big([-t_0, t_0], \operatorname{Hol}\big(V_\delta \cap \{|s| \geqslant 5\delta/4\}\big)\Big). \tag{6.5.17}$$

Since the flat trace coincides with the usual trace for operators of finite rank,

$$\begin{aligned}\operatorname{tr}_{\mathrm{s}}^{\flat} \vartheta_t \iota_{\dot{X}_t} Q_t(s) - \operatorname{tr}_{\mathrm{s},C^\bullet} \Pi_t \vartheta_t \iota_{\dot{X}_t} (\mathcal{L}_{X_t}^{\nabla} + s)^{-1} \\ = \operatorname{tr}_{\mathrm{s}}^{\flat} \vartheta_t \iota_{\dot{X}_t} \left(\mathcal{L}_{X_t}^{\nabla} + s\right)^{-1} (\mathrm{Id} - \Pi_t) \mathrm{e}^{-\varepsilon(\mathcal{L}_{X_t}^{\nabla}+s)} \\ + \operatorname{tr}_{\mathrm{s},C^\bullet} \Pi_t \vartheta_t \iota_{\dot{X}_t} (\mathcal{L}_{X_t}^{\nabla} + s)^{-1} \left(\mathrm{e}^{-\varepsilon(\mathcal{L}_{X_t}^{\nabla}+s)} - \mathrm{Id}\right).\end{aligned}$$

Then (6.5.16), (6.5.17) and (6.5.12) imply that the right hand side of the last equation is continuous with respect to t with values in holomorphic functions on $(V_\delta \cap \{|s| \geqslant 5\delta/4\}) \cup \{|s| \leqslant 3\delta/2\}$ (indeed $s \mapsto (\mathcal{L}_{X_t}^{\nabla} + s)^{-1}\left(\mathrm{e}^{-\varepsilon(\mathcal{L}_{X_t}^{\nabla}+s)} - \mathrm{Id}\right)$ is holomorphic of $C_t^\bullet$), and so is the left hand side. As a consequence, (6.5.10) shows that both members of (6.5.9) are holomorphic on this region and

$$\zeta_{X_t,\nabla}^{(\lambda,\infty)}(0) = \zeta_{X_0,\nabla}^{(\lambda,\infty)}(0) \exp\left(-\int_0^t \operatorname{tr}_{\mathrm{s},C_\tau^\bullet} \Pi_\tau \vartheta \iota_{\dot{X}_\tau} \mathrm{d}\tau\right)^{(-1)^{q+1}}.$$

Comparing this with Lemma 6.5.4 we obtain Theorem 6.5.1 by definition of the dynamical torsion, cf §6.4.4.

6.6 Variation of the connection

In this section we compute the variation of the dynamical torsion when the connection is perturbed. This formula will be crucial to compare the dynamical torsion and Turaev's refined combinatorial torsion.

6.6.1 Real-differentiable families of flat connections

Let $U \subset \mathbb{C}$ be some open set and consider $\nabla(z)$, $z \in U$, a family of flat connections on E. We will assume that the map $z \mapsto \nabla(z)$ is $\mathcal{C}^1$, that is, there exists continuous maps $z \mapsto \mu_z, \nu_z \in \Omega^1(M, \mathrm{End}(E))$ such that for any $z_0 \in U$ one has

$$\nabla(z) = \nabla(z_0) + \mathrm{Re}(z - z_0)\mu_{z_0} + \mathrm{Im}(z - z_0)\nu_{z_0} + o(z - z_0), \tag{6.6.1}$$

where $o(z - z_0)$ is understood in the Fréchet topology of $\Omega^1(M, \mathrm{End}(E))$. We will denote for any $\sigma \in \mathbb{C}$

$$\alpha_{z_0}(\sigma) = \mathrm{Re}(\sigma)\mu_{z_0} + \mathrm{Im}(\sigma)\nu_{z_0} \in \Omega^1(M, \mathrm{End}(E)). \tag{6.6.2}$$

Note that since the connections $\nabla(z)$ are assumed to be flat, we have

$$[\nabla(z), \alpha_z(\sigma)] = \nabla(z)\alpha_z(\sigma) + \alpha_z(\sigma)\nabla(z) = 0. \tag{6.6.3}$$

6.6.2 A cochain contraction induced by the Anosov flow

For $z \in U$ let

$$\left(\mathcal{L}_X^{\nabla(z)} + s\right)^{-1} = \sum_{j=1}^{J(0)} \frac{\left(-\mathcal{L}_X^{\nabla(z)}\right)^{j-1} \Pi_0(z)}{s^j} + Y(z) + \mathcal{O}(s) \tag{6.6.4}$$

be the development (8.2.10) for the resonance $s_0 = 0$. Let $C^\bullet(0;z) = \operatorname{ran} \Pi_0(z)$. Recall from §6.3.6 that since $\nabla(z)$ is acyclic, the complex $(C^\bullet(0;z), \nabla(z))$ is acyclic. Therefore there exists a cochain contraction $k(z) : C^\bullet(0;z) \to C^\bullet(0;z)$, i.e. a map of degree -1 such that

$$\nabla(z)k(z) + k(z)\nabla(z) = \mathrm{Id}_{C^\bullet(0;z)} \,. \tag{6.6.5}$$

We now define

$$K(z) = \iota_X Y(z)(\mathrm{Id} - \Pi_0(z)) + k(z)\Pi_0(z) \ : \ \Omega^\bullet(M,E) \to \mathcal{D}'^\bullet(M,E). \tag{6.6.6}$$

A crucial property of the operator K is that it satisfies the chain homotopy equation

$$\nabla(z)K(z) + K(z)\nabla(z) = \mathrm{Id}_{\Omega^\bullet(M,E)}, \tag{6.6.7}$$

as follows from the development (6.6.4).

6.6.3 The variation formula

For simplicity, we will set for every $z \in U$

$$\tau(z) = \tau_\vartheta(\nabla(z)).$$

The operators $K(z)$ defined above are involved in the variation formula of the dynamical torsion, as follows.

Proposition 6.6.1. *The map $z \mapsto \tau(z)$ is real differentiable ; we have for every $z \in U$ and $\varepsilon > 0$ small enough*

$$\mathrm{d}(\log \tau)_z \sigma = -\operatorname{tr}_{\mathrm{s}}^\flat \left(\alpha_z(\sigma) K(z) \mathrm{e}^{-\varepsilon \mathcal{L}_X^{\nabla(z)}} \right), \quad \sigma \in \mathbb{C}. \tag{6.6.8}$$

The proof of the previous proposition is similar of that of the last subsection, i.e. we compute the variation of each part of the dynamical torsion. The rest of this section is devoted to the proof of Proposition 6.6.1.

6.6.4 Anisotropic Sobolev spaces for a family of connections

Fix some $z_0 \in U$. Recall from §6.5.1 that we chose some anisotropic Sobolev spaces $\mathcal{H}_1^\bullet \subset \mathcal{H}^\bullet$. Notice that

$$\mathcal{L}_X^{\nabla(z)} = \mathcal{L}_X^{\nabla(z_0)} + \beta(z)(X), \tag{6.6.9}$$

where $\beta(z) \in \Omega^1(M, \mathrm{End}(E))$ is defined by

$$\nabla(z) = \nabla(z_0) + \beta(z).$$

Therefore (6.6.1) implies that $\mathcal{L}_X^{\nabla(z)} - \mathcal{L}_X^{\nabla(z_0)}$ is a $\mathcal{C}^1$ family of pseudo-differential operators of order 0, and thus forms a $\mathcal{C}^1$ family of bounded operators $\mathcal{H}^\bullet \to \mathcal{H}^\bullet$ and $\mathcal{H}_1^\bullet \to \mathcal{H}_1^\bullet$ by construction of the anisotropic spaces and standard rules of pseudo-differential calculus (see for example [FS11]). As a consequence and thanks to Proposition 6.5.3, we are in position to apply [Kat76, Theorem 3.11] ; thus if δ is small enough we have that

$$R_\rho = \left\{ (z,s) \in \mathbb{C}^2, \ |z - z_0| < \delta, \ s \in \Omega(c,\rho), \ s \notin \sigma_{\mathcal{H}^\bullet}(\mathcal{L}_X^{\nabla(z)}) \right\} \text{ is open,} \tag{6.6.10}$$

where $\sigma_{\mathcal{H}^\bullet}(\mathcal{L}_X^{\nabla(z)})$ denotes the resolvent set of $\mathcal{L}_X^{\nabla(z)}$ on $\mathcal{H}^\bullet$, and $\Omega(c,\rho)$ is defined in (6.5.3). Moreover (6.6.1) and (6.6.9) imply that for any open set $\mathcal{Z} \subset \Omega(c,\rho)$ such that $\operatorname{Res}\left(\mathcal{L}_X^{\nabla(z_0)}\right) \cap \overline{\mathcal{Z}} = \emptyset$, there exists $\delta_{\mathcal{Z}} > 0$ such that for any $j \in \{0,1\}$,

$$\left(\mathcal{L}_X^{\nabla(z)} + s\right)^{-1} \in \mathcal{C}^1\Big(\big\{|z-z_0| < \delta_{\mathcal{Z}}\big\},\ \operatorname{Hol}\big(\mathcal{Z}_s, \mathcal{L}\left(\mathcal{H}_j^\bullet, \mathcal{H}_j^\bullet\right)\big)\Big). \tag{6.6.11}$$

For all z, the map $s \mapsto \left(\mathcal{L}_X^{\nabla(z)} + s\right)^{-1}$ is meromorphic in the region $\Omega(c,\rho)$ with poles (of finite multiplicity) which coincide with the resonances of $\mathcal{L}_X^{\nabla(z)}$ in this region.

Moreover, the arguments from the proof of [DZ16, Proposition 3.4] can be made uniformly for the family $z \mapsto \left(\mathcal{L}_X^{\nabla(z)} + s\right)^{-1}$ to obtain that for some closed conic set $\Gamma \subset T^*(M \times M)$ not intersecting the conormal to the diagonal and any $\varepsilon > 0$ small enough, the map $(s,z) \mapsto \mathcal{K}(s,z)$ is bounded from $\mathcal{Z} \times \{|z-z_0| < \delta_{\mathcal{Z}}\}$ with values $\mathcal{D}'_\Gamma(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E)$, where $\mathcal{K}(s,z)$ is the Schwartz kernel of the shifted resolvent $\left(\mathcal{L}_X^{\nabla(z)} + s\right)^{-1} \mathrm{e}^{-\varepsilon \mathcal{L}_X^{\nabla(z)}}$.

6.6.5 A family of spectral projectors

Fix $\lambda \in (0,1)$ such that

$$\{s \in \mathbb{C},\ |s| \leqslant \lambda\} \cap \operatorname{Res}\left(\mathcal{L}_X^{\nabla(z_0)}\right) \subset \{0\}. \tag{6.6.12}$$

Thanks to (6.6.10), if z is close enough to z_0,

$$\{s \in \mathbb{C},\ |s| = \lambda\} \cap \operatorname{Res}\left(\mathcal{L}_X^{\nabla(z)}\right) = \emptyset, \tag{6.6.13}$$

by compacity of the circle. For $z \in U$ we will denote by

$$\Pi(z) = \frac{1}{2i\pi} \int_{|s|=\lambda} \left(\mathcal{L}_X^{\nabla(z)} + s\right)^{-1} \mathrm{d}s \tag{6.6.14}$$

the spectral projector of $\mathcal{L}_X^{\nabla(z)}$ on generalized eigenvectors for resonances in $\{s \in \mathbb{C},\ |s| \leqslant \lambda\}$, and $C^\bullet(z) = \operatorname{ran} \Pi(z)$. It follows from (6.6.11), (6.6.13) and (6.6.14) that the map

$$z \mapsto \Pi(z) \in \mathcal{L}(\mathcal{H}_j^\bullet, \mathcal{H}_j^\bullet)$$

is $\mathcal{C}^1$ for $j = 0,1$. We can therefore apply 6.10.3 to get, for δ small enough,

$$\Pi(z) \in \mathcal{C}^1\Big(\{|z-z_0| < \delta\}_z,\ \mathcal{L}(\mathcal{H}^\bullet, \mathcal{H}_1^\bullet)\Big). \tag{6.6.15}$$

6.6.6 Variation of the finite dimensional part

Because $(C^\bullet(z_0), \nabla(z_0))$ is acyclic, there exists a cochain contraction $k(z_0) : C^\bullet(z_0) \to C^{\bullet-1}(z_0)$, cf §6.2.6. The next lemma computes the variation of the finite dimensional part of the dynamical torsion.

Lemma 6.6.2. *The map $z \mapsto c(z) = \tau(C^\bullet(z), \Gamma)$ is real differentiable at $z = z_0$ and*

$$\mathrm{d}(\log c)_{z_0}\sigma = -\mathrm{tr}_{\mathrm{s},C^\bullet}\Pi(z_0)\alpha_{z_0}(\sigma)k(z_0), \quad \sigma \in \mathbb{C}.$$

Proof. By continuity of the family $z \mapsto \Pi(z)$, we have that $\Pi(z)|_{C^\bullet(z_0)} : C^\bullet(z_0) \to C^\bullet(z)$ is an isomorphism for $|z - z_0|$ small enough, of inverse denoted by $Q(z)$. For those z we denote by $\widehat{C}^\bullet(z)$ the graded vector space $C^\bullet(z_0)$ endowed with the differential

$$\widehat{\nabla}(z) = Q(z)\nabla(z)\Pi(z) : C^\bullet(z_0) \to C^\bullet(z_0).$$

Then because Γ commutes with every $\Pi(z)$ one has

$$\tau(\widehat{C}^\bullet(z), \Gamma) = \tau(C^\bullet(z), \Gamma) \tag{6.6.16}$$

By (6.6.15) we can apply (6.10.2) in the proof of Lemma 6.10.2 which gives for any h small enough

$$\widehat{\nabla}(z_0 + \sigma)\Pi(z_0) = \Pi(z_0)\nabla(z_0)\Pi(z_0) + \Pi(z_0)\alpha_{z_0}(\sigma)\Pi(z_0) + o_{C^\bullet(z_0)\to C^\bullet(z_0)}(\sigma).$$

Therefore the real differentiable version of Lemma 6.2.5 implies the desired result. □

6.6.7 Variation of the zeta part

We give a first Proposition which computes the variation of the Ruelle zeta function in its convergence region.

Proposition 6.6.3 (Variation of the dynamical zeta function)**.** *For* $\mathrm{Re}(s)$ *big enough, the map* $z \mapsto g_s(z) = \zeta_{X,\nabla(z)}(s)$ *is* $\mathcal{C}^1$ *near* $z = z_0$ *and we have for every* $\varepsilon > 0$ *small enough*

$$\mathrm{d}(\log g_s)_{z_0}\sigma = (-1)^{q+1}\mathrm{e}^{-\varepsilon s}\,\mathrm{tr}_\mathrm{s}^\flat\left(\alpha_{z_0}(\sigma)\iota_X\left(\mathcal{L}_X^{\nabla(z_0)} + s\right)^{-1}\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z_0)}}\right).$$

Proof. Let φ^t denote the flow of X. For $\gamma \in \mathcal{G}_X$, $\mathrm{d}\varphi^{-\tau(\gamma)}|\gamma$ will denote $\mathrm{d}\varphi^{-\tau(\gamma)}$ taken at any point of the image of γ ; this ambiguity will not stand long since another choice of base point will lead to a conjugated linear map, and we aim to take traces. We have the standard factorization, for $\mathrm{Re}(s)$ big enough and any z near z_0,

$$g_s(z) = \exp\sum_{k=0}^{n}(-1)^k k\sum_{\gamma\in\mathcal{G}_X}\frac{\ell^\#(\gamma)}{\tau(\gamma)}\,\mathrm{tr}\,\rho_{\nabla(z)}(\gamma)\mathrm{e}^{-s\tau(\gamma)}\frac{\mathrm{tr}\wedge^k(\mathrm{d}\varphi^{-\tau(\gamma)})|_\gamma}{\det(I - P_\gamma)}, \tag{6.6.17}$$

where $P_\gamma = \mathrm{d}\varphi^{-\tau(\gamma)}|_{E_u\oplus E_s}$ is the linearized Poincaré map of γ, and $\ell^\#(\gamma)$ is the primitive period of γ. Now (6.3.10) implies

$$\mathrm{tr}\,\rho_{\nabla(z_0+\sigma)}(\gamma) = \mathrm{tr}\,\rho_{\nabla(z_0)}(\gamma) - \mathrm{tr}\left(\rho_{\nabla(z_0)}(\gamma)\int_\gamma \alpha_{z_0}(\sigma)(X)\right) + o(\sigma)\tau(\gamma).$$

As a consequence, the sum in (6.6.17) is $\mathcal{C}^1$ near $z = z_0$ for $\mathrm{Re}(s)$ big enough, and $\mathrm{d}(\log g_s)_{z_0}\sigma$ is equal to

$$-\sum_{k=0}^{n}(-1)^k k\sum_{\gamma\in\mathcal{G}_X}\frac{\ell^\#(\gamma)}{\tau(\gamma)}\,\mathrm{tr}\left(\rho_{\nabla(z_0)}(\gamma)\int_\gamma\alpha_{z_0}(\sigma)(X)\right)\mathrm{e}^{-s\tau(\gamma)}\frac{\mathrm{tr}\wedge^k(\mathrm{d}\varphi^{-\tau(\gamma)})|_\gamma}{\det(I - P_\gamma)}.$$

Now a slight extension of Guillemin trace formula [Gui77] gives, in $\mathcal{D}'(\mathbb{R}_{>0})$,

$$\operatorname{tr}^\flat \alpha_{z_0}(\sigma)(X)\mathrm{e}^{-t\mathcal{L}_X^\nabla}\Big|_{\Omega^k(M,E)} = \sum_\gamma \frac{\ell^\#(\gamma)}{\tau(\gamma)} \operatorname{tr}\left(\rho_{\nabla(z_0)}(\gamma)\int_\gamma \alpha_{z_0}(\sigma)(X)\right)\frac{\operatorname{tr}\wedge^k \mathrm{d}\varphi^{-\tau(\gamma)}}{|\det(I-P_\gamma)|}\delta(t-\ell(\gamma)),$$

where δ is the Dirac distribution. But now recall from §6.3.5 that $|\det(I-P_\gamma)| = (-1)^q\det(I-P_\gamma)$. Therefore, if $\varepsilon>0$ satisfies $\varepsilon<\tau(\gamma)$ for all γ, arguing exactly as in [DZ16, §4], with (6.3.2) in mind,

$$\mathrm{d}(\log g_s)_{z_0}\sigma = \mathrm{e}^{-\varepsilon s}(-1)^{q+1}\operatorname{tr}^\flat_{\mathrm{gr}}\left(\alpha_{z_0}(\sigma)(X)\left(\mathcal{L}_X^{\nabla(z_0)}+s\right)^{-1}\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z_0)}}\right).$$

Now it remains to turn the graded trace $\operatorname{tr}^\flat_{\mathrm{gr}}$ into a super trace $\operatorname{tr}^\flat_{\mathrm{s}}$ keeping in mind the relation $\operatorname{tr}^\flat_{\mathrm{gr}} = \operatorname{tr}^\flat_{\mathrm{s}}(N\cdot)$ where N is the number operator, cf. §B.3.1. Note that $\alpha_{z_0}(\sigma)(X) = [\alpha_{z_0}(\sigma),\iota_X] = \alpha_{z_0}(\sigma)\circ\iota_X + \iota_X\circ\alpha_{z_0}(\sigma)$. We therefore have

$$\begin{aligned} N\alpha_{z_0}(\sigma)(X) &= N[\alpha_{z_0}(\sigma),\iota_X] \\ &= N\alpha_{z_0}(\sigma)\iota_X + \iota_X(N-1)\alpha_{z_0}(\sigma) \\ &= N\alpha_{z_0}(\sigma)\iota_X - (N-1)\alpha_{z_0}\iota_X + [(N-1)\alpha,\iota_X]. \end{aligned}$$

Since ι_X commutes with $\left(\mathcal{L}_X^{\nabla(z_0)}+s\right)^{-1}\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z_0)}}$ one finally obtains

$$\begin{aligned} N\alpha_{z_0}(\sigma)(X) = \alpha_{z_0}(\sigma)\iota_X&\left(\mathcal{L}_X^{\nabla(z_0)}+s\right)^{-1}\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z_0)}} \\ &+\left[(N-1)\alpha_{z_0}(\sigma)\left(\mathcal{L}_X^{\nabla(z_0)}+s\right)^{-1}\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z_0)}},\ \iota_X\right]. \end{aligned}$$

This concludes by cyclicity of the flat trace. □

The following lemma is a direct consequence of Lemma 6.10.2 and the fact that $\Pi_0(z_0) = \Pi(z_0)$ by (6.6.12).

Lemma 6.6.4. *For* $\mathrm{Re}(s)$ *big enough, the map*

$$z\mapsto h_s(z) = \det\nolimits_{\mathrm{gr},C^\bullet(z)}\left(\mathcal{L}_X^{\nabla(z)}+s\right)^{(-1)^{q+1}}$$

is $\mathcal{C}^1$ *near* $z=z_0$*, and*

$$\mathrm{d}(\log h_s)_{z_0}\sigma = (-1)^{q+1}\operatorname{tr}_{\mathrm{s},C^\bullet(z_0)}\left(\Pi_0(z_0)\alpha_{z_0}(\sigma)\iota_X\left(\mathcal{L}_X^{\nabla(z_0)}+s\right)^{-1}\right).$$

6.6.8 Proof of Proposition 6.6.1

Combining the two lemmas of the preceding subsection we obtain for $\mathrm{Re}(s)$ big enough, the map $z\mapsto\zeta^{(\lambda,\infty)}_{X,\nabla(z)}(s) = g_s(z)/h_s(z)$ is real differentiable at $z=z_0$ (and therefore on U since we may vary z_0). Moreover for every $\varepsilon > 0$ small enough

$$\mathrm{d}\left(\log \frac{g_s}{h_s}\right)_z \sigma = (-1)^{q+1}\Bigg(\mathrm{e}^{-\varepsilon s}\,\mathrm{tr}_\mathrm{s}^\flat\, \alpha_z(\sigma)\iota_X\left(\mathcal{L}_X^{\nabla(z)}+s\right)^{-1}\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z)}} - \mathrm{tr}_{\mathrm{s},C^\bullet(z)}\Pi_0(z)\alpha_z(\sigma)\iota_X\left(\mathcal{L}_X^{\nabla(z)}+s\right)^{-1}\Bigg). \tag{6.6.18}$$

This gives the variation of $\zeta_{X,\nabla(z)}^{(\lambda,\infty)}(s)$ for $\mathrm{Re}(s)$ big enough. To obtain the variation of $b(z) = \zeta_{X,\nabla(z)}^{(\lambda,\infty)}(0)$, we can reproduce the arguments made in §6.5.4 to obtain

$$(-1)^{q+1}\mathrm{d}\left(\log b\right)_z \sigma = \mathrm{tr}_\mathrm{s}^\flat\left(\alpha_z(\sigma)\iota_X Y(z)(\mathrm{Id}-\Pi_0(z))\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}\right) + \mathrm{tr}_{\mathrm{s},C^\bullet(z)}\Big(\Pi_0(z)\alpha_z(\sigma)\iota_X Q_z(\varepsilon)\Big),$$

where

$$Q_z(\varepsilon) = \sum_{n\geqslant 1}\frac{(-\varepsilon)^n}{n!}\left(\mathcal{L}_X^{\nabla(z)}\right)^{n-1} : C^\bullet(z)\to C^\bullet(z).$$

Recall that if $c(z) = \tau(C^\bullet(z),\Gamma)$ one has $\tau(z) = c(z)b(z)^{(-1)^q}$. Therefore Lemma 6.6.2 gives, with what precedes,

$$\mathrm{d}(\log\tau)_z\sigma = -\,\mathrm{tr}_\mathrm{s}^\flat\Big(\alpha_z(\sigma)K(z)\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}\Big) - \mathrm{tr}_{\mathrm{s},C^\bullet(z)}\left(\Pi_0(z)\alpha_z(\sigma)\Big(k(z)\left(\mathrm{Id}-\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}\right)+\iota_X Q_z(\varepsilon)\Big)\right). \tag{6.6.19}$$

We have $\mathrm{Id}-\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z)}} = -\mathcal{L}_X^{\nabla(z)}Q_z(\varepsilon)$, which leads to

$$\iota_X Q_z(\varepsilon) + k(z)\left(\mathrm{Id}-\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}\right) = \Big(\iota_X - k(z)\mathcal{L}_X^{\nabla(z)}\Big)Q_z(\varepsilon).$$

But now since $k(z)$ is a cochain contraction, we get

$$\iota_X - k(z)\mathcal{L}_X^{\nabla(z)} = [\nabla(z), k(z)\iota_X].$$

Because $\nabla(z)$ commutes with $\Pi_0(z)$ and $\mathcal{L}_X^{\nabla(z)}$, we obtain with (6.6.3)

$$\Big[\nabla(z),\Pi_0(z)\alpha_z(\sigma)k(z)\iota_X Q_z(\varepsilon)\Big] = \Pi_0(z)\alpha\Big(\iota_X Q_z(\varepsilon)+k(z)\left(\mathrm{Id}-\mathrm{e}^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}\right)\Big).$$

This concludes by (6.6.19) and the cyclicity of the trace.

6.7 Euler structures, Chern-Simons classes

The Turaev torsion is defined using *Euler structures*, introduced by Turaev [Tur90], whose purpose is to fix sign ambiguities of combinatorial torsions. We shall use however the representation in terms of vector fields used by Burghelea–Haller [BH06]. The goal of the present section is to introduce these Euler structures, in view of the definition of the Turaev torsion.

6.7.1 The Chern-Simons class of a pair of vector fields

If $X \in \mathcal{C}^\infty(M, TM)$ is a vector field with isolated non degenerate zeros, we define the singular 0-chain

$$\mathrm{div}(X) = -\sum_{x \in \mathrm{Crit}(X)} \mathrm{ind}_X(x)[x] \ \in \ C_0(M, \mathbb{Z}),$$

where $\mathrm{Crit}(X)$ is the set of critical points of X and $\mathrm{ind}_X(x)$ denotes the Poincaré-Hopf index of x as a critical point of X [4]. Note also that $\mathrm{div}\,(-X) = -\mathrm{div}(X)$ since M is odd dimensional.

Let X_0, X_1 be two vector fields with isolated non degenerate zeros. Let $p : M \times [0,1] \to M$ be the projection over the first factor and choose a smooth section H of the bundle $p^*TM \to M \times [0,1]$, transversal to the zero section, such that H restricts to X_i on $\{i\} \times M$ for $i = 0, 1$. Then the set $H^{-1}(0) \subset M \times [0,1]$ is an oriented smooth submanifold of dimension 1 with boundary (it is oriented because M and $[0,1]$ are), and we denote by $[H^{-1}(0)]$ its fundamental class.

Definition 6.7.1. The class

$$p_*[H^{-1}(0)] \in C_1(M, \mathbb{Z})/\partial C_2(M, \mathbb{Z}),$$

where p_* is the pushforward by p, does not depend on the choice of the homotopy H relating X_0 and X_1, cf. [BH06, §2.2]. This is the *Chern-Simons class* of the pair (X_0, X_1), denoted by $\mathrm{cs}(X_0, X_1)$.

We have the fundamental formulae

$$\begin{aligned} \partial \mathrm{cs}(X_0, X_1) &= \mathrm{div}(X_1) - \mathrm{div}(X_0), \\ \mathrm{cs}(X_0, X_1) + \mathrm{cs}(X_1, X_2) &= \mathrm{cs}(X_0, X_2), \end{aligned} \tag{6.7.1}$$

for any other vector field with non degenerate zeros X_2. Notice also that if X_0 and X_1 are nonsingular vector fields, then $\mathrm{cs}(X_0, X_1)$ defines a homology class in $H_1(M, \mathbb{Z})$.

6.7.2 Euler structures.

Let X be a smooth vector field on M with non degenerate zeros. An *Euler chain* for X is a singular one-chain $e \in C_1(M, \mathbb{Z})$ such that $\partial e = \mathrm{div}(X)$. Euler chains for X always exist because M is odd-dimensional and thus $\chi(M) = 0$.

Two pairs (X_0, e_0) and (X_1, e_1), with X_i a vector field with non degenerate zeros and e_i an Euler chain for X_i, $i = 0, 1$, will be said to be equivalent if

$$e_1 = e_0 + \mathrm{cs}(X_0, X_1) \ \in \ C_1(M, \mathbb{C})/\partial C_2(M, \mathbb{Z}). \tag{6.7.2}$$

Definition 6.7.2. An *Euler structure* is an equivalence class $[X, e]$ for the relation (6.7.2). We will denote by $\mathrm{Eul}(M)$ the set of Euler structures.

There is a free and transitive action of $H_1(M, \mathbb{Z})$ on $\mathrm{Eul}(M)$ given by

$$[X, e] + h = [X, e + h], \quad h \in H_1(M, \mathbb{Z}).$$

4. $\mathrm{ind}_X(x) = (-1)^{\dim E_s(x)}$ if x is hyperbolic and $E_s(x) \subset T_xM$ is the stable subspace of x.

6.7.3 Homotopy formula relating flows

Let X_0, X_1 be two vector fields with non degenerate zeros. Let H be a smooth homotopy between X_0 and X_1 as in §6.7.1 and set $X_t = H(t, \cdot) \in \mathcal{C}^\infty(M, TM)$. For $\varepsilon > 0$ we define $\Phi_\varepsilon : M \times [0,1] \to M \times M \times [0,1]$ via

$$\Phi_\varepsilon(x,t) = \left(\mathrm{e}^{-\varepsilon X_t}(x), x, t\right), \quad x \in M, \quad t \in [0,1].$$

Set also, with notations of §B.2, $H_\varepsilon = \mathrm{Gr}(\Phi_\varepsilon) \subset M \times M \times \mathbb{R}$. Then H_ε is a submanifold with boundary of $M \times M \times \mathbb{R}$ which is oriented (since M and $\mathbb{R}$ are). Define

$$[H_\varepsilon] = (\Phi_\varepsilon)_* \left([M] \times [[0,1]]\right) \in \mathcal{D}'^n(M \times M \times \mathbb{R})$$

to be the associated integration current, cf. §B.2. Let g be any metric on M and let $\rho > 0$ be smaller than its injectivity radius. Then for any $x, y \in M$ with $\mathrm{dist}(x,y) \leqslant \rho$, we denote by $P(x,y) \in \mathrm{Hom}(E_x, E_y)$ the parallel transport by ∇ along the minimizing geodesic joining x to y. Then $P \in \mathcal{C}^\infty(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E)$ and we can define

$$\mathcal{R}_\varepsilon = -\pi_*[H_\varepsilon] \otimes P \in \mathcal{D}'^{n-1}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E),$$

where $\pi : M \times M \times \mathbb{R} \to M \times M$ is the projection over the two first factors. Note that $\mathcal{R}_\varepsilon$ is well defined if ε is small enough so that

$$\mathrm{dist}\left(x, \mathrm{e}^{-sX_t}(x)\right) \leqslant \rho, \quad s \in [0, \varepsilon], \quad t \in [0,1], \quad x \in M, \tag{6.7.3}$$

which implies $\mathrm{supp}\, \pi_*[H_\varepsilon] \subset \{(x,y),\ \mathrm{dist}(x,y) \leqslant \rho\}$. Now, let

$$R_\varepsilon : \Omega^\bullet(M, E) \to \mathcal{D}'^{\bullet-1}(M, E)$$

be the operator of degree -1 whose Schwartz kernel is $\mathcal{R}_\varepsilon$.

Lemma 6.7.3. *We have the following homotopy formula*

$$[\nabla, R_\varepsilon] = \nabla R_\varepsilon + R_\varepsilon \nabla = \mathrm{e}^{-\varepsilon \mathcal{L}^\nabla_{X_1}} - \mathrm{e}^{-\varepsilon \mathcal{L}^\nabla_{X_0}}. \tag{6.7.4}$$

Proof. First note that because M is odd dimensional, the boundary (computed with orientations) of the manifold H_ε is

$$\partial H_\varepsilon = \mathrm{Gr}(\mathrm{e}^{-\varepsilon X_0}) \times \{0\} - \mathrm{Gr}(\mathrm{e}^{-\varepsilon X_1}) \times \{1\}.$$

Therefore we have, cf. (B.2.1),

$$(-1)^n \mathrm{d}^{M \times M} \pi_*[H_\varepsilon] = \pi_*[\partial H_\varepsilon] = \left[\mathrm{Gr}(\mathrm{e}^{-\varepsilon X_0})\right] - \left[\mathrm{Gr}(\mathrm{e}^{-\varepsilon X_1})\right]$$

where $\left[\mathrm{Gr}(\mathrm{e}^{-\varepsilon X_i})\right]$ denotes the integration current on the manifold $\mathrm{Gr}(\mathrm{e}^{-\varepsilon X_i})$ for $i = 0, 1$. Now note that we have by construction $\nabla^{E^\vee \boxtimes E} P = 0$. Therefore

$$\nabla^{E^\vee \boxtimes E} \mathcal{R}_\varepsilon = (-1)^n \Big(\left[\mathrm{Gr}(\mathrm{e}^{-\varepsilon X_1})\right] - \left[\mathrm{Gr}(\mathrm{e}^{-\varepsilon X_0})\right] \Big) \otimes P.$$

Note that by definition of $\mathrm{e}^{-\mathcal{L}^\nabla_{X_i}}$ (cf §6.3.3), the formula (6.7.3) and the flatness of ∇ imply that the Schwartz kernel of $\mathrm{e}^{-\varepsilon \mathcal{L}^\nabla_{X_i}}$ is $\left[\mathrm{Gr}(\mathrm{e}^{-\varepsilon X_i})\right] \otimes P$. This concludes because the Schwartz kernel of $[\nabla, R_\varepsilon]$ is $(-1)^n \nabla^{E^\vee \boxtimes E} \mathcal{R}_\varepsilon$, cf. [HLJ01, Lemma 2.2]. □

The next formula follows from the definition of the flat trace and the Chern-Simons classes. It will be crucial for the topological interpretation of the variation formula obtained in §6.6.

Lemma 6.7.4. *We have for any* $\alpha \in \Omega^\bullet(M, \mathrm{End}(E))$ *such that* $\mathrm{tr}\,\alpha$ *is closed and* $\varepsilon > 0$ *small enough*

$$\mathrm{tr}_\mathrm{s}^\flat\, \alpha R_\varepsilon = \big\langle \mathrm{tr}\,\alpha,\ \mathrm{cs}(X_0, X_1)\big\rangle. \tag{6.7.5}$$

Note that because H is transverse to the zero section, we have

$$\mathrm{WF}(\mathcal{R}_\varepsilon) \cap N^*\Delta = \emptyset, \tag{6.7.6}$$

where $N^*\Delta$ denotes the conormal to the diagonal Δ in $M \times M$, so that the above flat trace is well defined.

Proof. We denote by $i : M \hookrightarrow M \times M$ the diagonal inclusion. Note that the Schwartz kernel of αR_ε is $(-1)^n \pi_2^*\alpha \wedge \mathcal{R}_\varepsilon = -\pi_2^*\alpha \wedge \mathcal{R}_\varepsilon$ since n is odd. From the definition of the super flat trace $\mathrm{tr}_\mathrm{s}^\flat$, we find that

$$\mathrm{tr}_\mathrm{s}^\flat\, \alpha R_\varepsilon = \Big\langle \mathrm{tr}\, i^* \left(\pi_2^*\alpha \wedge \pi_*[H_\varepsilon] \otimes P\right),\ 1\Big\rangle, \tag{6.7.7}$$

where $\pi_2 : M \times M \to M$ is the projection over the second factor. Of course we have $i^*P = \mathrm{Id}_E \in \mathcal{C}^\infty(M, \mathrm{End}(E))$. We therefore have

$$\mathrm{tr}\, i^* \left(\pi_2^*\alpha \wedge \pi_*[H_\varepsilon] \otimes P\right) = \mathrm{tr}\,\alpha \wedge i^*\pi_*[H_\varepsilon] = \mathrm{tr}\,\alpha \wedge p_* j^*[H_\varepsilon]$$

where $j : M \times [0,1] \hookrightarrow M \times M \times [0,1]$, $(x,t) \mapsto (x,x,t)$. This leads to

$$\mathrm{tr}_\mathrm{s}^\flat\, \alpha R_\varepsilon = \big\langle \mathrm{tr}\,\alpha \wedge p_* j^*[H_\varepsilon], 1\big\rangle = \big\langle p^*\,\mathrm{tr}\,\alpha, j^*[H_\varepsilon]\big\rangle.$$

Now if ε is small enough, we can see that $j^*[H_\varepsilon] = [H^{-1}(0)]$. Therefore

$$\mathrm{tr}_\mathrm{s}^\flat\, \alpha R_\varepsilon = \big\langle \mathrm{tr}\,\alpha, p_*[H^{-1}(0)]\big\rangle = \big\langle \mathrm{tr}\,\alpha, \mathrm{cs}(X_0, X_1)\big\rangle.$$

□

6.8 Morse theory and variation of Turaev torsion.

We introduce here the Turaev torsion which is defined in terms of CW decompositions. In the spirit of the seminal work of Bismut–Zhang [BZ92] based on geometric constructions of Laudenbach [Lau92], we use a CW decomposition which comes from the unstable cells of a Morse-Smale gradient flow induced by a Morse function. This allows us to interpret the variation of the Turaev torsion as a supertrace on the space of generalized resonant states for the Morse-Smale flow. This interpretation will be convenient for the comparison of the Turaev torsion with the dynamical torsion.

6.8.1 Morse theory and CW-decompositions

Let f be a Morse function on M and $\widetilde{X} = -\operatorname{grad}_g f$ be its associated gradient vector field with respect to some Riemannian metric g (the tilde notation is used to make the difference with the Anosov flows we studied until now). For any $a \in \operatorname{Crit}(f)$, we denote by

$$W^s(a) = \left\{ y \in M,\ \lim_{t\to\infty} \mathrm{e}^{t\widetilde{X}} y = a \right\}, \quad W^u(a) = \left\{ y \in M,\ \lim_{t\to\infty} \mathrm{e}^{-t\widetilde{X}} y = a \right\},$$

the stable and unstable manifolds of a. Then it is well known that $W^s(a)$ (resp. $W^u(x)$) is a smooth embedded open disk of dimension $n - \operatorname{ind}_f(a)$ (resp. $\operatorname{ind}_f(a)$), where $\operatorname{ind}_f(a)$ is the index of a as a critical point of f, that is, in a Morse chart $(z_1, \dots, z_n)$ near a,

$$f(z_1, \dots, z_n) = f(a) - z_1^2 - \cdots - z_{\operatorname{ind}_f(a)}^2 + z_{\operatorname{ind}_f(a)+1}^2 + \cdots + z_n^2.$$

For simplicity, we will denote

$$|a| = \operatorname{ind}_f(a) = \dim W^u(a),$$

and we fix an orientation of every $W^u(a)$.

We assume that $\widetilde{X}$ satisfies the Morse-Smale condition, that is, for any $a, b \in \operatorname{Crit}(f)$, the manifolds $W^s(a)$ and $W^u(b)$ are transverse. Also, we assume that for every $a \in \operatorname{Crit}(f)$, the metric g is flat near a. Let us summarize some results from [Qin10, Theorems 3.2,3.8,3.9] ensured by the unstable manifolds of f. We would like to mention that such results can be found in slightly different form in the work of Laudenbach [Lau92] and are used in [BZ92] [5].

First, $W^u(a)$ admits a compactification to a smooth $|a|$-dimensional manifold with corner $\overline{W}^u(a)$, endowed with a smooth map $e_a : \overline{W}^u(a) \to M$ that extends the inclusion $W^u(a) \subset M$. Then the collection $W = \left\{ \overline{W}^u(a) \right\}_{a \in \operatorname{Crit}(f)}$ and the applications e_a induce a CW-decomposition on M. Moreover, the boundary operator of the cellular chain complex is given by

$$\partial \overline{W}^u(a) = \sum_{|b|=|a|-1} \#\mathcal{L}(a,b) \overline{W}^u(b),$$

where $\mathcal{L}(a,b)$ is the moduli space of gradient lines joining a to b and $\#\mathcal{L}(a,b)$ is the sum of the orientations induced by the orientations of the unstable manifolds of (a,b), see [Qin10, Theorem 3.9].

6.8.2 The Thom-Smale complex

We set $C_\bullet(W, E^\vee) = \bigoplus_{k=0}^n C_k(W, E^\vee)$ where

$$C_k(W, E^\vee) = \bigoplus_{\substack{a \in \operatorname{Crit}(f) \\ |a| = k}} E_a^\vee, \quad k = 0, \dots, n.$$

5. A difference is that Laudenbach only needs to compactify the unstable cells as C^1–manifolds with conical singularities whereas Qin proves smooth compactification as manifolds with corners.

We endow the complex $C_\bullet(W, E^\vee)$ with the boundary operator $\partial^{\nabla^\vee}$ defined by

$$\partial^{\nabla^\vee} u = \sum_{|b|=|a|-1} \sum_{\gamma \in \mathcal{L}(a,b)} \varepsilon_\gamma P_\gamma(u), \quad a \in \mathrm{Crit}(f), \quad u \in E_a^\vee,$$

where for $\gamma \in \mathcal{L}(a,b)$, $P_\gamma \in \mathrm{End}(E_a^\vee, E_b^\vee)$ is the parallel transport of $\nabla^\vee$ along the curve γ and $\varepsilon_\gamma = \pm 1$ is the orientation number of $\gamma \in \mathcal{L}(a,b)$.

Then by [Lau92] (see also [DR19b] for a different approach), there is a canonical isomorphism

$$H_\bullet(M, \nabla^\vee) \simeq H_\bullet(W, \nabla^\vee),$$

where $H_\bullet(M, \nabla^\vee)$ is the singular homology of flat sections of $(E^\vee, \nabla^\vee)$ and $H_\bullet(W, \nabla^\vee)$ denotes the homology of the complex $C_\bullet(W, E^\vee)$ endowed with the boundary map $\partial^{\nabla^\vee}$. Therefore this complex is acyclic since ∇ (and thus $\nabla^\vee$) is.

6.8.3 The Turaev torsion

Fix some base point $x_\star \in M$ and for every $a \in \mathrm{Crit}(f)$, let γ_a be some path in M joining $x_\star$ to a. Define

$$e = \sum_{a \in \mathrm{Crit}(f)} (-1)^{|a|} \gamma_a \in C_1(M, \mathbb{Z}). \tag{6.8.1}$$

Note that the Poincaré-Hopf index of $\widetilde{X}$ near $a \in \mathrm{Crit}(f)$ is $-(-1)^{|a|}$ so that

$$\partial e = \mathrm{div}(\widetilde{X}) \tag{6.8.2}$$

because $\sum_{a \in \mathrm{Crit}(f)} (-1)^{|a|} = \chi(M) = 0$ by the Poincaré-Hopf index theorem. Therefore e is an Euler chain for $\widetilde{X}$ and

$$\mathfrak{e} = [\widetilde{X}, e]$$

defines an Euler structure. Choose some basis $u_1, \ldots, u_d$ of $E_{x_\star}^\vee$. For each $a \in \mathrm{Crit}(f)$, we propagate this basis via the parallel transport of ∇ along γ_a to obtain a basis $u_{1,a}, \ldots, u_{d,a}$ of E_a. We choose an ordering of the cells $\left\{\overline{W^u}(a)\right\}$; this gives us a cohomology orientation $\mathfrak{o}$ (see [Tur90, §6.3]). Moreover this ordering and the chosen basis of $E_a^\vee$ give us (using the wedge product) an element $c_k \in \det C_k(W, E^\vee)$ for each k, and thus an element $c \in \det C_\bullet(W, E^\vee)$.

The *Turaev torsion* of ∇ with respect to the choices $\mathfrak{e}, \mathfrak{o}$ is then defined by [FT00, §9.2 p. 218]

$$\tau_{\mathfrak{e},\mathfrak{o}}(\nabla)^{-1} = \varphi_{C_\bullet(W,\nabla^\vee)}(c) \in \mathbb{C} \setminus 0,$$

where $\varphi_{C_\bullet(W,\nabla^\vee)}$ is the homology version of the isomorphism (6.2.1). Note that $\nabla^\vee$ (and not ∇) is involved in the definition of $\tau_{\mathfrak{e},\mathfrak{o}}(\nabla)$. Indeed, we use here the cohomological version of Turaev's torsion, which is more convenient for our purposes, and which is consistent with [BK07b], [BK+08, p. 252].

6.8.4 Resonant states of the Morse-Smale flow

In [DR19b], it has been shown that we can define Ruelle resonances for the Morse-Smale gradient flow $\mathcal{L}_{\widetilde{X}}^{\nabla}$ as described in §6.3 in the context of Anosov flows. More precisely, we have that the resolvent

$$\left(\mathcal{L}_{\widetilde{X}}^{\nabla}+s\right)^{-1}: \ \Omega^{\bullet}(M,E)\to\mathcal{D}'^{\bullet}(M,E),$$

is well defined for $\mathrm{Re}(s)\gg 0$, has a meromorphic continuation to all $s\in\mathbb{C}$. The poles of this continuation are the Ruelle resonances of $\mathcal{L}_{\widetilde{X}}^{\nabla}$ and the set of those will be denoted by $\mathrm{Res}(\mathcal{L}_{\widetilde{X}}^{\nabla})$. In fact, the set $\mathrm{Res}(\mathcal{L}_{\widetilde{X}}^{\nabla})$ does not depend on the flat vector bundle (E,∇). Let $\lambda>0$ be such that $\mathrm{Res}(\mathcal{L}_{\widetilde{X}}^{\nabla})\cap\{|s|\leqslant\lambda\}\subset\{0\}$; set

$$\widetilde{\Pi}=\frac{1}{2\pi i}\int_{|s|=\lambda}\left(\mathcal{L}_{\widetilde{X}}^{\nabla}+s\right)^{-1}\mathrm{d}s \tag{6.8.3}$$

the spectral projector associated to the resonance 0, and denote by

$$\widetilde{C}^{\bullet}=\mathrm{ran}\ \widetilde{\Pi}\subset\mathcal{D}'^{\bullet}(M,E)$$

the associated space of generalized eigenvectors for $\mathcal{L}_{\widetilde{X}}^{\nabla}$. Since ∇ and $\mathcal{L}_{\widetilde{X}}^{\nabla}$ commute, ∇ induces a differential on the complex $\widetilde{C}^{\bullet}$. Moreover, $\widetilde{\Pi}$ maps $\mathcal{D}'^{\bullet}_{\Gamma}(M,E)$ to itself continuously where

$$\Gamma=\bigcup_{a\in\mathrm{Crit}(f)}\overline{N^*W^u(a)}\subset T^*M.$$

6.8.5 A variation formula for the Turaev torsion

Assume that we are given a $\mathcal{C}^1$ family of acyclic connections $\nabla(z)$ on E as in §6.6. We denote by $\widetilde{\Pi}_-(z)$ the spectral projector (6.8.3) associated to $\nabla(z)$ and $-\widetilde{X}$, and set $\widetilde{C}^{\bullet}_-(z)=\mathrm{ran}\ \widetilde{\Pi}_-(z)$. By [DR19b] we have that all the complexes $(\widetilde{C}^{\bullet}(z),\nabla(z))$ are acyclic and there exists cochain contractions $\tilde{k}_-(z):\widetilde{C}^{\bullet}_-(z)\to\widetilde{C}^{\bullet-1}_-(z)$. As in §6.6.3 we have a variation formula for the Turaev torsion.

Proposition 6.8.1. *The map $z\mapsto\tilde{\tau}(z)=\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$ is real differentiable on U and for any $z\in U$*

$$\mathrm{d}(\log\tilde{\tau})_z\sigma=-\mathrm{tr}_{\mathrm{s},\widetilde{C}^{\bullet}(z)}\left(\widetilde{\Pi}_-(z)\alpha_z(\sigma)\tilde{k}_-(z)\right)-\int_e\mathrm{tr}\,\alpha_z(\sigma),\quad\sigma\in\mathbb{C}$$

where $\alpha_z(\sigma)$ is given by (6.6.2) and e is given by (6.8.1).

The rest of this section is devoted to the proof of Proposition 6.8.1. For convenience, we will first study the variation of $z\mapsto\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)^{\vee})$.

6.8.6 A preferred basis

Let $a\in\mathrm{Crit}(f)$ and $k=|a|$. We denote by $[W^u(a)]\in\mathcal{D}'^{n-k}_{\Gamma}(M)$ the integration current over the unstable manifold $W^u(a)$ of $\widetilde{X}$, it is a well defined current far from

$\partial W^u(a)$. We also pick a cut-off function $\chi_a \in \mathcal{C}^\infty(M)$ valued in $[0,1]$ with $\chi_a \equiv 1$ near a and χ_a is supported in a small neighborhood Ω_a of a, with $\overline{\Omega_a} \cap \partial W^u(a) = \emptyset$. Recall from 6.8.3 that we have a basis $u_{1,a}, \ldots, u_{d,a}$ of E_a. Using the parallel transport of ∇, we obtain flat sections of E over $W^u(a)$ that we will still denote by $u_{1,a}, \ldots, u_{d,a}$. Define

$$\tilde{u}_{j,a} = \widetilde{\Pi}\Big(\chi_a[W^u(a)] \otimes u_{j,a}\Big) \in \widetilde{C}^{n-k}, \quad j = 1, \ldots, d. \tag{6.8.4}$$

By [DR20c] we have that $\big\{\tilde{u}_{j,a},\ a \in \mathrm{Crit}(f),\ 1 \leqslant j \leqslant d\big\}$ is a basis of $\widetilde{C}^\bullet$. Adapting the proof of [DR17, Theorem 2.6] to the bundle case, we obtain the following proposition which will allow us to compute the Turaev torsion with the help of the complex $\widetilde{C}^\bullet$.

Proposition 6.8.2. *The map* $\Phi : C_\bullet(W, \nabla) \to \widetilde{C}^{n-\bullet}$ *defined by*

$$\Phi\big(u_{j,a}\big) = \tilde{u}_{j,a}, \quad a \in \mathrm{Crit}(f), \quad j = 1, \ldots, d,$$

is an isomorphism and satisfies[6]

$$\Phi \circ \partial^\nabla = (-1)^\bullet \nabla \circ \Phi.$$

An immediate corollary is that (using the notation of §6.2.2)

$$\tau_{\mathfrak{e},\mathfrak{o}}(\nabla^\vee) = \varphi_{C_\bullet(W,\nabla)}(u)^{-1} = \tau(\widetilde{C}^\bullet, \tilde{u}), \tag{6.8.5}$$

where $u \in \det C_\bullet(W,\nabla)$ (resp. $\tilde{u} \in \det \widetilde{C}^\bullet$) is the element given by the basis $\{u_{j,a}\}$ (resp. $\{\tilde{u}_{j,a}\}$) and the ordering of the cells $W^u(a)$.

6.8.7 Proof of Proposition 6.8.1

For any $a \in \mathrm{Crit}(f)$ we denote by $P_{\gamma_a}(z) \in \mathrm{Hom}(E_{x_\star}, E_a)$ the parallel transport of $\nabla(z)$ along γ_a. We set

$$u_{j,a}(z) = P_{\gamma_a}(z) P_{\gamma_a}(z_0)^{-1} u_{j,a}$$

and

$$\tilde{u}_{j,a}(z) = \widetilde{\Pi}(z)\Big(\chi_a[W^u(a)] \otimes u_{j,a}(z)\Big),$$

where again we consider $u_{j,a}(z)$ as a $\nabla(z)$-flat section of E over $W^u(a)$ using the parallel transport of $\nabla(z)$. The construction of Ruelle resonances for Morse-Smale gradient flow follows from the construction of anisotropic Sobolev spaces

$$\Omega^\bullet(M,E) \subset \widetilde{\mathcal{H}}_1^\bullet \subset \widetilde{\mathcal{H}}^\bullet \subset \mathcal{D}'^\bullet(M,E),$$

see [DR19a], on which $\mathcal{L}_X^\nabla + s$ is a holomorphic family of Fredholm operators of index 0 in the region $\{\mathrm{Re}(s) > -2\}$, and such that $\nabla(z)$ is bounded $\widetilde{\mathcal{H}}_1^\bullet \to \widetilde{\mathcal{H}}^\bullet$. Every argument made in §6.6.4 also stand here and $z \mapsto \widetilde{\Pi}(z)$ is a $\mathcal{C}^1$ family of bounded operators $\widetilde{\mathcal{H}}^\bullet \to \widetilde{\mathcal{H}}_1^\bullet$.

6. $(-1)^\bullet$ comes from $\partial = (-1)^{\deg +1}\mathrm{d}$ comparing the boundary ∂ and De Rham differential d

Note that by continuity, $\widetilde{\Pi}(z)$ induces an isomorphism $\widetilde{C}^\bullet(z_0) \to \widetilde{C}^\bullet(z)$ for z close enough to zero. Let $\tilde{u}(z) \in \det \widetilde{C}^\bullet(z)$ be the element given by the basis $\{\tilde{u}_{j,a}(z)\}$ and the ordering of the cells $W^u(a)$. Then by (6.8.5) and (6.2.5) we have

$$\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)^\vee) = \tau\left(\widetilde{C}^\bullet(z), \tilde{u}(z)\right) = \left[\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)\right]\tau\left(\widetilde{C}^\bullet(z), \widetilde{\Pi}(z)\tilde{u}(z_0)\right), \quad (6.8.6)$$

where $\widetilde{\Pi}(z)\tilde{u}(z_0) \in \det \widetilde{C}^\bullet(z)$ is the image of $\tilde{u}$ by the isomorphism $\det \widetilde{C}^\bullet(z_0) \to \det \widetilde{C}^\bullet(z)$ induced by $\widetilde{\Pi}(z)$, and $\tilde{u}(z) = \left[\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)\right]\widetilde{\Pi}(z)\tilde{u}(z_0)$. Doing exactly as in §6.6.6, we obtain that $z \mapsto \hat{\tau}(z) = \tau\left(\widetilde{C}^\bullet(z), \widetilde{\Pi}(z)\tilde{u}\right)$ is $\mathcal{C}^1$ and

$$\mathrm{d}(\log\hat{\tau})_{z_0}\sigma = -\mathrm{tr}_{\mathrm{s},\widetilde{C}^\bullet}\widetilde{\Pi}(z_0)\alpha_{z_0}(\sigma)\tilde{k}(z_0). \quad (6.8.7)$$

Therefore it remains to compute the variation of $\left[\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)\right]$. This is the purpose of the next formula.

Lemma 6.8.3. *We have*

$$\left[\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)\right] = \prod_{a\in\mathrm{Crit}(f)} \det\left(P_{\gamma_a}(z)P_{\gamma_a}(z_0)^{-1}\right)^{(-1)^{n-|a|}}.$$

Proof. By definition of the basis $\{u_{a,j}\}$ in §6.8.3 it suffices to show that for z small enough

$$\widetilde{\Pi}(z)\tilde{u}_{a,i} = \sum_{j=1}^{d} A^j_{a,i}(z)\tilde{u}_{a,j}(z), \quad a \in \mathrm{Crit}(f), \quad 1 \leqslant i,j \leqslant d, \quad (6.8.8)$$

where the coefficients $A^j_{a,i}(z)$ are defined by $u_{a,i}(z_0)(a) = \sum_{j=1}^{d} A^j_{a,i}(z)u_{a,j}(z)(a)$.

Consider the dual operator $\mathcal{L}^{\nabla(z)^\vee}_{-\widetilde{X}} : \Omega^\bullet(M, E^\vee) \to \Omega^\bullet(M, E^\vee)$. The above constructions, starting from a dual basis $s_1, \dots, s_d \in E^\vee_{x_\star}$ of $u_1, \dots, u_d$, give a basis $\{s_{a,i}(z)\}$ of each $\Gamma(W^s(a), \nabla(z)^\vee)$ (the space of flat section of $\nabla(z)^\vee$ over $W^s(a)$), since the unstable manifolds of $-\widetilde{X}$ are the stable ones of $\widetilde{X}$. Let $\widetilde{C}^\bullet_\vee(z)$ be the range of the spectral projector $\widetilde{\Pi}^\vee(z)$ from (6.8.3) associated to the vector field $-\widetilde{X}$ and the connection $\nabla(z)^\vee$. We have a basis $\{\tilde{s}_{a,i}(z)\}$ of $\widetilde{C}^\bullet_\vee(z)$ given by

$$\tilde{s}_{a,i}(z) = \widetilde{\Pi}^\vee(z)\Big(\chi_a[W^s(a)] \otimes s_{a,i}(z)\Big).$$

We will prove that for any $a, b \in \mathrm{Crit}(f)$ with same Morse index we have for any $1 \leqslant i,j \leqslant d$,

$$\left\langle \tilde{s}_{a,j}(z),\ \tilde{u}_{a,i}(z_0)\right\rangle = \begin{cases} \left\langle s_{a,j}(z)(a), u_{a,i}(z_0)(a)\right\rangle_{E^\vee_a, E_a} & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases}. \quad (6.8.9)$$

First assume that $a \neq b$. Then $W^u(a) \cap W^s(b) = \emptyset$ by the transversality condition, since a and b have same Morse index. Therefore for any $t_1, t_2 \geqslant 0$, we have

$$\left\langle \mathrm{e}^{-t_1\mathcal{L}^{\nabla(z)^\vee}_{-\widetilde{X}}}\Big(\chi_b[W^s(b)] \otimes s_{b,j}(z)\Big),\ \mathrm{e}^{-t_2\mathcal{L}^{\nabla(z_0)}_{\widetilde{X}}}\Big(\chi_a[W^u(a)] \otimes u_{a,i}(z)\Big)\right\rangle = 0, \quad (6.8.10)$$

since the currents in the pairing have disjoint support because they are respectively contained in $W^s(b)$ and $W^u(a)$. Now notice that for $\mathrm{Re}(s)$ big enough, one has

$$\left(\mathcal{L}_{-\widetilde{X}}^{\nabla(z)^\vee}+s\right)^{-1}=\int_0^\infty \mathrm{e}^{-t\mathcal{L}_{-\widetilde{X}}^{\nabla(z)^\vee}}\mathrm{e}^{-ts}\mathrm{d}t \quad \text{and} \quad \left(\mathcal{L}_{\widetilde{X}}^{\nabla(z_0)}+s\right)^{-1}=\int_0^\infty \mathrm{e}^{-t\mathcal{L}_{\widetilde{X}}^{\nabla(z_0)}}\mathrm{e}^{-ts}\mathrm{d}t.$$

Therefore the representation (6.8.3) of the spectral projectors and the analytic continuation of the above resolvents imply with (6.8.10) that $\left\langle \tilde{s}_{b,j}(z),\ \tilde{u}_{a,i}\right\rangle = 0$.

Next assume that $a=b$. Then $W^u(a)\cap W^s(a)=\{a\}$. Since the support of $\tilde{s}_{a,i}(z)$ (resp. $\tilde{u}_{a,i}(z_0)$) is contained in the closure of $W^s(a)$ (resp. $W^u(a)$), we can compute

$$\begin{aligned}\Big\langle \widetilde{\Pi}^\vee(z)\Big(\chi_a[W^s(a)]\otimes s_{a,j}(z)\Big),\ \widetilde{\Pi}\Big(\chi_a[W^u(a)]\otimes u_{a,i}(z_0)\Big)\Big\rangle &\\ = \Big\langle \chi_a[W^s(a)]\otimes s_{a,j}(z),\ \chi_a[W^u(a)]\otimes u_{a,i}(z_0)\Big\rangle&\\ = \Big\langle [a], \langle s_{a,j}(z), u_{a,i}(z_0)\rangle_{E^\vee,E}\Big\rangle&,\end{aligned}$$

where the first equality stands because $\tilde{s}_a(z)=[W^s(a)]\otimes s_{a,j}(z)$ near a by [DR20c, Proposition 7.1]. This gives (6.8.9).

This identity immediately yields (6.8.8) with $A^j_{a,i}(z)=\big\langle s_{a,j}(z)(a), u_{a,i}(z_0)(a)\big\rangle_{E_a^\vee,E_a}$ since we have

$$\widetilde{\Pi}(z)=\sum_{a,i}\langle \tilde{s}_{a,j}(z),\cdot\rangle \tilde{u}_{a,j}(z) \tag{6.8.11}$$

□

Using the lemma, we obtain, if $\mu(z)=\left[\tilde{u}(z):\widetilde{\Pi}(z)\tilde{u}(z_0)\right]$,

$$\mathrm{d}(\log\mu)_{z_0}\sigma=\sum_{a\in\mathrm{Crit}(f)}(-1)^{n-|a|}\operatorname{tr}\Big(A_{\gamma_a}(z_0,\sigma)P_{\gamma_a}(z_0)^{-1}\Big)$$

where $A_{\gamma_a}(z_0,\sigma)=\mathrm{d}\left(P_{\gamma_a}\right)_{z_0}\sigma$. Since n is odd, we obtain by definition of e and (6.3.10)

$$\mathrm{d}(\log\mu)_{z_0}\sigma=\sum_{a\in\mathrm{Crit}(f)}(-1)^{|a|}\int_{\gamma_a}\operatorname{tr}\alpha_{z_0}(\sigma)=\int_e \operatorname{tr}\alpha_{z_0}(\sigma).$$

This equation combined with (6.8.6) and (6.8.7) yields, if $\tilde{\tau}^\vee(z)=\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)^\vee)$

$$\mathrm{d}(\log\tilde{\tau}^\vee)_{z_0}\sigma=-\mathrm{tr}_{\mathrm{s},\widetilde{C}^\bullet}\widetilde{\Pi}(z_0)\alpha_{z_0}(\sigma)\tilde{k}(z_0)+\int_e\operatorname{tr}\alpha_{z_0}(\sigma).$$

The proof is almost finished. But since we need to formulate our results in terms of the cohomological torsion, we still have to make some tedious formal manipulations to pass to the cohomological formalism. The first step is to replace ∇ by the dual connection $\nabla^\vee$ in the above formula. We also introduce some notation. The operator $\widetilde{\Pi}$ was the spectral projector on the kernel of $\mathcal{L}_{\widetilde{X}}^{\nabla}$. Now we need to work with the spectral projector on $\ker\left(\mathcal{L}_{\widetilde{X}}^{\nabla(z_0)^\vee}\right)$ (resp. $\mathcal{L}_{-\widetilde{X}}^{\nabla(z_0)}$), which we denote by $\widetilde{\Pi}_+^\vee(z_0)$ (resp

$\widetilde{\Pi}_-(z_0)$) where the $+$ (resp $-$) sign emphasizes the fact that we deal with $+\widetilde{X}$ (resp $-\widetilde{X}$). Now note that

$$\nabla(z)^\vee = \nabla(z_0)^\vee - {}^T\big(\alpha_{z_0}(z-z_0)\big) + o(z-z_0).$$

Therefore, applying what precedes to $\tilde{\tau}(z)$ we get

$$\mathrm{d}(\log\tilde{\tau})_{z_0}\sigma = -\mathrm{tr}_{\mathrm{s},\widetilde{C}^\bullet_{\vee,+}}\left(\widetilde{\Pi}^\vee_+(z_0)\big(-{}^T\alpha_{z_0}(\sigma)\big)\tilde{k}^\vee_+(z_0)\right) + \int_e \mathrm{tr}\left(-{}^T\alpha_{z_0}(\sigma)\right), \quad (6.8.12)$$

where $\widetilde{\Pi}^\vee_+(z_0)$ is the spectral projector (6.8.3) associated to $\nabla(z_0)^\vee$ and $+\widetilde{X}$, $\widetilde{C}^\bullet_{\vee,+} = \operatorname{ran}\widetilde{\Pi}^\vee_+(z_0)$, and $\tilde{k}^\vee_+(z_0)$ is any cochain contraction on the complex $(\widetilde{C}^\bullet_{\vee,+}, \nabla(z_0)^\vee)$. Now, we have the identification

$$\left(\widetilde{C}^k_{\vee,+}\right)^\vee \simeq \widetilde{C}^{n-k}_-,$$

where $\widetilde{C}^\bullet_-$ is the range of $\widetilde{\Pi}_-(z_0)$, the spectral projector (6.8.3) associated to $\nabla(z_0)$ and $-\widetilde{X}$. It is easy to show that under this identification, one has

$$\left(\widetilde{\Pi}^\vee_+\big({}^T\alpha_{z_0}(\sigma)\big)\tilde{k}(z_0)\right)^\vee = \widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma)k_-(z_0) + \left[\widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma), k_-(z_0)\right],$$

where for any $j \in \{0,\dots,n\}$, we set

$$k_-(z_0)|_{\widetilde{C}^{n-j}_-} = (-1)^{j+1}\big(\tilde{k}^\vee_+(z_0)|_{\tilde{C}^{j+1}}\big)^\vee : \widetilde{C}^{n-j}_- \to \widetilde{C}^{n-j-1}_-.$$

Then $k_-(z_0)$ is a cochain contraction on the complex $(\widetilde{C}^\bullet_-, \nabla(z_0))$. As a consequence, since n is odd,

$$\mathrm{tr}_{\mathrm{s},\widetilde{C}^\bullet_{\vee,+}}\left(\widetilde{\Pi}^\vee_+(z_0)\big(-{}^T\alpha_{z_0}(\sigma)\big)\tilde{k}^\vee_+(z_0)\right) = \mathrm{tr}_{\mathrm{s},\widetilde{C}^\bullet_-}\widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma)k_-(z_0).$$

This concludes by (6.8.12) since $\mathrm{tr}(-{}^T\beta) = -\operatorname{tr}\beta$ for any $\beta \in \Omega^1(M, \mathrm{End}(E))$.

6.9 Comparison of the dynamical torsion with the Turaev torsion

In this section we see the dynamical torsion and the Turaev torsion as functions on the space of acyclic representations. This is an open subset of a complex affine algebraic variety. Therefore we can compute the derivative of $\tau_\vartheta/\tau_{\mathfrak{e},\mathfrak{o}}$ along holomorphic curves, using the variation formulae obtained in §§6.6,6.8. From this computation we will deduce Theorem 6.1.3.

6.9.1 The algebraic structure of the representation variety

We describe here the analytic structure of the space

$$\mathrm{Rep}(M,d) = \mathrm{Hom}(\pi_1(M), \mathrm{GL}(\mathbb{C}^d))$$

of complex representations of degree d of the fundamental group. Since M is compact, $\pi_1(M)$ is generated by a finite number of elements $c_1, \ldots, c_L \in \pi_1(M)$ which satisfy finitely many relations. A representation $\rho \in \mathrm{Rep}(M, d)$ is thus given by $2L$ invertible $d \times d$ matrices $\rho(c_1), \ldots, \rho(c_L)$, $\rho(c_1^{-1}), \ldots \rho(c_L^{-1})$ with complex coefficients satisfying finitely many polynomial equations. Therefore the set $\mathrm{Rep}(M, d)$ has a natural structure of a complex affine algebraic set. We will denote the set of its singular points by $\Sigma(M, d)$. In what follows, we will only consider the classical topology of $\mathrm{Rep}(M, d)$.

We will say that a representation $\rho \in \mathrm{Rep}(M, d)$ is acyclic if ∇_ρ is acyclic. We denote by $\mathrm{Rep}_{\mathrm{ac}}(M, d) \subset \mathrm{Rep}(M, d)$ the space of acyclic representations. This is an open set (in the Zariski topology, thus in the classical one) in $\mathrm{Rep}(M, d)$, see [BH06, §4.1]. For any $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M, d)$ we set

$$\tau_\vartheta(\rho) = \tau_\vartheta(\nabla_\rho), \quad \tau_{\mathfrak{e},\mathfrak{o}}(\rho) = \tau_{\mathfrak{e},\mathfrak{o}}(\nabla_\rho),$$

for any Euler structure $\mathfrak{e}$ and any cohomological orientation $\mathfrak{o}$.

6.9.2 Holomorphic families of acyclic representations

Let $\rho_0 \in \mathrm{Rep}_{\mathrm{ac}}(M, d) \setminus \Sigma(M, d)$ be a regular point. Take $\delta > 0$ and $\rho(z)$, $|z| < \delta$, a holomorphic curve in $\mathrm{Rep}_{\mathrm{ac}}(M, d) \setminus \Sigma(M, d)$ such that $\rho(0) = \rho_0$. Theorems 6.1.3 and 6.1.4 will be a consequence of the following

Proposition 6.9.1. *Let X be a contact Anosov vector field on M. Let $\mathfrak{e} = [\widetilde{X}, e]$ be the Euler structure defined in §6.8.3. Note that $-\mathrm{cs}(-\widetilde{X}, X) + e$ is a cycle and defines a homology class $h \in H_1(M, \mathbb{Z})$. Then $z \mapsto \tau_\vartheta(\rho(z))/\tau_{\mathfrak{e},\mathfrak{o}}(\rho(z))$ is complex differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\tau_\vartheta(\rho(z))}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho(z))}\langle \det \rho(z), h\rangle\right) = 0$$

for any cohomological orientation $\mathfrak{o}$.

Proposition 6.9.1 relies on the variation formulae given by Propositions 6.6.1 and 6.8.1, and Lemma 6.7.4 which gives a topological interpretation of those.

6.9.3 An adapted family of connections

Following [BV17, §4.1], there exists a flat vector bundle E over M and a $\mathcal{C}^1$ family of connections $\nabla(z)$, $|z| < \delta$, in the sense of §6.6.1, such that [7]

$$\rho_{\nabla(z)} = \rho(z) \tag{6.9.1}$$

for every z; we can moreover ask the family $\nabla(z)$ to be complex differentiable at $z = 0$, that is,

$$\nabla(z) = \nabla + z\alpha + o(z), \tag{6.9.2}$$

where $\nabla = \nabla(0)$ and $\alpha \in \Omega^1(M, \mathrm{End}(E))$. Note that flatness of $\nabla(z)$ implies

$$[\nabla, \alpha] = \nabla\alpha + \alpha\nabla = 0.$$

7. It is actually stated in [BV17, §4.1] that one can find a $\mathcal{C}^1$ family of connections satisfying (6.9.1); however looking carefully at the proofs one can choose the family $\nabla(z)$ to be $\mathcal{C}^1$ in z.

6.9.4 A cochain contraction induced by the Morse-Smale gradient flow

Let

$$\left(\mathcal{L}^{\nabla}_{-\widetilde{X}} + s\right)^{-1} = \frac{\widetilde{\Pi}_-}{s} + \widetilde{Y} + \mathcal{O}(s)$$

be the Laurent expansion of $\left(\mathcal{L}^{\nabla}_{-\widetilde{X}} + s\right)^{-1}$ near $s = 0$. The fact that $s = 0$ is a simple pole comes from [DR19a]. As in 6.6.2, we consider the operator

$$\widetilde{K} = \iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-) + \tilde{k}_-\widetilde{\Pi}_- \ : \ \Omega^{\bullet}(M, E) \to \mathcal{D}'^{\bullet}(M, E),$$

where $\tilde{k}_-$ is any cochain contraction on $\widetilde{C}^{\bullet}_- = \operatorname{ran} \widetilde{\Pi}_-$. Note that we have the identity

$$[\nabla, \widetilde{K}] = \nabla\widetilde{K} + \widetilde{K}\nabla = \mathrm{Id}\,. \tag{6.9.3}$$

The next proposition will allow us to interpret the term $\operatorname{tr}_{\mathrm{s},\widetilde{C}^{\bullet}}\widetilde{\Pi}_-(z)\alpha_z(\sigma)\tilde{k}_-(z)$ appearing in Proposition 6.8.1 as a flat trace similar to the one appearing in Proposition 6.6.1. This will be crucial for the comparison between τ_ϑ and $\tau_{\mathfrak{e},\mathfrak{o}}$.

Proposition 6.9.2. *For $\varepsilon > 0$ small enough, the wavefront set of the Schwartz kernel of the operator $\iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-)\mathrm{e}^{-\varepsilon\mathcal{L}^{\nabla}_{-\tilde{X}}}$ does not meet the conormal to the diagonal in $M \times M$ and we have for any $\alpha \in \Omega^1(M, \mathrm{End}(E))$*

$$\operatorname{tr}^{\flat}_{\mathrm{s}} \alpha\iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-)\mathrm{e}^{-\varepsilon\mathcal{L}^{\nabla}_{-\tilde{X}}} = 0.$$

We refer to Section §6.12 for the proof. An immediate corollary is the formula

$$\operatorname{tr}_{\mathrm{s},\widetilde{C}^{\bullet}_-}\widetilde{\Pi}_-\alpha\tilde{k}_- = \operatorname{tr}^{\flat}_{\mathrm{s}} \alpha\widetilde{K}\mathrm{e}^{-\varepsilon\mathcal{L}^{\nabla}_{-\tilde{X}}}. \tag{6.9.4}$$

Indeed, since $\mathcal{L}^{\nabla}_{-\widetilde{X}}\widetilde{\Pi}_- = 0$, we have $\widetilde{\Pi}_-\mathrm{e}^{-\varepsilon\mathcal{L}^{\nabla}_{-\tilde{X}}} = \widetilde{\Pi}_-$. Moreover, since the trace of finite rank operators coincides with the flat trace, we have $\operatorname{tr}_{\mathrm{s},\widetilde{C}^{\bullet}_-}\widetilde{\Pi}_-\alpha\tilde{k}_- = \operatorname{tr}_{\mathrm{s},\widetilde{C}^{\bullet}_-}\widetilde{\Pi}_-\alpha\tilde{k}_-\mathrm{e}^{-\varepsilon\mathcal{L}^{\nabla}_{-\tilde{X}}} = \operatorname{tr}^{\flat}_{\mathrm{s}} \alpha\tilde{k}_-\widetilde{\Pi}_-\mathrm{e}^{-\varepsilon\mathcal{L}^{\nabla}_{-\tilde{X}}}$. Therefore we obtain with Proposition 6.9.2

$$\operatorname{tr}_{\mathrm{s},\widetilde{C}^{\bullet}}\widetilde{\Pi}_-\alpha\tilde{k}_- = \operatorname{tr}^{\flat}_{\mathrm{s}} \alpha\iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-)\mathrm{e}^{-\varepsilon\mathcal{L}^{\nabla}_{-\tilde{X}}} + \operatorname{tr}^{\flat}_{\mathrm{s}} \alpha\tilde{k}_-\widetilde{\Pi}_-\mathrm{e}^{-\varepsilon\mathcal{L}^{\nabla}_{-\tilde{X}}},$$

which gives (6.9.4).

6.9.5 Proof of Proposition 6.9.1

Note that we have by (6.9.1)

$$\tau_\vartheta(\rho(z)) = \tau_\vartheta(\nabla(z)), \quad \tau_{\mathfrak{e},\mathfrak{o}}(\rho(z)) = \tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)).$$

We will set $f(z) = \tau_\vartheta(\nabla(z))/\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$ for simplicity. Now we apply Proposition 6.6.1, Proposition 6.8.1 to obtain that $z \mapsto f(z)$ is real differentiable (since $z \mapsto \nabla(z)$

is) ; moreover it is complex differentiable at $z = 0$ by (6.9.2) and for $\varepsilon > 0$ small enough we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} \log f(z) = -\operatorname{tr}_{\mathrm{s}}^\flat \alpha K \mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} + \operatorname{tr}_{\mathrm{s}}^\flat \alpha \widetilde{K} \mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} + \langle \operatorname{tr} \alpha, e \rangle, \tag{6.9.5}$$

where we used (6.9.4). Let

$$\Delta = \nabla\nabla^\star + \nabla^\star\nabla \ : \ \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$$

be the Hodge-Laplace operator induced by any metric on M and any Hermitian product on E. Because ∇ is acyclic, Δ is invertible and Hodge theory gives that its inverse Δ^{-1} is a pseudo-differential operator of order -2. Define

$$J = \nabla^\star \Delta^{-1} \ : \ \mathcal{D}'^\bullet(M,E) \to \mathcal{D}'^{\bullet-1}(M,E).$$

We have of course

$$[\nabla, J] = \nabla J + J\nabla = \mathrm{Id}_{\mathcal{D}'^\bullet(M,E)} \, . \tag{6.9.6}$$

Let R_ε be the interpolator at time ε defined in §6.7.3 for the pair of vector fields $(-\widetilde{X}, X)$. This implies with (6.7.4)

$$[\nabla, R_\varepsilon] = \mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla}. \tag{6.9.7}$$

Now define

$$G_\varepsilon = J\left(K\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K}\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - R_\varepsilon\right) : \Omega^\bullet(M,E) \to \mathcal{D}'^{\bullet-2}(M,E).$$

Let us compute, having (6.9.6) in mind,

$$\begin{aligned}
[\nabla, G_\varepsilon] &= \nabla J\left(K\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K}\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - R_\varepsilon\right) - J\left(K\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K}\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - R_\varepsilon\right)\nabla \\
&= (\mathrm{Id} - J\nabla)\left(K\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K}\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - R_\varepsilon\right) \\
&\qquad\qquad - J\left(K\nabla\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K}\nabla\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - R_\varepsilon\nabla\right) \\
&= K\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K}\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - R_\varepsilon \\
&\qquad\qquad - J\left([\nabla, K]\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - [\nabla, \widetilde{K}]\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - [\nabla, R_\varepsilon]\right),
\end{aligned}$$

where we used that $\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla}$ and $\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla}$ commute with ∇. Now note that (6.6.7), (6.7.4) and (6.9.3) imply

$$[\nabla, K]\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - [\nabla, \widetilde{K}]\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - [\nabla, R_\varepsilon] = \mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - \left(\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla}\right) = 0.$$

Therefore we obtained

$$[\nabla, G_\varepsilon] = K\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K}\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - R_\varepsilon.$$

Because $[\nabla, \alpha] = 0$ we have

$$[\nabla, \alpha G_\varepsilon] = -\alpha\left(K\mathrm{e}^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K}\mathrm{e}^{-\varepsilon \mathcal{L}_{-\tilde{X}}^\nabla} - R_\varepsilon\right).$$

Using the notations of §B.3.1, we have that $\mathrm{WF}(J), \mathrm{WF}(\alpha)$ and $\mathrm{WF}(\nabla)$ are contained in the conormal bundle of the diagonal $N^*\Delta$ since J, α, ∇ are pseudodifferential operators ; moreover, equation (6.7.6) shows that

$$\mathrm{WF}\left(K\mathrm{e}^{-\varepsilon\mathcal{L}_X^\nabla} - \widetilde{K}\mathrm{e}^{-\varepsilon\mathcal{L}_{-\tilde{X}}^\nabla} - R_\varepsilon\right) \cap N^*\Delta = \emptyset.$$

It follows from wave front composition [Hör90, Theorem 8.2.14] that $\mathrm{WF}(\alpha G_\varepsilon) \cap N^*\Delta = \emptyset$. The operators $\nabla, \alpha G_\varepsilon$ satisfy the assumptions of Proposition B.3.1 which gives $\mathrm{tr}_\mathrm{s}^\flat\,[\nabla, \alpha G_\varepsilon] = 0$ and therefore (6.9.5) reads

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=0} \log f(z) = -\,\mathrm{tr}_\mathrm{s}^\flat\, \alpha R_\varepsilon + \big\langle \mathrm{tr}\,\alpha, e\big\rangle. \tag{6.9.8}$$

The identity $[\nabla, \alpha] = 0$ also implies that $\mathrm{d}\,\mathrm{tr}\,\alpha = \mathrm{tr}\,\nabla^{E\otimes E^\vee}\alpha = \mathrm{tr}[\nabla, \alpha] = 0$. As a consequence we can apply (6.7.5) to obtain

$$\mathrm{tr}_\mathrm{s}^\flat\, \alpha R_\varepsilon = \big\langle \mathrm{tr}\,\alpha,\ \mathrm{cs}(-\widetilde{X}, X)\big\rangle.$$

Now note that $\partial\big(-\mathrm{cs}(-\widetilde{X}, X) + e\big) = -\big(\mathrm{div}(X) - \mathrm{div}(-\widetilde{X})\big) + \mathrm{div}(\widetilde{X}) = 0$ by (6.7.1) and (6.8.2) since X is non singular. Therefore we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=0} \log f(z) = \big\langle \mathrm{tr}\,\alpha, h\big\rangle$$

where $h = [-\mathrm{cs}(-\widetilde{X}, X) + e] \in H_1(M, \mathbb{Z})$. Finally, let us note that by (6.3.10),

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=0} \log\big\langle \det \rho(z), h\big\rangle = -\big\langle \mathrm{tr}\,\alpha, h\big\rangle,$$

since $\rho(z) = \rho_{\nabla(z)}$. Therefore the proposition is proved for $z = 0$. However the same argument holds for every z close enough to 0, which concludes.

6.9.6 Proof of Theorems 6.1.3 and 6.1.4

By Hartog's theorem and Proposition 6.9.1, we have that the map

$$\rho \mapsto \frac{\tau_\vartheta(\rho)}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho)}\langle \det \rho, h\rangle \tag{6.9.9}$$

is locally constant on $\mathrm{Rep}_\mathrm{ac}(M, d) \setminus \Sigma(M, d)$.

Moreover, we can reproduce all the arguments we made in the continuous category to obtain that $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is actually continuous on $\mathrm{Rep}_\mathrm{ac}(M, d)$. Because $\mathrm{Rep}_\mathrm{ac}(M, d) \setminus \Sigma(M, d)$ is open and dense in $\mathrm{Rep}_\mathrm{ac}(M, d)$, we get that the map 6.9.9 is locally constant on $\mathrm{Rep}_\mathrm{ac}(M, d)$.

By [FT00, p. 211] we have, if $\mathfrak{e}'$ is another Euler structure, $\tau_{\mathfrak{e}',\mathfrak{o}}(\rho) = \langle \det \rho, \mathfrak{e}' - \mathfrak{e}\rangle \tau_{\mathfrak{e},\mathfrak{o}}(\rho)$. As a consequence, if we set $\mathfrak{e}_\vartheta = [-X, 0]$ which defines an Euler structure since X is nonsingular (see §6.7.2), we have $\mathfrak{e} - \mathfrak{e}_\vartheta = h$ and we obtain that $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\rho)$ is locally constant on $\mathrm{Rep}_\mathrm{ac}(M, d)$.

Now let η be another contact form inducing an Anosov Reeb flow and denote by X_η its Reeb flow. Then if $\mathfrak{e}_\eta = [-X_\eta, 0]$, we have

$$\mathfrak{e}_\eta - \mathfrak{e}_\vartheta = \mathrm{cs}(X, X_\eta)$$

by definition. Therefore

$$\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\rho) = \tau_{\mathfrak{e}_\eta,\mathfrak{o}}(\rho)\langle \det \rho, \mathfrak{e}_\vartheta - \mathfrak{e}_\eta\rangle = \tau_{\mathfrak{e}_\eta,\mathfrak{o}}(\rho)\langle \det \rho, \mathrm{cs}(X_\eta, X)\rangle$$

and we obtain that

$$\rho \mapsto \frac{\tau_\vartheta(\rho)}{\tau_\eta(\rho)}\langle \det \rho, \mathrm{cs}(X, X_\eta)\rangle$$

is locally constant on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$. By Theorem 6.5.1 we thus obtain Theorem 6.1.4.

Finally assume that $\dim M = 3$ and $b_1(M) \neq 0$. Take $\mathcal{R}$ a connected component of $\mathrm{Rep}_{\mathrm{ac}}(M, d)$ and assume that it contains an acyclic and unitary representation ρ_0. We invoke [DGRS18, Theorem 1] and the Cheeger-Müller theorem [Che79, Mül78] to obtain that $0 \notin \mathrm{Res}(\mathcal{L}_X^{\nabla_{\rho_0}})$ and

$$|\tau_\vartheta(\rho_0)| = |\zeta_{X,\nabla_{\rho_0}}(0)|^{-1} = \tau_{\mathrm{RS}}(\rho_0),$$

where the first equality comes from (6.4.8) (we have $q = 1$ since $\dim M = 3$) and $\tau_{\mathrm{RS}}(\rho_0)$ is the Ray-Singer torsion of (M, ρ_0), cf. [RS71]. On the other hand, we have by [FT00, Theorem 10.2] that $\tau_{\mathrm{RS}}(\rho_0) = |\tau_{\mathfrak{e},\mathfrak{o}}(\rho_0)|$ since ρ_0 is unitary. Therefore the map $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\rho)$ is of modulus one on $\mathcal{R}$. This concludes the proof of Theorem 6.1.3.

6.10 Projectors of finite rank

6.10.1 Traces on variable finite dimensional spaces

In what follows, we consider two Hilbert spaces $\mathcal{G} \subset \mathcal{H}$, the inclusion being dense and continuous. We will denote by $\mathcal{L}(\mathcal{H}, \mathcal{G})$ the space of bounded linear operators $\mathcal{H} \to \mathcal{G}$ endowed with the operator norm. Let $\delta > 0$ and Π_t, $|t| \leqslant \delta$, be a family of finite rank projectors on $\mathcal{H}$ such that $\mathrm{ran}\,\Pi_t \subset \mathcal{G}$. Assume that $t \mapsto \Pi_t$ is differentiable at $t = 0$ as a family of bounded operators $\mathcal{H} \to \mathcal{G}$, that is,

$$\Pi_t = \Pi + tP + o_{\mathcal{H}\to\mathcal{G}}(t) \tag{6.10.1}$$

for some $P \in \mathcal{L}(\mathcal{H}, \mathcal{G})$, where $\Pi = \Pi_0$. Denote $C_t = \mathrm{ran}\,\Pi_t$ and $C = \mathrm{ran}\,\Pi$. Note that by continuity, $\Pi_t|_C : C \to C_t$ is invertible for $|t|$ small enough; we denote by $Q_t : C_t \to C$ its inverse.

Lemma 6.10.1. *We have*

(i) $P = \Pi P + P\Pi$,

(ii) $Q_t\Pi_t = \Pi\Pi_t + o_{\mathcal{H}\to\mathcal{G}}(z)$.

Proof. Using (6.10.3) and $\Pi_t^2 = \Pi_t$ we obtain *(i)*. This implies

$$\begin{aligned}\Pi_t \circ \Pi \circ \Pi_t &= \Big(\Pi + tP + o(t)\Big)\Pi\Big(\Pi + tP + o(t)\Big)\\ &= \Pi + t\Big(P\Pi + \Pi P\Big) + o(t)\\ &= \Pi + tP + o(t)\\ &= \Pi_t + o(t),\end{aligned}$$

where all the $o(t)$ are taken in $\mathcal{L}(\mathcal{H},\mathcal{G})$. Therefore $Q_t \circ \Pi_t \circ \Pi \circ \Pi_t = Q_t\Pi_t + o(t)$. Since $Q_t \circ \Pi_t \circ \Pi = \Pi$ by definition, one obtains

$$Q_t \circ \Pi_t = \Pi \circ \Pi_t + o(t),$$

which proves the first part of the Lemma. The second part is very similar. □

Lemma 6.10.2. *Let A_t, $|t| \leqslant \delta$, be a $\mathcal{C}^1$ family of bounded operators $\mathcal{G} \to \mathcal{H}$ such that A_t commutes with Π_t for every t. Denote $A = A_0$. Then $t \mapsto \operatorname{tr}_{C_t}(A_t)$ is real differentiable at $t = 0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \operatorname{tr}_{C_t}(A_t) = \operatorname{tr}_C\big(\Pi\dot{A}\big),$$

where $\dot{A}_t = \frac{\mathrm{d}}{\mathrm{d}t}A_t$. If moreover A is invertible on C, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \log\det_{C_t}(A_t) = \operatorname{tr}_C\left(\Pi\dot{A}(A|_C)^{-1}\right).$$

Proof. We start from

$$\operatorname{tr}_{C_t}(A_t) = \operatorname{tr}_C(Q_tA_t\Pi_t).$$

Now since A_t commutes with Π_t we have by the second part Lemma 6.10.1

$$\begin{aligned}Q_tA_t\Pi_t\Pi &= \Pi\Pi_tA_t\Pi + o_{C\to C}(t)\\ &= \Pi A\Pi + t\Pi\Big(\dot{A} + PA\Pi + \Pi AP\Big)\Pi + o_{C\to C}(t).\end{aligned}$$

But now the first part of Lemma 6.10.1 gives $\Pi P\Pi = 0$. We therefore obtain, because A and Π commute,

$$Q_tA_t\Pi_t\Pi = \Pi A\Pi + t\Pi\dot{A}\Pi + o_{C\to C}(t), \tag{6.10.2}$$

which concludes. □

6.10.2 Gain of regularity

Assume that we are given four Hilbert spaces $\mathcal{E} \subset \mathcal{F} \subset \mathcal{G} \subset \mathcal{H}$ with continuous and dense inclusions. Let Π_t, $|t| < \delta$ be a family of finite rank projectors on $\mathcal{H}$ which is differentiable at $t = 0$ as family of bounded operators $\mathcal{G} \to \mathcal{H}$ (note that this differs from the last subsection where we had $\mathcal{H} \to \mathcal{G}$ instead), that is

$$\Pi_t = \Pi + tP + o_{\mathcal{G}\to\mathcal{H}}(t) \tag{6.10.3}$$

for some $P \in \mathcal{L}(\mathcal{G},\mathcal{H})$. We will denote $C_t = \operatorname{ran}(\Pi_t) \subset \mathcal{H}$ and $C = \operatorname{ran}(\Pi)$.

Lemma 6.10.3. *Under the above assumptions, assume that Π_t is bounded $\mathcal{E} \to \mathcal{F}$ and that Π_t is differentiable at $t = 0$ as a family of $\mathcal{L}(\mathcal{E}, \mathcal{F})$. Assume also that* $\operatorname{rank} \Pi_t$ *does not depend on t. Then P is actually bounded $\mathcal{G} \to \mathcal{F}$ and*

$$\Pi_t = \Pi + tP + o_{\mathcal{G}\to\mathcal{F}}(t).$$

Proof. Because $\mathcal{E}$ is dense in $\mathcal{H}$ we know that $C \subset \mathcal{F}$. There exists $\varphi^1, \dots, \varphi^m \in \mathcal{E}$ such that $\varphi_t^1, \dots, \varphi_t^m$ is a basis of C_t for t small enough where we set $\varphi_t^j = \Pi_t(\varphi^j) \in \mathcal{F}$. Denote $\tilde{\varphi}_t^j = \Pi(\varphi_t^j) \in C$. This family $t \mapsto \tilde{\varphi}_t^j \in C$ is differentiable at $t = 0$. Let $\nu_t^1, \dots, \nu_t^m \in C^*$ be the dual basis of $\tilde{\varphi}_t^1, \dots, \tilde{\varphi}_t^m$. Because C is finite dimensional, Π is actually bounded $\mathcal{H} \to \mathcal{F}$. As a consequence the map

$$t \mapsto \ell_t^j = \nu_t^j \circ \Pi \circ \Pi_t \in \mathcal{G}'$$

is differentiable at $t = 0$. Noting that

$$\Pi_t = \sum_{j=1}^m \varphi_t^j \otimes \ell_t^j : \mathcal{G} \to \mathcal{F},$$

we finally obtain that $t \mapsto \Pi_t \in \mathcal{L}(\mathcal{G}, \mathcal{F})$ is differentiable at $t = 0$. □

6.11 Continuity of the Pollicott-Ruelle spectrum

We describe here the spaces used in §§6.5,6.6. In what follows, M is a compact manifold, (E, ∇) a flat vector bundle on M and X_0 is a vector field on M generating an Anosov flow, cf. §7.1.3. We denote by $T^*M = E_{u,0}^* \oplus E_{s,0}^* \oplus E_{0,0}^*$ its Anosov decomposition of T^*M.

6.11.1 Bonthonneau's uniform weight function

We state here a lemma from Bonthonneau which is [Bon20, Lemma 3]. This gives us an escape function having uniform good properties for a family of vector fields. A consequence is that one can define some uniform anisotropic Sobolev spaces on which each vector field of the family has good spectral properties. In what follows, $|\cdot|$ is a smooth norm on T^*M.

Lemma 6.11.1. *There exists conical neighborhoods N_u and N_s of $E_{u,0}^*$ and $E_{s,0}^*$, some constants $C, \beta, T, \eta > 0$, and a weight function $m \in \mathcal{C}^\infty(T^*M, [0,1])$ such that the following holds. Let X be any vector field satisfying $\|X - X_0\|_{\mathcal{C}^1} < \eta$, and denote by Φ^t its induced flow on T^*M and by E_u^* and E_s^* its (dual) unstable and stable bundles. Then*

1. *$E_\bullet^* \subset N_\bullet$, for $\bullet = s, u$ and for any $t > 0$, $\xi_u \in E_u^*$ and $\xi_s \in E_s^*$ one has*
$$|\Phi^t(\xi_u)| \geqslant \frac{1}{C} e^{\beta t} |\xi_u|, \quad |\Phi^{-t}(\xi_s)| \geqslant \frac{1}{C} e^{\beta t} |\xi_s|.$$
2. *For every $t \geqslant T$ it holds*
$$\Phi^t \left(\complement N_s \cap X^\perp\right) \subset N_u, \quad \Phi^{-t} \left(\complement N_u \cap X^\perp\right) \subset N_s,$$
where $X^\perp = \{\xi \in T^*M,\ \xi \cdot X = 0\}$.
3. *If $\mathbf{X}$ is the Lie derivative induced by Φ^t, then*
$$m \equiv 1 \text{ near } N_s, \quad m \equiv -1 \text{ near } N_u, \quad \mathbf{X}.m \geqslant 0.$$

6.11.2 Anisotropic Sobolev spaces

Take the weight function m of Lemma 6.11.1. Define the escape function g by

$$g(x,\xi) = m(x,\xi)\log(1+|\xi|), \quad (x,\xi) \in T^*M.$$

We set $G = \mathrm{Op}(g) \in \Psi^{0+}(M)$ for any quantization procedure Op. Then by [Zwo12, §§8.3,9.3,14.2] we have $\exp(\pm\mu G) \in \Psi^{\mu+}(M)$ for any $\mu > 0$. For any $\mu > 0$ and $j \in \mathbb{Z}$ we define the spaces

$$\mathcal{H}^\bullet_{\mu G,j} = \exp(-\mu G) H^j(M, \wedge^\bullet \otimes E) \subset \mathcal{D}'^\bullet(M,E),$$

where $H^j(M, \wedge^\bullet \otimes E)$ is the usual Sobolev space of order j on M with values in the bundle $\wedge^\bullet \otimes E$. Note that any pseudo-differential operator of order m is bounded $\mathcal{H}^\bullet_{\mu G,j} \to \mathcal{H}^\bullet_{\mu G,j-m}$ for any μ, m, j.

6.11.3 Uniform parametrices

Let us consider a smooth family of vector fields X_t, $|t| < \varepsilon$, perturbing X_0. For any $c, \rho > 0$ we will denote

$$\Omega(c,\rho) = \{\mathrm{Re}(s) > c\} \cup \{|s| \leqslant \rho\} \subset \mathbb{C}.$$

The spaces defined in the last subsection yields an uniform version of [DZ16, Proposition 3.4], as follows.

Proposition 6.11.2. *[Bon20, Lemma 9] Let Q be a pseudo-differential operator micro-locally supported near the zero section in T^*M and elliptic there. There exists $c, \varepsilon_0 > 0$ such that for any $\rho > 0$ and $J \in \mathbb{N}$, there is $\mu_0, h_0 > 0$ such that the following holds. For each $\mu \geqslant \mu_0$, $0 < h < h_0$, $j \in \mathbb{Z}$ such that $|j| \leqslant J$ and $s \in \Omega(c,\rho)$ the operator*

$$\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s : \mathcal{H}^\bullet_{\mu G,j+1} \to \mathcal{H}^\bullet_{\mu G,j}$$

is invertible for $|t| \leqslant \varepsilon_0$ and the inverse is bounded $\mathcal{H}^\bullet_{\mu G,j} \to \mathcal{H}^\bullet_{\mu G,j}$ independently of t.

6.11.4 Continuity of the Pollicott-Ruelle spectrum

We fix $\rho, J \geqslant 4$ and μ_0, μ, h_0, h, j as in Proposition 6.11.2. We first observe that

$$\left(\mathcal{L}^\nabla_{X_t} + s\right)\left(\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s\right)^{-1} = \mathrm{Id} + h^{-1}Q\left(\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s\right)^{-1}. \qquad (6.11.1)$$

Since Q is supported near 0 in T^*M, it is smoothing and thus trace class on any $\mathcal{H}^\bullet_{\mu G,j}$. By analytic Fredholm theory, the family $s \mapsto K(t,s) = h^{-1}Q\left(\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s\right)^{-1}$ is a holomorphic family of trace class operators on $\mathcal{H}^\bullet_{\mu G,j}$ in the region $\Omega(c,\rho)$. We can therefore consider the Fredholm determinant

$$D(t,s) = \det_{\mathcal{H}^\bullet_{\mu G,j}}\left(\mathrm{Id} + K(t,s)\right).$$

It follows from [Sim05, Corollary 2.5] that for each t, $s \mapsto D(t,s)$ is holomorphic on $\Omega(c,\rho)$. Moreover (6.11.1) shows that its zeros coincide, on $\Omega(c,\rho)$, with the Pollicott-Ruelle resonances of $\mathcal{L}^{\nabla}_{X_t}$. In addition, we have for any $s \in \Omega(c,\rho)$,

$$\begin{aligned}\left(\mathcal{L}^{\nabla}_{X_t} - h^{-1}Q + s\right)^{-1} - \left(\mathcal{L}^{\nabla}_{X_{t'}} - h^{-1}Q + s\right)^{-1} \\ = -\left(\mathcal{L}^{\nabla}_{X_t} - h^{-1}Q + s\right)^{-1}\left(\mathcal{L}^{\nabla}_{X_t} - \mathcal{L}^{\nabla}_{X_{t'}}\right)\left(\mathcal{L}^{\nabla}_{X_{t'}} - h^{-1}Q + s\right)^{-1}.\end{aligned} \tag{6.11.2}$$

We have

$$\frac{\mathcal{L}^{\nabla}_{X_t} - \mathcal{L}^{\nabla}_{X_{t'}}}{t-t'} \underset{t\to t'}{\longrightarrow} \mathcal{L}^{\nabla}_{\dot{X}_t} \text{ in } \mathcal{L}(\mathcal{H}^{\bullet}_{\mu G,j+1}, \mathcal{H}^{\bullet}_{\mu G,j}). \tag{6.11.3}$$

where $\dot{X}_t = \frac{\mathrm{d}}{\mathrm{d}t}X_t$ and $\mathcal{L}(\mathcal{H}^{\bullet}_{\mu G,j+1}, \mathcal{H}^{\bullet}_{\mu G,j})$ is the space of bounded linear operators $\mathcal{H}^{\bullet}_{\mu G,j+1} \to \mathcal{H}^{\bullet}_{\mu G,j}$ endowed with the operator norm. We therefore obtain by Proposition 6.11.2 and because Q is smoothing (and thus trace class $\mathcal{H}^{\bullet}_{\mu G,j} \to \mathcal{H}^{\bullet}_{\mu G,j'}$ for any μ, j, j') that $K(t',s) \to K(t,s)$ as $t' \to t$ in $\mathcal{L}^1(\mathcal{H}^{\bullet}_{\mu G,0})$ locally uniformly in s, where $\mathcal{L}^1(\mathcal{H}^{\bullet}_{\mu G,0})$ is the space of trace class operators on $\mathcal{H}^{\bullet}_{\mu G,0}$ endowed with its usual norm. As a consequence, we obtain with [Sim05, Corollary 2.5]

$$D(t,s) \in \mathcal{C}^0\big([-\varepsilon_0,\varepsilon_0]_t, \mathrm{Hol}\big(\Omega(c,\rho)_s\big)\big). \tag{6.11.4}$$

6.11.5 Regularity of the resolvent

Let $\mathcal{Z}$ be an open set of $\mathbb{C}$ whose closure is contained in the interior of $\Omega(c,\rho)$. We assume that $\overline{\mathcal{Z}} \cap \mathrm{Res}(\mathcal{L}^{\nabla}_{X_0}) = \emptyset$. Up to taking ε_0 smaller, Rouché's theorem and (6.11.4) imply that there exists $\delta > 0$ such that $\mathrm{dist}\left(\mathcal{Z},\ \mathrm{Res}(\mathcal{L}^{\nabla}_{X_t})\right) > \delta$ for any $|t| \leqslant \varepsilon_0$. As a consequence, we obtain that for every $|j| \leqslant J$, the map $\left(\mathcal{L}^{\nabla}_{X_t} + s\right)^{-1} : \mathcal{H}^{\bullet}_{\mu G,j} \to \mathcal{H}^{\bullet}_{\mu G,j}$ is bounded independently of $(t,s) \in [-\varepsilon_0,\varepsilon_0] \times \mathcal{Z}$. Noting that

$$\frac{\left(\mathcal{L}^{\nabla}_{X_t} + s\right)^{-1} - \left(\mathcal{L}^{\nabla}_{X_{t'}} + s\right)^{-1}}{t-t'} = -\left(\mathcal{L}^{\nabla}_{X_t} + s\right)^{-1}\frac{\mathcal{L}^{\nabla}_{X_t} - \mathcal{L}^{\nabla}_{X_{t'}}}{t-t'}\left(\mathcal{L}^{\nabla}_{X_{t'}} + s\right)^{-1}, \tag{6.11.5}$$

we obtain by (6.11.3) that $t' \mapsto \left(\mathcal{L}^{\nabla}_{X_{t'}} + s\right)^{-1}$ is continuous in $\mathcal{L}(\mathcal{H}^{\bullet}_{\mu G,j+1}, \mathcal{H}^{\bullet}_{\mu G,j})$. Therefore, applying (6.11.5) again, we get that

$$\left(\mathcal{L}^{\nabla}_{X_t} + s\right)^{-1} \in \mathcal{C}^1\big([-\varepsilon_0,\varepsilon_0]_t, \mathrm{Hol}(\mathcal{Z}_s,\ \mathcal{L}(\mathcal{H}^{\bullet}_{\mu G,j+1}, \mathcal{H}^{\bullet}_{\mu G,j-2})\big). \tag{6.11.6}$$

Note that here we need $|j-2|, |j+1| \leqslant J$.

6.11.6 Regularity of the spectral projectors

Let $0 < \lambda < 1$ such that $\{|s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}^{\nabla}_{X_0}) = \emptyset$. Applying the last subsection with $\mathcal{Z} = \{|s| = \lambda\}$, we get $\{|s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}^{\nabla}_{X_t}) = \emptyset$ for any $|t| \leqslant \varepsilon_0$. We can therefore define for those t

$$\Pi_t = \frac{1}{2\pi i}\int_{|s|=\lambda}\left(\mathcal{L}^{\nabla}_{X_t} + s\right)^{-1}\mathrm{d}s : \mathcal{H}^{\bullet}_{\mu G,j} \to \mathcal{H}^{\bullet}_{\mu G,j}.$$

Then (6.11.6) gives that $\Pi_t \in \mathcal{C}^1\big([-\varepsilon_0,\varepsilon_0]_t, \mathcal{Z}_s,\ \mathcal{L}(\mathcal{H}^\bullet_{\mu G,j+1}, \mathcal{H}^\bullet_{\mu G,j-2}\big)$. This is true for $j=3$ and $j=-1$ because $J \geqslant 4$. Moreover by Rouché's theorem, the number m of zeros of $s \mapsto D(t,s)$ does not depend on t. Noting that

$$\partial_s K(t,s)(1+K(t,s))^{-1} = -K(t,s)\left(\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s\right)^{-1}(1+K(t,s))^{-1},$$

we obtain by [DZ19, Theorem C.11] and the cyclicity of the trace that m is equal to

$$\begin{aligned}\frac{1}{2\pi i}\operatorname{tr}&\int_{|s|=\lambda}\partial_s K(t,s)(1+K(t,s))^{-1}\mathrm{d}s\\ &= -\frac{1}{2\pi i}\operatorname{tr}\int_{|s|=\lambda}\left(\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s\right)^{-1}(1+K(t,s))^{-1}K(t,s)\mathrm{d}s\\ &= \frac{1}{2\pi i}\operatorname{tr}\int_{|s|=\lambda}\left(\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s\right)^{-1}(1+K(t,s))^{-1},\end{aligned}$$

where we used that $s \mapsto \left(\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s\right)^{-1}$ is holomorphic on $\{|s| \leqslant \lambda\}$. The last integral is equal to $\operatorname{tr}\Pi_t = \operatorname{rank}\Pi_t$ by (6.11.1). As a consequence we can apply Lemma 6.10.3 to obtain that

$$\Pi_t \in \mathcal{C}^1\big([-\varepsilon_0,\varepsilon_0]_t, \mathcal{L}(\mathcal{H}^\bullet_{\mu G,0}, \mathcal{H}^\bullet_{\mu G,1})\big). \tag{6.11.7}$$

6.11.7 Wavefront set of the spectral projectors

Let $(E,\nabla^\vee)$ be the dual bundle of (E,∇). Then (6.3.2) implies, for any $\operatorname{Re}(s) \gg 0$, $u \in \Omega^k(M,E)$ and $v \in \Omega^{n-k}(M,E^\vee)$,

$$\left\langle \left(\mathcal{L}^\nabla_{X_t} + s\right)^{-1} u, v\right\rangle = \left\langle u, \left(\mathcal{L}^{\nabla^\vee}_{-X_t} + s\right)^{-1} v\right\rangle, \tag{6.11.8}$$

where $\langle\cdot,\cdot\rangle$ is the pairing from §??. This shows that $\operatorname{Res}(\mathcal{L}^{\nabla^\vee}_{-X_t}) = \operatorname{Res}(\mathcal{L}^\nabla_{X_t})$. Therefore we can apply the preceding construction with the escape function g replaced by $-g$ (the unstable bundle of $-X_t$ is the stable one of X_t and reciprocally) and we obtain that

$$\Pi_t^\vee = \frac{1}{2\pi i}\int_{|s|=\lambda}\left(\mathcal{L}^{\nabla^\vee}_{-X_t} + s\right)^{-1}\mathrm{d}s \ \in\ \mathcal{C}^1\big([-\varepsilon_0,\varepsilon_0]_t, \mathcal{L}(\mathcal{H}^\bullet_{-\mu G,0}, \mathcal{H}^\bullet_{-\mu G,1})\big).$$

Note that (6.11.8) implies

$$\langle \Pi_t u, v\rangle = \langle u, \Pi_t^\vee v\rangle, \quad u \in \Omega^k(M,E), \quad v \in \Omega^{n-k}(M,E^\vee). \tag{6.11.9}$$

We denote $C_t^\bullet = \operatorname{ran}\Pi_t$, $C_t^{\vee\bullet} = \operatorname{ran}\Pi_t^\vee$ and $m = \operatorname{rank}\Pi_t = \operatorname{rank}\Pi_t^\vee$. Take $\varphi^1,\dots,\varphi^m$ and $\psi^1,\dots,\psi^m$ some elements of $\Omega^\bullet(M,E)$ such that $\Pi_0(\varphi^1),\dots\Pi_0(\varphi^m)$ is a basis of $C_0^\bullet$ and $\langle \Pi_0\varphi^i,\psi^j\rangle = 0$ if $i \neq j$ and $\langle \Pi_0\varphi^i,\psi_j\rangle = 1$ otherwise. For t small enough we set

$$\varphi^i_t = \Pi_t\varphi^i, \quad \psi^t_j = \Pi_t^\vee\psi^j.$$

Like in the proof of Lemma 6.10.3, (6.11.9) implies that

$$\Pi_t = \sum_{i=1}^m m_{ij}(t)\varphi^i_t\langle\psi^j_t,\cdot\rangle, \tag{6.11.10}$$

where $t \mapsto m_{ij}(t)$ is continuous near $t = 0$ and $m_{ij}(0) = \delta_{ij}$.

Next we show that there exists open conic neighborhoods of N_u and N_s such that, uniformly in $t \in [-\varepsilon_0, \varepsilon_0]$,

$$\mathrm{WF}(\varphi_t^i) \subset W_u, \quad \mathrm{WF}(\psi_t^i) \subset W_s, \quad W_u \cap W_s = \emptyset, \quad i = 1, \dots, m. \tag{6.11.11}$$

This means that the map $[-\varepsilon_0, \varepsilon_0] \ni t \mapsto \varphi_t^i$ (resp. ψ_t^i) is bounded in $\mathcal{D}'^{\bullet}_{W_u}(M, E)$ (resp. $\mathcal{D}'^{\bullet}_{W_s}(M, E^\vee)$). To proceed, we note that we can construct two weight functions m_u, m_s satisfying the properties of Lemma 6.11.1 such that $\{m_u \leqslant 0\} \cap \{m_s \geqslant 0\} = \emptyset$ (for example by choosing well the χ from [Bon20, p. 6]). Let $G_u, G_s \in \Psi^{0+}(M)$ be the associated operators from §6.11.2. Up to choosing ε_0 smaller, we obtain with (6.11.7) that the map $t \mapsto \varphi_t^i$ is bounded in $\mathcal{H}^{\bullet}_{\mu G_u, 0}$ for $\mu > 0$ big enough. For any $\chi \in \mathcal{C}^\infty(T^*M, [0,1])$ such that $\operatorname{supp} \chi \subset \{m_u \geqslant \delta\}$ for some $\delta > 0$, we have by classical rules of pseudo-differential calculus

$$\| \operatorname{Op}(\chi) \varphi_t^i \|_{H^{\delta\mu}(M, \wedge^\bullet \otimes E)} \leqslant C_\mu \|\varphi_t^i\|_{\mathcal{H}^{\bullet}_{\mu G_u, 0}} \leqslant C'_\mu, \quad t \in [-\varepsilon_0, \varepsilon_0],$$

for some constants C_μ, C'_μ independent of t. As a consequence, we obtain (for example using [DR17, Lemma 7.4]) that $[-\varepsilon_0, \varepsilon_0] \ni t \mapsto \varphi_t^i$ is bounded in $\mathcal{D}'^{\bullet}_{W_u}(M, E)$ where $W_u = \{m_u \leqslant 0\}$. Doing exactly the same with $-m_s$ and $-X_t$ we obtain that $[-\varepsilon_0, \varepsilon_0] \ni t \mapsto \psi_t^i$ is bounded in $\mathcal{D}'^{\bullet}_{W_s}(M, E^\vee)$ with $W_s = \{-m_s \geqslant 0\}$. This shows (6.11.11).

6.12 The wave front set of the Morse-Smale resolvent

The purpose of this section is to prove Proposition 6.9.2. For simplicity we prove it for $\widetilde{X}$ instead of $-\widetilde{X}$. We will denote by $\widehat{\Pi}$ the spectral projector (6.8.3) for the trivial bundle $(\mathbb{C}, \mathrm{d})$. Recall that $\mathcal{D}'_\Gamma(M \times M)$ denotes distributions whose wave front set is contained in the closed conic set $\Gamma \subset T^\bullet(M \times M)$. A family $(f_t)_{t \geqslant 0}$ of distributions will be $\mathcal{O}_{\mathcal{D}'_\Gamma}(1)$ if it is bounded in $\mathcal{D}'_\Gamma$ in the sense of [?, p. 31]. We will need the following

Lemma 6.12.1. *Let $\varepsilon > 0$ and $a \in \operatorname{Crit}(f)$. There exists $c > 0$, a closed conic set $\Gamma \subset T^*(M \times M)$ with $\Gamma \cap N^*\Delta(T^*M) = \emptyset$ and $\chi \in \mathcal{C}^\infty(M, [0,1])$ such that $\chi \equiv 1$ near a such that*

$$\mathcal{K}_{\chi, t+\varepsilon} = \mathcal{O}_{\mathcal{D}'^{n}_{\Gamma}(M \times M)}(\mathrm{e}^{-tc}),$$

where for $t \geqslant 0$, $\mathcal{K}_{\chi,t}$ is the Schwartz kernel of the operator $\chi \mathrm{e}^{-t\mathcal{L}_{\widetilde{X}}} \left(\mathrm{Id} - \widehat{\Pi} \right) \chi$.

Proof. Because $\widetilde{X}$ is $\mathcal{C}^\infty$-linearizable, we can take $U \subset \mathbb{R}^n$ to be a coordinate patch centered in a so that, in those coordinates, $\mathrm{e}^{-t\widetilde{X}}(x) = \mathrm{e}^{-tA}(x)$ where A is a matrix whose eigenvalues have nonvanishing real parts. Denoting $(x^1, \dots, x^n)$ the coordinates of the patch, $\widetilde{X}$ reads

$$\widetilde{X} = \sum_{1 \leqslant i,j \leqslant n} A_i^j x^i \partial_j.$$

We have a decomposition $\mathbb{R}^n = W^u \oplus W^s$ stable by A such that $A|_{W^u}$ (resp. $A|_{W^s}$) have eigenvalues with positive (resp. negative) real parts, $d_{u/s} = \dim W^{u/s}$, this induces a decomposition of the coordinates $x = (x_s, x_u)$. We will denote by $A_u = A|_{W^u} \oplus 0_{W^s}$, $A_s = 0_{W^u} \oplus A|_{W^s}$ and $c > 0$ such that

$$c < \inf_{\lambda \in \mathrm{sp}(A)} |\operatorname{Re}(\lambda)|$$

where $\mathrm{sp}(A)$ is the spectrum of A.

Let $\chi_1, \chi_2 \in \Omega^\bullet(M)$ such that $\operatorname{supp} \chi_i \subset \operatorname{supp} \chi$ for $i = 1, 2$. For simplicity, we identify e^{-tA} and its action on differential forms and currents given by the pull-back, $\delta^d(x)$ denotes the Dirac δ distribution at $0 \in \mathbb{R}^d$, π_1, π_2 are the projections $M \times M \mapsto M$ on the first and second factor respectively.

$$\begin{aligned}
\langle \mathcal{K}_{\chi,t}, \pi_1^*\chi_1 \wedge \pi_2^*\chi_2 \rangle &= \langle \chi_2,\ \mathrm{e}^{-tA}(\mathrm{Id} - \widehat{\Pi})\chi_1 \rangle \\
&= \left\langle \chi_2,\ \mathrm{e}^{-tA}\left(\chi_1 - \delta^{d_u}(x_u)\mathrm{d}x_u \int_{W^s} \pi_{s,0}^*\chi_1\right)\right\rangle \\
&= \left\langle \mathrm{e}^{tA_s}\chi_2,\ \mathrm{e}^{-tA_u}\chi_1 \right\rangle - \left(\int_{W^u} \pi_{u,0}^*\chi_2\right)\left(\int_{W^s} \pi_{s,0}^*\chi_1\right) \\
&= \int_0^1 \int_U \partial_\tau \left(\mathrm{e}^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge \mathrm{e}^{-tA_u}\pi_{s,\tau}^*\chi_1\right) \mathrm{d}\tau,
\end{aligned}$$

where $\pi_{u,\tau}, \pi_{s,\tau} : U \to U$ are defined by $\pi_{u,\tau}(x_u, x_s) = (x_u, \tau x_s)$ and $\pi_{s,\tau}(x_u, x_s) = (\tau x_u, x_s)$. Now write $\chi_2 = \sum_{|I|=k} \beta_I \mathrm{d}x_s^{I_s} \wedge \mathrm{d}x_u^{I_u}$. We have

$$\begin{aligned}
\partial_\tau \pi_{u,\tau}^*\chi_2(x_u, x_s) &= \partial_\tau \sum_I \tau^{|I_s|}\beta_I(x_u, \tau x_s)\mathrm{d}x_u^{I_u} \wedge \mathrm{d}x_s^{I_s} \\
&= \sum_I |I_s|\tau^{|I_s|-1}\beta_I(x_u, \tau x_s)\mathrm{d}x_u^{I_u} \wedge \mathrm{d}x_s^{I_s} \\
&\qquad + \sum_I \tau^{|I_s|}\left(\partial_{x_s}\beta_I\right)_{(x_u,\tau x_s)}(x_s)\mathrm{d}x_u^{I_u} \wedge \mathrm{d}x_s^{I_s}.
\end{aligned}$$

Therefore

$$\partial_\tau \mathrm{e}^{tA_s}\pi_{u,\tau}^*\chi_2 = \sum_I \left(|I_s|\tau^{|I_s|-1}\beta_I(x_u, \tau \mathrm{e}^{tA_s}x_s) + \tau^{|I_s|}\left(\partial_{x_s}\beta_I\right)_{(x_u,\tau x_s)}\left(\mathrm{e}^{tA_s}x_s\right)\right)\mathrm{e}^{tA_s}\mathrm{d}x^I.$$

Because $|\mathrm{e}^{tA_s}x_s| = \mathcal{O}(\mathrm{e}^{-tc})$ and $\mathrm{e}^{tA_s}\mathrm{d}x^I = \mathcal{O}(\mathrm{e}^{-ct|I_s|})$, $I = (I_s, I_u)$ is a multi–index and repeating the same argument for $\partial_\tau \mathrm{e}^{-tA_u}\pi_{s,\tau}^*\chi_1$, we obtain the bound :

$$\partial_\tau\left(\mathrm{e}^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge \mathrm{e}^{-tA_u}\pi_{s,\tau}^*\chi_1\right) = \mathcal{O}_{\chi_1,\chi_2}(\mathrm{e}^{-tc}). \tag{6.12.1}$$

Replacing χ_1 and χ_2 by $\chi_1 \mathrm{e}^{i\langle\xi,\cdot\rangle}$ and $\chi_2 \mathrm{e}^{i\langle\eta,\cdot\rangle}$ with $\xi, \eta \in \mathbb{R}^n$, one gets

$$\begin{aligned}
&\left\langle \mathcal{K}_{\chi,t}, \pi_1^*\left(\chi_1 \mathrm{e}^{i\langle\xi,\cdot\rangle}\right) \wedge \pi_2^*\left(\chi_2 \mathrm{e}^{i\langle\eta,\cdot\rangle}\right)\right\rangle \\
&\quad = \int_0^1 \int_U \partial_\tau\left(\mathrm{e}^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge \mathrm{e}^{-tA_u}\pi_{s,\tau}^*\chi_1\right)\mathrm{e}^{i\langle \mathrm{e}^{tA_s}(x_u,\tau x_s),\eta\rangle}\mathrm{e}^{i\langle \mathrm{e}^{-tA_u}(\tau x_u, x_s),\xi\rangle}\mathrm{d}\tau \\
&\qquad + \int_0^1 \int_U \mathrm{e}^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge \mathrm{e}^{-tA_u}\pi_{s,\tau}^*\chi_1 \partial_\tau\left(\mathrm{e}^{i\langle \mathrm{e}^{tA_s}(x_u,\tau x_s),\eta\rangle}\mathrm{e}^{i\langle \mathrm{e}^{-tA_u}(\tau x_u, x_s),\xi\rangle}\right)\mathrm{d}\tau.
\end{aligned}$$

Denoting $g(\tau, x_u, x_s) = \mathrm{e}^{i\langle \mathrm{e}^{tA_s}(x_u, \tau x_s), \eta\rangle} \mathrm{e}^{i\langle \mathrm{e}^{-tA_u}(\tau x_u, x_s), \xi\rangle}$ we have

$$\partial_\tau g(\tau, x_u, x_s) = i\left(\langle \mathrm{e}^{tA_s} x_s, \eta_s\rangle + \langle \mathrm{e}^{-tA_u} x_u, \xi_u\rangle\right) g(\tau, x_u, x_s) = \mathcal{O}_{\mathcal{C}^\infty(M)}(\mathrm{e}^{-tc}),$$

because $|\mathrm{e}^{tA_s} x_s|, |\mathrm{e}^{-tA_u} x_u| = \mathcal{O}(\mathrm{e}^{-tc})$. Repeating the process that led to (6.12.1) but for derivatives of χ_1, χ_2 as test forms with successive integration by parts, we therefore obtain for any $N \in \mathbb{N}$:

$$\begin{aligned}\left|\left\langle \mathcal{K}_{\chi,t}, \pi_1^*\left(\chi_1 \mathrm{e}^{i\langle \xi_1, \cdot\rangle}\right) \wedge \pi_2^*\left(\chi_2 \mathrm{e}^{i\langle \xi_2, \cdot\rangle}\right)\right\rangle\right| & \\ \leqslant C_{N,\chi_1,\chi_2} \mathrm{e}^{-tc} & \left(1 + |\mathrm{e}^{tA_s}\eta_s| + |\mathrm{e}^{-tA_u}\xi_u|\right) \\ & \times \int_0^1 \left(1 + |\tau \mathrm{e}^{tA_s}\eta_s + \xi_s| + |\tau \mathrm{e}^{-tA_u}\xi_u + \eta_u|\right)^{-N} \mathrm{d}\tau,\end{aligned}$$

where $\xi = (\xi_u, \xi_s)$ and $\eta = (\eta_u, \eta_s)$. Now assume (ξ, η) is close to $N^*\Delta(T^*M)$, say

$$\left|\frac{\xi}{|\xi|} + \frac{\eta}{|\eta|}\right| < \nu \text{ and } 1 - \nu < \frac{|\xi|}{|\eta|} < 1 + \nu$$

for some $\nu > 0$. Then we have for any $\tau \in [0, 1]$:

$$|\tau \mathrm{e}^{tA_s}\eta_s + \xi_s| + |\tau \mathrm{e}^{-tA_u}\xi_u + \eta_u| \geqslant \left(1 - \mathrm{e}^{-tc}(1 + \nu)\right)(|\xi_s| + |\eta_u|).$$

As a consequence, if $\nu > 0$ is small enough so that $(1+\nu)\mathrm{e}^{-(t+\varepsilon)c} < 1$, for every $t \geqslant 0$, we obtain

$$\left|\left\langle \mathcal{K}_{\chi,t+\varepsilon}, \pi_1^*\left(\chi_1 \mathrm{e}^{i\langle \xi, \cdot\rangle}\right) \wedge \pi_2^*\left(\chi_2 \mathrm{e}^{i\langle \eta, \cdot\rangle}\right)\right\rangle\right| \leqslant C'_{N,\chi_1,\chi_2}(1 + |\xi| + |\eta|)^{-N},$$

which concludes. □

Proof of Proposition 6.9.2. Fix $\varepsilon > 0$. For $a \in \mathrm{Crit}(f)$, take c_a, Γ_a, χ_a as in Lemma 6.12.1. The proof of Lemma 6.12.1 actually shows that for $\mathrm{Re}(s) > -c_a$, the integral

$$G_{\chi_a,\varepsilon,s} = \int_0^\infty \mathrm{e}^{-ts} \chi_a \mathrm{e}^{-(t+\varepsilon)\widetilde{X}} (\mathrm{Id} - \widehat{\Pi}) \chi_a \mathrm{d}t$$

converges as an operator $\Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$. Moreover, its Schwartz kernel $\mathcal{G}_{\chi_a,\varepsilon,s}$ is locally bounded in $\mathcal{D}'^n_{\Gamma_a}(M \times M)$ in the region $\{\mathrm{Re}(s) > -c_a\}$. We will need the following lemma.

Lemma 6.12.2. *For any $\mu > 0$, there is $\nu > 0$ with the following property. For every $x \in M$ such that $\mathrm{dist}(x, \mathrm{Crit}(f)) \geqslant \mu$, it holds*

$$\mathrm{dist}\left(x, \mathrm{e}^{-(t+\varepsilon)\widetilde{X}}(x)\right) \geqslant \nu, \quad t \geqslant 0.$$

Proof. We proceed by contradiction. Suppose that there is $\mu > 0$ and sequences $x_m \in M$ and $t_m \geqslant \varepsilon$ such that $\mathrm{dist}\left(x_m, \mathrm{e}^{-t_m\widetilde{X}}(x_m)\right) \to 0$ as $m \to \infty$ and $\mathrm{dist}(x_m, \mathrm{Crit}(f)) \geqslant \mu$. Extracting a subsequence we may assume that $x_m \to x$, $t_m \to \infty$ (indeed if $t_m \to t_\infty < \infty$ then x is a periodic point for $\widetilde{X}$, which does not exist) and for any m,

$$\mathrm{e}^{-t\widetilde{X}}(x_m) \to a \text{ and } \mathrm{e}^{t\widetilde{X}}(x_m) \to b \text{ as } t \to \infty,$$

for some $a, b \in \mathrm{Crit}(f)$. Since the space of broken curves $\overline{\mathcal{L}}(a,b)$ is compact (see [AD]), we may assume that the sequence of curves $\gamma_m = \left\{ \mathrm{e}^{t\widetilde{X}}(x_m),\ t \in \mathbb{R} \right\}$ converges to a broken curve $\ell = (\ell^1, \dots, \ell^q) \in \overline{\mathcal{L}}(a,b)$ with $\ell^j \in \mathcal{L}(c_{j-1}, c_j)$ for some $c_0, \dots, c_q \in \mathrm{Crit}(f)$ with $c_0 = a$ and $c_q = b$. Because $x_m \to x$, the proof of [AD, Theorem 3.2.2] implies $x \in \ell^j$ for some j so that $\mathrm{e}^{-t\widetilde{X}} x \to c_{j-1}$ as $t \to \infty$. Therefore replacing x by $\mathrm{e}^{-t\widetilde{X}}(x)$ for t big enough, we may assume that x is contained in a Morse chart $\Omega(c_{j-1})$ near c_{j-1}. Then $c_{j-1} \neq a$. Indeed if it was not the case then we would have $\mathrm{e}^{-t_m\widetilde{X}} x_m \to a$ as $m \to \infty$ (since x_m would be contained in $\Omega(a) \cap W^u(a)$ for big enough m and $t_m \to \infty$), which is not the case since $\mathrm{dist}(x, \mathrm{Crit}(f)) \geqslant \mu \implies x \neq a$ and $\mathrm{dist}\left(x_m, \mathrm{e}^{-t_m\widetilde{X}}(x_m)\right) \to 0$ as $m \to \infty$. Therefore the flow line of x_m exits $\Omega(c_{j-1})$ in the past. We therefore obtain, since $\mathrm{e}^{-t_m\widetilde{X}} x_m \to x$, that there is $i < j-1$ so that $c_i = c_{j-1}$. This is absurd since the sequence $\left(\mathrm{ind}_f(c_i)\right)_{i=0,\dots,q}$ is strictly decreasing. $\square$

By (6.8.11) we have $\mathrm{supp}\, \mathcal{K}_{\widehat{\Pi}} \cap \Delta = \mathrm{Crit}(f)$, where $\mathcal{K}_{\widehat{\Pi}}$ is the Schwartz kernel of $\widehat{\Pi}$ and Δ is the diagonal in $M \times M$; the same holds for $\mathrm{e}^{-(t+\varepsilon)\widetilde{X}}\widehat{\Pi} = \widehat{\Pi}$ (see [DR17]). Moreover, Lemma 6.12.2 implies that if $\chi \in \mathcal{C}^\infty(M, [0,1])$ satisfies $\chi \equiv 1$ near Δ and has support close enough to Δ, we have

$$\chi \mathrm{e}^{-(t+\varepsilon)\widetilde{X}} \chi = \sum_a \chi_a \mathrm{e}^{-(t+\varepsilon)\widetilde{X}} \chi_a.$$

Let $c = \min_{a \in \mathrm{Crit}(f)} c_a$. For $\mathrm{Re}(s) > -c$,

$$G_{\chi,\varepsilon,s} = \int_0^\infty \mathrm{e}^{-ts} \chi \mathrm{e}^{-(t+\varepsilon)\widetilde{X}} (\mathrm{Id} - \widehat{\Pi}) \chi \mathrm{d}t$$

defines an operator $\Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$, whose Schwartz kernel $\mathcal{G}_{\chi,\varepsilon,s}$ is locally bounded in $\mathcal{D}'^n_\Gamma(M \times M)$ in the region $\{\mathrm{Re}(s) > -c\}$, where $\Gamma = \bigcup_{a \in \mathrm{Crit}(f)} \Gamma_a$.

Now for $\mathrm{Re}(s) \gg 0$, we have as a consequence of the Hille–Yosida Theorem applied to $\mathcal{L}_{\widetilde{X}}$ acting on suitable anisotropic spaces [DR17, 3.2.3] :

$$\left(\mathcal{L}_{\widetilde{X}} + s\right)^{-1} = \int_0^\infty \mathrm{e}^{-ts} \mathrm{e}^{-t\widetilde{X}} \mathrm{d}t : \Omega^\bullet(M) \mapsto \mathcal{D}'^\bullet(M).$$

Therefore for $\mathrm{Re}(s) \gg 0$, it holds

$$G_{\chi,\varepsilon,s} = \chi \left(\mathcal{L}_{\widetilde{X}} + s\right)^{-1} (\mathrm{Id} - \widehat{\Pi}) \mathrm{e}^{-\varepsilon\widetilde{X}} \chi.$$

Since both members are holomorphic in the region $\{\mathrm{Re}(s) > -c\}$ and coincide for $\mathrm{Re}(s) \gg 0$, they coincide in the region $\mathrm{Re}(s) > -c$. Let $\beta \in \Omega^1(M)$. We can compute for $\mathrm{Re}(s) \gg 0$, since $\iota_{\widetilde{X}} \widehat{\Pi} = 0$ by [DR17],

$$\begin{aligned}
\mathrm{tr}_{\mathrm{s}}^\flat\ \beta \iota_{\widetilde{X}} \left(\mathcal{L}_{\widetilde{X}} + s\right)^{-1} (\mathrm{Id} - \widehat{\Pi}) \mathrm{e}^{-\varepsilon\mathcal{L}_{\widetilde{X}}} &= \mathrm{tr}_{\mathrm{s}}^\flat\ \beta \iota_{\widetilde{X}} G_{\chi,\varepsilon,s} \\
&= \int_0^\infty \mathrm{e}^{-ts}\, \mathrm{tr}_{\mathrm{s}}^\flat\ \beta \iota_{\widetilde{X}} \mathrm{e}^{-(t+\varepsilon)\widetilde{X}} (\mathrm{Id} - \widehat{\Pi}) \\
&= \int_0^\infty \mathrm{e}^{-ts}\, \mathrm{tr}_{\mathrm{s}}^\flat \beta \iota_{\widetilde{X}} \mathrm{e}^{-(t+\varepsilon)\widetilde{X}},
\end{aligned}$$

where we could interchange the integral and the flat trace thanks to the bound obtained in Lemma 6.12.1. Now the Atiyah-Bott trace formula [AB67] gives

$$\mathrm{tr}_{\mathrm{s}}^{\flat}\, \beta \iota_{\widetilde{X}} \mathrm{e}^{-(t+\varepsilon)\widetilde{X}} = 0$$

since $\widetilde{X}$ vanishes at its critical points. By holomorphy this holds true for any s such that $\mathrm{Re}(s) > -c$. In particular if $\lambda > 0$ is small enough

$$\mathrm{tr}_{\mathrm{s}}^{\flat}\, \beta \iota_{\widetilde{X}} \widehat{Y}(\mathrm{Id} - \widehat{\Pi})\mathrm{e}^{-\varepsilon \widetilde{X}} = \frac{1}{2i\pi} \int_{|s|=\lambda} \mathrm{tr}_{\mathrm{s}}^{\flat}\, \beta \iota_{\widetilde{X}} \frac{\left(\mathcal{L}_{\widetilde{X}} + s\right)^{-1}}{s} (\mathrm{Id} - \widehat{\Pi})\mathrm{e}^{-\varepsilon \mathcal{L}_{\widetilde{X}}} \mathrm{d}s = 0,$$

where $\left(\mathcal{L}_{\widetilde{X}} + s\right)^{-1} = \widehat{Y} + \dfrac{\widehat{\Pi}}{s} + \mathcal{O}(s)$. Therefore Proposition 6.9.2 is proved in the case where (E, ∇) is the trivial bundle. The general case is handled similarly. □

Troisième partie

Orbites périodiques et diffusion par des obstacles

Chapitre 7

Prescription des rebonds

Dans ce chapitre, nous étendons un résultat de comptage sous contrainte obtenu au chapitre 4 au cadre des flots de billard associés à une famille d'obstacles convexes du plan euclidien. Ce chapitre contient l'article *Closed billiard trajectories with prescribed bounces* [Chaa].

Sommaire

Introduction

Consider $D_0, D_1, \dots, D_r \subset \mathbb{R}^2$ ($r \geqslant 3$) some compact and strictly convex open sets, with smooth boundaries $\partial D_0, \dots, \partial D_r$. We assume that $D_i \cap D_j = \emptyset$ whenever $i \neq j$. We moreover assume that the billiard $\mathbf{B}' = \{D_0, D_1, \dots, D_r\}$ satisfies the non-eclipse condition, that is,

$$\operatorname{conv}(D_i \cup D_j) \cap D_k = \emptyset, \quad k \neq i, j,$$

where $\operatorname{conv}(A)$ denotes the convex hull of a set A. We will denote $D = \bigcup_j D_j$. A *billiard trajectory* is a piecewise Euclidian trajectory [1] $\gamma : I \to \mathbb{R}^2 \setminus D^\circ$ (here $I \subset \mathbb{R}$ is an interval) which rebounds on each ∂D_j according to Fresnel Descartes' law (see Figure 7.1). A trajectory $\gamma : [0, \tau] \to \mathbb{R}^2 \setminus D^\circ$ will be said to be *closed* if $\gamma(0) = \gamma(\tau)$

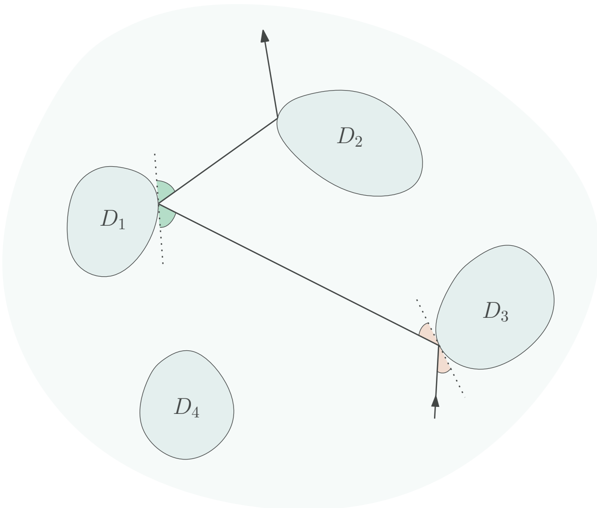

FIGURE 7.1 – A billiard trajectory

and $\gamma'(0) = \gamma'(\tau)$; a closed trajectory will be said to be *primitive* if $\gamma|_{[0,\tau']}$ is not closed for every $\tau' < \tau$. We will identify two closed trajectories $\gamma_j : \mathbb{R}/\tau_j\mathbb{Z} \to \mathbb{R}^2 \setminus D^\circ$ ($j = 1, 2$) whenever $\tau_1 = \tau_2$ and $\gamma_1(\cdot) = \gamma_2(\cdot + \tau)$ for some $\tau \in \mathbb{R}$. Denote by $\mathcal{P}_{\mathbf{B}'}$ the set of primitive closed trajectories of the billiard table $\mathbf{B}'$. Then a result of Morita [Mor91] states that there is $h'_{\mathbf{B}} > 0$ such that

$$\sharp\{\gamma \in \mathcal{P}_{\mathbf{B}'} \ : \ \tau(\gamma) \leqslant t\} \sim \frac{e^{h_{\mathbf{B}'} t}}{h_{\mathbf{B}'} t}, \quad t \to \infty, \tag{7.0.1}$$

where $\tau(\gamma)$ denotes the period of a periodic trajectory γ.

The purpose of the present paper is to give the asymptotic growth of the number of primitive closed trajectories of $\mathbf{B}'$ when we additionnaly prescribe their number of rebounds on D_0. More precisely, for $\gamma \in \mathcal{P}_{\mathbf{B}'}$ we denote by $r(\gamma)$ the number of rebounds of γ on D_0 ; we have the following result.

Theorem 7.0.1. *There are $c, h_{\mathbf{B}} > 0$ such that for every $n \geqslant 1$, it holds*

$$\sharp\{\gamma \in \mathcal{P}_{\mathbf{B}'} \ : \ \tau(\gamma) \leqslant t, \ r(\gamma) = n\} \sim \frac{(ct)^n}{n!} \frac{e^{h_{\mathbf{B}} t}}{h_{\mathbf{B}} t}, \quad t \to \infty. \tag{7.0.2}$$

1. By "Euclidian" we mean trajectories going in a straight line with constant speed 1.

Moreover $h_{\mathbf{B}}$ *depends only on the billiard table* $\mathbf{B} = \{D_1, \dots, D_r\}$.

As we will see in §7.4, by using the symbolic representation of the billiard flow and (7.0.1), one can prove that for some constants $a, b > 0$ we have

$$a t^{n-1} \exp(h_{\mathbf{B}} t) \leqslant \sharp\{\gamma \in \mathcal{P}_{\mathbf{B}'} \ : \ \tau(\gamma) \leqslant t,\ r(\gamma) = n\} \leqslant b t^{n-1} \exp(h_{\mathbf{B}} t)$$

provided t is large enough ; yet this method do not *a priori* provide the more precise asymptotics (7.0.2).

Our approach for proving (7.0.2) is reminiscent of that of Chapter 4 about the asymptotic growth of the number of closed geodesics on negatively curved surfaces for which certain intersection numbers are prescribed. In particular we make use of the work of Dyatlov–Guillarmou [DG16] about the existence of Pollicott–Ruelle resonances for open hyperbolic systems (the recent work of Küster–Schütte–Weich [KSW21] details how a hyperbolic billiard flow can be described by the framework of [DG16]). This allows to obtain a microlocal description of the transfer operator $\mathcal{T}(s)$ associated to the first return map (of the billiard flow) to $\pi^{-1}(\partial D_0)$ (here $\pi : S\mathbb{R}^2 \to \mathbb{R}^2$ is the natural projection), weighted by $\exp(-s t_0(\cdot))$ where $t_0(\cdot)$ is the first return time to $\pi^{-1}(\partial D_0)$ (see §7.2), and to apply a Tauberian theorem of Delange to the (transversal) trace of the composition [2] $\mathcal{T}(s)^n$ (which is linked to some dynamical zeta function involving the periodic orbits rebounding n times on ∂D_0).

Similar asymptotics for open dispersive billiards in $\mathbb{R}^d$ ($d \geqslant 3$) could also be obtained with our methods ; however here we restrict ourselves to the case $d = 2$ for the sake of simplicity.

Related works

In [Mor91] Morita proves the asymptotics (7.0.1) by constructing a symbolic coding of the billiard flow and by using the work of Parry-Pollicott [PP83]. Later, Stoyanov [Sto12] proved the more precise asymptotics

$$\sharp\{\gamma \in \mathcal{P}_{\mathbf{B}'} \ : \ \tau(\gamma) \leqslant t\} = \int_2^{\exp(h_{\mathbf{B}'} t)} \frac{\mathrm{d}u}{\log u} + O(\mathrm{e}^{ct}), \quad t \to +\infty,$$

for some $c \in \,]0, h_{\mathbf{B}}[$, by proving some non-integrability condition over the non-wandering set and by using Dolgopyat-type estimates (see also [PS12] for an asymptotics of the number of primitive closed trajectories with periods lying in exponentially shrinking intervals.

Organization of the chapter

This chapter is organized as follows. In §7.1 we present some geometrical and dynamical tools. In §7.2 we introduce the weighted transfer operator associated to the first return map to ∂D_0 and we compute its Attiyah-Bott transversal trace. In §7.3 we make use of a Tauberian argument. In §7.4 we prove some *a priori* estimates on $\sharp\{\gamma \in \mathcal{P}_{\mathbf{B}'} \ : \ \tau(\gamma) \leqslant t,\ r(\gamma) = n\}$. Finally in §7.5 we combine the results of §§7.3,7.4 to prove Theorem 7.0.1.

2. Actually, we compute the trace of $(\varrho \mathcal{T}(s))^n$ for some cutoff function $\varrho \in C^\infty(\pi^{-1}(\partial D_0), [0, 1])$.

7.1 Preliminaries

In this section we introduce the billiard flow associated to convex obstacles in the Euclidian space $\mathbb{R}^d$, and we recall the construction of a smooth model given by [KSW21].

7.1.1 The billiard flow

Let $D_1, \dots, D_r \subset \mathbb{R}^d$ be pairwise disjoint compact convex obstacles, satisfying the condition (8.1.1), where $r \in \mathbb{N}_{\geqslant 3}$. We denote by $S\mathbb{R}^d$ the unit tangent bundle of $\mathbb{R}^d$ and by $\pi : S\mathbb{R}^d \to \mathbb{R}^d$ the natural projection. For $x \in \partial D_j$, we denote by $n_j(x)$ the outward unit normal vector to ∂D_j at the point x pointing into $\mathbb{R}^d \setminus D_j$. Set $D = \bigcup_j D_j$ and

$$\mathcal{D} = \{(x, v) \in S\mathbb{R}^d \ : \ x \in \partial D\}.$$

We will say that $(x, v) \in T_{\partial D_j}\mathbb{R}^d$ is incoming (resp. outgoing) if $\langle v, n_j(x)\rangle > 0$ (resp. $\langle v, n_j(x)\rangle < 0$), and introduce

$$\mathcal{D}_{\text{in}} = \{(x, v) \in \mathcal{D} \ : \ (x, v) \text{ is incoming}\},$$
$$\mathcal{D}_{\text{out}} = \{(x, v) \in \mathcal{D} \ : \ (x, v) \text{ is outgoing}\}.$$

We define the grazing set $\mathcal{D}_{\text{g}} = T(\partial D) \cap \mathcal{D}$. We have

$$\mathcal{D} = \mathcal{D}_{\text{g}} \sqcup \mathcal{D}_{\text{in}} \sqcup \mathcal{D}_{\text{out}}.$$

The billiard flow $(\phi_t)_{t\in\mathbb{R}}$ is the complete flow acting on $S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ which is defined as follows. For $(x, v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ we set

$$\tau_\pm(x, v) = \pm \inf\{t > 0 : x \pm tv \in \partial D\}$$

and for $(x, v) \in \mathcal{D}_{\text{in/out/g}}$ we denote by $v' \in \mathcal{D}_{\text{out/in/g}}$ the image of v by the reflexion with respect to $T_x\partial D$ at $x \in \partial D$, that is

$$v' = v - 2\langle v, n_j(x)\rangle n_j(x), \quad v \in S_x\mathbb{R}^d, \quad x \in \partial D_j.$$

Then for $(x, v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\text{g}}$ we define

$$\phi_t(x, v) = (x + tv, v), \quad t \in [\tau_-(x, v), \tau_+(x, v)],$$

while for $(x, v) \in \mathcal{D}_{\text{in/out}}$, we set

$$\phi_t(x, v) = (x + tv, v) \quad \text{if} \quad \begin{cases} \ \ (x, v) \in \mathcal{D}_{\text{in}}, \ t \in [0, \tau_+(x, v)]\,, \\ \text{or } (x, v) \in \mathcal{D}_{\text{out}}, \ t \in [\tau_-(x, v), 0]\,, \end{cases}$$

and

$$\phi_t(x, v) = (x + tv', v') \quad \text{if} \quad \begin{cases} \ \ (x, v) \in \mathcal{D}_{\text{out}}, \ t \in \,]0, \tau_+(x, v)]\,, \\ \text{or } (x, v) \in \mathcal{D}_{\text{in}}, \ t \in [\tau_-(x, v), 0[\,. \end{cases}$$

Next we extend (ϕ_t) to a complete flow (which we still denote by (ϕ_t)) satisfying the property

$$\phi_{t+s}(x, v) = (\phi_t \circ \phi_s)(x, v), \quad t, s \in \mathbb{R}, \quad (x, v) \in S\mathbb{R}^d \setminus \pi^{-1}(D).$$

Strictly speaking, (ϕ_t) is not a flow, since the above flow property does not hold in full generality for $(x, v) \in \mathcal{D}_{\text{in/out}}$. However we arrange it considering an appropriate quotient space (see §7.1.2 below).

7.1.2 A smooth model for the non-grazing billiard flow

In this paragraph, we briefly recall the construction of [KSW21] which allows to obtain a smooth model for the non-grazing billiard flow. We first define the (non-grazing) billiard table M as

$$M = B/\sim, \quad B = S\mathbb{R}^d \setminus \left(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\right),$$

where $(x,v) \sim (y,w)$ if and only if $(x,v) = (y,w)$ or

$$x = y \in \partial D \quad \text{and} \quad w = v'.$$

The set M is endowed with the quotient topology. We will change the notation and pass from ϕ_t to the non-grazing flow φ_t, which is defined on M as follows. For $(x,v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{in}}$ we define

$$\varphi_t([(x,v)]) = [\phi_t(x,v)], \quad t \in \left]\tau_-^{\mathrm{g}}(x,v), \tau_+^{\mathrm{g}}(x,v)\right[,$$

where $[z]$ denotes the equivalence class of the vector $z \in B$ for the relation $\sim$, and

$$\tau_\pm^{\mathrm{g}}(x,v) = \pm \sup\{t > 0 : \phi_{\pm t}(x,v) \in \mathcal{D}_{\mathrm{g}}\}.$$

Note that this formula indeed defines a flow on M since each $(x,v) \in B$ has a unique representative in $(S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{in}}$. The flow φ_t is continuous but not complete and for times $t \notin \left]\tau_-^{\mathrm{g}}(x,v), \tau_+^{\mathrm{g}}(x,v)\right[$, the flow is not defined.

Following [KSW21], we define smooth charts on $M = B/\sim$ as follows. Introduce the surjection map $\pi_M : B \to M$ by $\pi_M(x,v) = [(x,v)]$ and note that by the definition of φ_t one has

$$\varphi_t \circ \pi_M = \pi_M \circ \phi_t. \tag{7.1.1}$$

We set $\mathring{B} = S\mathbb{R}^d \setminus \pi^{-1}(D)$. Then $\pi_M : \mathring{B} \to M$ is a homeomorphism onto its image $\mathcal{O}$. Let $\mathcal{G} = \pi_M(\mathcal{D}_{\mathrm{in}})$ be the gluing region. We consider the map $\pi_M^{-1} : \mathcal{O} \to \mathring{B}$ as a chart. Next we wish to define charts in an open neighborhood of $\mathcal{G}$. For every point $z_\star = (x_\star, v_\star) \in \mathcal{D}_{\mathrm{in}}$ let

$$F_{z_\star} : U_{z_\star} \times U_{z_\star} \to \mathcal{D}_{\mathrm{in}}$$

be a local smooth parameterization of $\mathcal{D}_{\mathrm{in}}$, where $U_{z_\star}$ is an open small neighborhood of 0 in $\mathbb{R}^{d-1}$. For small $\varepsilon_{z_\star} > 0$, we may define the map $\psi_{z_\star} : \left]-\varepsilon_{z_\star}, \varepsilon_{z_\star}\right[\times U_{z_\star} \times U_{z_\star} \to M$ by

$$\psi_{z_\star}(t,y,w) = (\pi_M \circ \phi_t \circ F_{z_\star})(y,w). \tag{7.1.2}$$

Shrinking $U_{z_\star}$ and taking $\varepsilon_{z_\star}$ smaller, $\psi_{z_\star}$ is a homeomorphism onto its image $\mathcal{O}_{z_\star} \subset M$, (see Corollary 4.3 in [KSW21]). It is easy to prove this. To see that $\psi_{z_\star}$ is injective, let $F_{z_\star}(y_k, w_k) = (x_k, v_k) \in \mathcal{D}_{\mathrm{in}}$, $k = 1,2$, and assume that $\pi_M \phi_{t_1}(x_1, v_1) = \pi_M \phi_{t_2}(x_2, v_2)$. Since vectors in $\mathcal{D}_{\mathrm{in}}$ are transversal to ∂D, we see that for each $z \in \mathcal{O}_{z_\star}$, there is a unique $t \in \left]-\varepsilon_{z_\star}, \varepsilon_{z_\star}\right[$ such that $\varphi_t(z) \in \mathcal{G}$. In particular, we have $t_1 = 0$ if and only if $t_2 = 0$. If that is the case, then $(x_1, v_1) = (x_2, v_2)$ since $\pi_M : \mathcal{D}_{\mathrm{in}} \to \mathcal{G}$ is injective. If $t_1 \neq 0, t_2 \neq 0$, then t_1 and t_2 have the same sign and by infectivity of $\pi_M : \mathring{B} \to M$ and the definition of ϕ_t, we have

$$\begin{cases} (x_1 + t_1 v_1, v_1) = (x_2 + t_2 v_2, v_2) & \text{if} \quad t_1, t_2 > 0, \\ (x_1 + t_1 v_1', v_1') = (x_2 + t_2 v_2', v_2') & \text{if} \quad t_1, t_2 < 0, \end{cases}$$

where v_k' is the reflexion of v_k with respect to $T_{x_k}\partial D$ for $k = 1, 2$. Thus one concludes that $(t_1, x_1, v_1) = (t_2, x_2, v_2)$. As mentioned above, the directions in $\mathcal{D}_{\mathrm{in}}$ are transversal to the boundary ∂D. This implies that the maps $\psi_{z_\star}$ are open ones. In particular, $\psi_{z_\star}$ realizes a homeomorphism onto its image and we declare the map $\psi_{z_\star}^{-1} : \mathcal{O}_{z_\star} \to]-\varepsilon_{z_\star}, \varepsilon_{z_\star}[\times U_{z_\star} \times U_{z_\star}$ as a chart. Hence we obtain an open covering

$$\mathcal{G} \subset \bigcup_{z_\star \in \mathcal{D}_{\mathrm{in}}} \mathcal{O}_{z_\star}.$$

Note that $\mathcal{O} \cap \mathcal{O}_{z_\star} \neq \emptyset$ for any $z_\star$, and clearly the map

$$(t, x, v) \mapsto (\pi_M^{-1} \circ \psi_{z_\star})(t, x, v) = (\phi_t \circ F_{z_\star})(x, v)$$

is smooth on $\psi_{z_\star}^{-1}(\mathcal{O} \cap \mathcal{O}_{z_\star})$. On the other hand, assume that $\mathcal{O}_{z_\star} \cap \mathcal{O}_{z_\star'} \neq \emptyset$ for some $z_\star, z_\star' \in \mathcal{D}_{\mathrm{in}}$. If $\pi_M(\phi_t(F_{z_\star}(x, v))) = \pi_M(\phi_s(F_{z_\star'}(y, w))) \in \mathcal{O}_{z_\star} \cap \mathcal{O}_{z_\star'}$, then as above this yields $t = s$, $F_{z_\star}(x, v) = F_{z_\star'}(y, w)$ and we conclude that

$$\begin{aligned}
(\psi_{z_\star}^{-1} \circ \psi^{\Lambda}{}_{z_\star'})(t, y, w) &= \left(\psi_{z_\star}^{-1} \circ \pi_M \circ \phi_t \circ F_{z_\star'}\right)(y, w) \\
&= \left(\psi_{z_\star}^{-1} \circ \pi_M \circ \phi_t \circ F_{z_\star}\right)\Big((F_{z_\star}^{-1} \circ F_{z_\star'})(y, w)\Big) \\
&= \left(t, (F_{z_\star}^{-1} \circ F_{z_\star})(y, w)\right).
\end{aligned} \tag{7.1.3}$$

This shows that the change of coordinates $\psi_{z_\star}^{-1} \circ \psi^{\Lambda}{}_{z_\star'}$ is smooth on the set $\psi^{\Lambda}{}_{z_\star'}^{-1}(\mathcal{O}_{z_\star} \cap \mathcal{O}_{z_\star'})$, and these charts endow M with a smooth structure. Now it is easy to see that the non-complete flow (φ_t) is smooth on M. Indeed, this is obvious far from the gluing region $\mathcal{G}$. Now let $z \in \mathcal{G}$ and $z_\star \in \mathcal{D}_{\mathrm{in}}$ be such that $\pi_M(z_\star) = z$. Then for $s, t \in \mathbb{R}$, with $|t| + |s|$ small, and $(y, w) \in U_{z_\star} \times U_{z_\star}$, we have

$$\begin{aligned}
\left(\psi_{z_\star}^{-1} \circ \varphi_s \circ \psi_{z_\star}\right)(t, y, w) &= \left(\psi_{z_\star}^{-1} \circ \varphi_s \circ \pi_M \circ \phi_t \circ F_{z_\star}\right)(y, w) \\
&= \left(\psi_{z_\star}^{-1} \circ \pi_M \circ \phi_{t+s} \circ F_{z_\star}\right)(y, w) \\
&= (s + t, y, w).
\end{aligned}$$

Consequently, the flow (φ_t) is also smooth near $\mathcal{G}$ and we obtain a smooth non-complete flow on M. We will denote by $X \in C^\infty(M, TM)$ the generator of this flow.

We conclude this paragraph by noting that the flow (φ_t) is actually a contact flow. Indeed, let $\alpha \in \Omega^1(S\mathbb{R}^d)$ be defined by

$$\langle \alpha(x, v), W \rangle = \langle v, \mathrm{d}\pi(z) W \rangle, \quad (x, v) \in S\mathbb{R}^d, \quad W \in T_{(x,v)} S\mathbb{R}^d.$$

Then it is not hard to see that the form α induces a one-form on M (still denoted by α) which satisfies that $\alpha \wedge (\mathrm{d}\alpha)^{d-1}$ is volume form and

$$\iota_X \alpha = 1, \quad \iota_X \mathrm{d}\alpha = 0,$$

where ι_X denotes the interior product.

7.1.3 Uniform hyperbolicity of the flow (φ_t)

From now on, we will work exclusively with the flow (φ_t) defined on the smooth model described in §7.1.2. The trapped set K of (φ_t) is defined as the set of points $z \in M$ which satisfy $-\tau_-^{\mathrm{g}}(z) = \tau_+^{\mathrm{g}}(z) = +\infty$ and

$$\sup A(z) = -\inf A(z) = +\infty, \quad \text{where} \quad A(z) = \{t \in \mathbb{R} \ : \ \pi(\varphi_t(z)) \in \partial D\}.$$

By definition, $\varphi_t(z)$ is defined for all $t \in \mathbb{R}$ whenever $z \in K$. The flow (φ_t) is called uniformly hyperbolic on K, if for each $z \in K$ there exists a decomposition

$$T_zM = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{7.1.4}$$

which is $\mathrm{d}\varphi_t$-invariant (in the sense that $\mathrm{d}\varphi_t(E_\bullet(z)) = E_\bullet(\varphi_t(z))$ for $\bullet = u, s$), with $\dim E_s(z) = \dim E_u(z) = d-1$. The spaces $E_s(z)$ and $E_u(z)$ depend continuously on z and for some constants $C, \nu > 0$ independent of $z \in K$, and some smooth norm $\|\cdot\|$ on TM, we have

$$\|\mathrm{d}\varphi_t(z)\cdot v\| \leqslant \begin{cases} C\mathrm{e}^{-\nu t}\|v\|, & v \in E_s(z), \quad t \geqslant 0, \\ C\mathrm{e}^{-\nu|t|}\|v\|, & v \in E_u(z), \quad t \leqslant 0. \end{cases} \tag{7.1.5}$$

We may define the trapped set K_e for the flow (φ_t). Then $K = \pi_M(K_e)$. The uniform hyperbolicity on K_e of the flow (ϕ_t) in the Euclidean metric can be defined by the splitting of the tangent space $T_z(\mathbb{R}^d)$ for $z \in \mathring{B} \cap K_e$ (see Definition 2.10 in [KSW21] and Appendix). Following this definition, one avoids the points $(x, v) \in K_e \cap \mathcal{D}_{\mathrm{in}}$. The uniform hyperbolicity of (ϕ_t) in the Euclidean metric implies the uniform hyperbolicity of (φ_t) in the smooth model (see [KSW21, Proposition 3.8]). Thus, to obtain (7.1.4) and (8.6.1), we may apply the uniform hyperbolcity of (ϕ_t) in the Euclidean metric on $\mathring{B} \cap K_e$ established for $d = 2$ in [Mor91] and [CM06, §4.4]. For $d \geqslant 2$, the same could be obtained by applying the results in [BCST03, §4]. For the sake of completeness, we present in §8.6 a proof of the uniform hyperbolicity of (ϕ_t) in the Euclidean metric as well as a construction of $E_s(z)$ and $E_u(z)$ for $z \in \mathring{B} \cap K_e$.

7.1.4 Symbolic coding

We define the first (future and the past) return times to ∂D by

$$t_\pm(z) = \inf\{t > 0 \ : \ \pi(\varphi_{\pm t}(z)) \in \partial D\}, \quad z \in M.$$

Set $\Lambda = \pi^{-1}(\partial D) \cap K$, and define

$$B_\pm(z) = \varphi_{\pm t_\pm(z)}(z), \quad z \in \Lambda.$$

By [?], the map $B_\pm : \Lambda \to \Lambda$ is Hölder conjugated to a subshift of finite type. More precisely, let $\mathcal{A} = \{1, \dots, r\}$ and

$$\Sigma_\mathcal{A} = \{(u_n) \in \mathcal{A}^\mathbb{Z} \ : \ u_n \neq u_{n+1}, \ n \in \mathbb{Z}\}.$$

Let $\sigma_\pm : \Sigma_\mathcal{A} \to \Sigma_\mathcal{A}$ be the map $(u_n) \mapsto (u_{n\pm1})$. We endow $\Sigma_\mathcal{A}$ with the topology coming from the distance

$$\mathrm{d}_{\Sigma_\mathcal{A}}(u, v) = \sum_{n\in\mathbb{Z}} 2^{-|n|}|u_n - v_n|.$$

Then there is a homeomorphism $\psi^\Lambda : \Lambda \to \Sigma_{\mathcal{A}}$, which is Hölder continuous, such that

$$\sigma_\pm \circ \psi^\Lambda = \psi^\Lambda \circ B_\pm.$$

In fact, ψ^Λ is simply given by

$$\psi^\Lambda(z)_n = j, \quad B_+^n(z) \in \partial D_j, \quad z \in \Lambda, \quad n \in \mathbb{Z}.$$

Of course the billiard flow (φ_t) is conjugated to the suspension of $\sigma_\pm$ associated to the time return map $t_\pm \circ {\psi^\Lambda}^{-1}$. This means that we have a Hölder homeomorphism

$$\Psi^\Lambda : K \to (\Lambda \times \mathbb{R}_+)/\sim$$

where $(z, t_+(z)) \sim (B_+(z), 0)$ for $z \in \Lambda$. In the coordinates (z, t), X is simply represented by ∂_t. In what follows, we will denote by $\psi_n^\Lambda(z)$ the n-th component of the sequence $\psi^\Lambda(z)$. An immediate consequence of the existence of a conjugacy Ψ^Λ as above is the following

Lemma 7.1.1. *There is $C > 0$ and $\beta > 1$ such that the following holds. Assume that $z, z' \in \Lambda$ satisfy*

$$\psi_n^\Lambda(z) = \psi_n^\Lambda(z'), \quad |n| \leqslant N.$$

Then $\mathrm{d}(z, z') \leqslant C\beta^{-N}$.

7.1.5 Isolating blocks

In this subsection we show that we can work with the framework of [DG16] (see also [KSW21, §5] for a more detailed exposition). We have

$$\Lambda = \bigcap_{t \in \mathbb{R}} \varphi_t(V \setminus TD)$$

where $V = \{z \in M \ : \ T_-(z) \neq \emptyset \text{ and } T_+(z) \neq \emptyset\} \subset M$. Here we set

$$T_\pm(z) = \{t \in T(z) \ : \ \pm t > 0\}.$$

By [CE71, Theorem 1.5], Λ is the maximal invariant set in some isolating block. More precisely, there exists a relatively compact neighborhood $U \subset M$ of Λ such that ∂U is smooth and

$$\partial_0 U = \{(x, v) \in \partial U \ : \ v \in T_x \partial U\}$$

is a smooth submanifold of ∂U of codimension 1, and with the property that for some $\varepsilon > 0$ one has

$$z \in \partial_0 U \quad \Longrightarrow \quad \forall |t| \in \,]0, \varepsilon[\,, \ \varphi_t(z) \notin U.$$

By proceeding as in [GMT17, Lemma 2.3], we may find a vector field Y on $M \setminus TD$ such that $X - Y$ is supported in an arbitrary small neighborhood of $\partial_0 U$, which is arbitrarily small in the C^∞ topology and such that for any boundary defining function $\rho : U \to \mathbb{R}_{\geqslant 0}$ of ∂U [3], we have, for any $z \in \partial U$,

$$Y\rho(z) = 0 \quad \Longrightarrow \quad Y^2\rho(z) < 0.$$

3. This means that $\rho > 0$ on U, $\rho = 0$ on ∂U and $\mathrm{d}\rho \neq 0$ on ∂U.

Moreover, it holds $\Gamma_\pm^X(U) = \Gamma_\pm^Y(U)$ where we set

$$\Gamma_\pm^X(U) = \{z \in U \ : \ \varphi_t(z) \in U, \ \mp t \geqslant 0\},$$
$$\Gamma_\pm^Y(U) = \{z \in U \ : \ \psi_t(z) \in U, \ \mp t \geqslant 0\}.$$

Here (ψ_t) denotes the non-complete flow generated by Y. Note also that it holds $\mathrm{dist}(\Gamma_\pm^X(U), \partial_0 U) > 0$. For simplicity, we will denote $\Gamma_\pm = \Gamma_\pm^X(U)$.

By [DG16, Lemma 2.10], there are two vector subbundles $E_\pm \subset T_{\Gamma_\pm} U$ with the following properties :

1. $E_+|_\Lambda = E_u$, $E_-|_\Lambda = E_s$ and $E_\pm(z)$ depends continuously on $z \in \Gamma_\pm$;
2. For some constants $C', \nu' > 0$ we have
$$\|\mathrm{d}\psi_{\mp t}(z)v\| \leqslant C'\mathrm{e}^{-\nu' t}\|v\|, \quad v \in E_\pm(z), \quad z \in \Gamma_\pm, \quad t \geqslant 0;$$
3. If $z \in \Gamma_\pm$ and $v \in T_z U$ satisfy $\langle \alpha(z), v\rangle = 0$ and $v \notin E_\pm(z)$, then as $t \to \mp\infty$
$$\|\mathrm{d}\psi_t(z)v\| \to \infty, \quad \frac{\mathrm{d}\psi_t(z)v}{\|\mathrm{d}\psi_t(z)v\|} \to E_\mp|_\Lambda.$$

7.1.6 The resolvent of the billiard flow

For $\mathrm{Re}(s) \gg 1$, we define the (future and past) resolvents $R_\pm(s) : \Omega_c^\bullet(U) \to \mathcal{D}'^\bullet(U)$ by

$$R_\pm(s)\omega(z) = \pm \int_0^{t_{\mp,U}(z)} \psi_{\mp t}^* \omega(z) \mathrm{e}^{-ts}\mathrm{d}t, \quad \omega \in \Omega_c^\bullet(U), \quad z \in U,$$

where we set

$$t_{\pm,U}(z) = \inf\{t > 0 \ : \ \psi_{\pm t}(z) \in \partial U\}, \quad z \in U.$$

Here $\Omega_c^\bullet(U)$ denotes the space of smooth differential forms which are compactly supported in U while $\mathcal{D}'^\bullet(U)$ denotes the space of currents in U (that is $\mathcal{D}'^k(U)$ is the dual space of $\Omega_c^{3-k}(U)$ for $k = 0, \dots, 3$). Note that

$$(\mathcal{L}_Y \pm s)\, R_\pm(s) = R_\pm(s)\,(\mathcal{L}_Y \pm s) = \mathrm{Id}_{\Omega_c^\bullet(U)}.$$

Then by [DG16], the family $s \mapsto R_\pm(s)$ extends to a family of operators meromorphic in the parameter $s \in \mathbb{C}$, whose poles have residues of finite rank. Denote by $\mathrm{Res}(Y)$ the set of those poles. Near any $s_0 \in \mathrm{Res}(Y)$ we have for some finite rank projector $\Pi_\pm(s_0) : \Omega^\bullet \to \mathcal{D}'^\bullet$

$$R_\pm(s) = H_\pm(s) + \sum_{j=1}^{J(s_0)} \frac{(X \pm s)^{j-1}\Pi_\pm(s_0)}{(s - s_0)^j}$$

where $s \mapsto H_\pm(s)$ is holomorphic near s_0. Moreover we have $\mathrm{supp}(\Pi_\pm(s_0)) \subset \Gamma_\pm \times \Gamma_\mp$ and

$$\mathrm{WF}'(H_\pm(s)) \subset \Delta(T^*U) \cup \Upsilon_\pm \cup (E_\pm^* \times E_\mp^*), \quad \mathrm{WF}'(\Pi_\pm(s_0)) \subset E_\pm^* \times E_\mp^*, \tag{7.1.6}$$

where $\Delta(T^*U) = \{(\xi, \xi), \ \xi \in T^*U\} \subset T^*(U \times U)$ and

$$\Upsilon_\pm = \{(\Phi_t(z,\xi), (z,\xi)), \ \pm t \geqslant 0, \ \langle \xi, X(z)\rangle = 0, \ z \in U \ \psi_t(z) \in U\}.$$

Here Φ_t denotes the symplectic lift of φ_t on T^*U, that is

$$\Phi_t(z,\xi) = (\varphi_t(z), (\mathrm{d}_z\varphi_t)^{-\top}\xi), \quad (z,\xi) \in T^*U, \quad \varphi_t(z) \in U,$$

and the subbundles $E_\pm^* \subset T^*_{\Gamma_\pm}U$ are defined by $E_\pm^*(\mathbb{R}X(z) \oplus E_\pm) = 0$. Also we denoted

$$\mathrm{WF}'(R_\pm(s)) = \{(z,\xi,z',\xi') \in T^*(U \times U),\ (z,\xi,z',-\xi') \in \mathrm{WF}(R_\pm(s))\},$$

where $\mathrm{WF}(R_\pm(s)) \subset T^*(U \times U)$ is the Hörmander wavefront set of (the Schwartz kernel of) $R_\pm(s)$, see [Hör90, §8], and

$$\mathrm{WF}'(R_\pm(s)) = \{(z,\xi,z',\xi')\ :\ (z,\xi,z,-\xi') \in \mathrm{WF}(R_\pm(s))\}.$$

7.1.7 The scattering operator

We define

$$\partial_\pm = \{z \in \partial U\ :\ \mp Y\rho(z) > 0\} \text{ and } \partial_0 = \{z \in \partial U\ :\ Y\rho(z) = 0\}.$$

The scattering map $S_\pm : \partial_\mp \setminus \Gamma_\mp \to \partial_\pm \setminus \Gamma_\pm$ is defined by

$$S_\pm(z) = \psi_{t_{\pm,U}(z)}(z), \quad z \in \partial_\mp \setminus \Gamma_\mp$$

(see Figure 7.2). The Scattering operator $\mathcal{S}_\pm(s) : \Omega_c^\bullet(\partial_\mp \setminus \Gamma_\mp) \to \Omega_c^\bullet(\partial_\pm \setminus \Gamma_\pm)$ is then defined by

$$\mathcal{S}_\pm(s)\omega = \left(S_\mp^*\omega\right)\mathrm{e}^{-st_{\mp,U}(\cdot)}, \quad \omega \in \Omega_c^\bullet\left(\partial_\mp \backslash \Gamma_\mp\right).$$

Note that for $\mathrm{Re}(s) \gg 1$, $\mathcal{S}_\pm(s)$ extends as an operator $C_c(\partial_\mp, \wedge^\bullet T^*\partial_\mp) \to C_c(\partial_\pm, \wedge^\bullet T^*\partial_\pm)$, where $C_c(\partial_\pm, \wedge^\bullet T^*\partial_\pm)$ is the space of compactly supported continuous forms on $\partial_\pm$, since for any $w \in \Omega^\bullet(U)$ and $t \in \mathbb{R}$ we have

$$\|\varphi_t^* w\|_\infty \leqslant C\mathrm{e}^{C|t|}\|w\|_\infty.$$

In what follows we let $\iota_\pm : \partial_\pm \to U$ be the inclusion and $(\iota_\pm)_* : \Omega_c^\bullet(\partial_\pm) \to \mathcal{D}'^{\bullet+1}(U)$ be the pushforward operator, which is defined by

$$\int_U (\iota_\pm)_* u \wedge v = \int_{\partial_\pm} u \wedge \iota_\pm^* v, \quad u \in \Omega_c^\bullet(\partial_\pm), \quad v \in \Omega^\bullet(U).$$

Proposition 7.1.2. *We have*

$$\mathrm{WF}(R_\pm(s)) \cap N^*(\partial_\pm \times \partial_\mp) = \emptyset. \tag{7.1.7}$$

In particular by [Hör90, Theorem 8.2.4], the operator $(\iota_\pm)^*\iota_X R_\pm(s)(\iota_\mp)_*$ is well defined. Moreover, for $\mathrm{Re}(s) \gg 1$ large enough, we have

$$\mathcal{S}_\pm(s) = (-1)^N(\iota_\pm)^*\iota_Y R_\pm(s)(\iota_\mp)_* : \Omega_c^\bullet(\partial_\mp) \to \mathcal{D}'^\bullet(\partial_\pm), \tag{7.1.8}$$

where $N : \mathcal{D}'^\bullet \to \mathcal{D}'^\bullet$ is the number operator[4].

4. That is, $N(\omega) = k\omega$ for $\omega \in \Omega_c^k(\partial_\pm)$.

Proof. By definition $Y(z)$ is transverse to $T_z\partial_\pm$ for $z \in \partial_\pm$. In particular, if $(z,\xi) \in T^*\partial_\pm$ satisfies $\langle \xi, Y(z)\rangle = 0$ and $\langle \xi, T_z\partial_\pm\rangle = 0$ then $\xi = 0$. As $\partial_+ \cap \partial_- = \emptyset$ we obtain (7.1.7) by (7.1.6).

Now let $W_\pm \subset \partial_\pm$ be open sets such that $\overline{W}_\pm \subset \partial_\pm$. As $\overline{W}_\pm \cap \partial_0 = \emptyset$, there is $\varepsilon > 0$ such that $t_{\mp,U}(z) > \varepsilon$ for every $z \in W_\pm$. In particular, the proof of Lemma 4.3.4 applies and leads to the fact that (7.1.8) holds when $\mathcal{S}_\pm(s)$ is seen as an operator $\Omega^\bullet_c(\partial_\mp \setminus \Gamma_\mp) \to \mathcal{D}'^\bullet(\partial_\pm \setminus \Gamma_\pm)$. By [BR75, Theorem 5.6], as Λ is not an attractor, we have $\mu(\Gamma_\pm) = 0$ where μ is the measure $|\alpha \wedge \mathrm{d}\alpha|$. Take $U_\pm \subset \partial_\pm$ a small neighborhood of $\Gamma_\pm$ in $\partial_\pm$ and $\delta > 0$ small enough. Since $\Gamma_\pm \cap \partial_0 = \emptyset$, we may assume that the map

$$U_\pm \times [0,\delta) \to U, \ (y,t) \mapsto \varphi_{\mp t}(y)$$

realizes a smooth diffeomorphism onto its image. In particular, because $\varphi_{\mp t}(\Gamma_\pm) \subset \Gamma_\pm$ for $t > 0$, we have $\mu_{\partial_\pm}(\Gamma_\pm \cap \partial_\pm) = 0$ where $\mu_{\partial_\pm}$ corresponds to the measure $|\iota_\pm^* \mathrm{d}\alpha|$. Thus we may proceed by similar arguments given in the proof of Proposition 7.1.2 to obtain that (7.1.8) holds when $\mathcal{S}_\pm(s)$ is seen as an operator $\Omega^\bullet_c(\partial_\mp) \to \mathcal{D}'^\bullet(\partial_\pm)$. □

7.2 Adding an obstacle

In this section we add an other obstacle D_0 and we will consider some weighted transfer operator associated to the first return map to $\pi^{-1}(\partial D_0)$; we will use the description of its microlocal structure to define and compute its flat trace.

7.2.1 Notations

We add another convex obstacle D_0, and we assume that the billiard table $(D_0, D_1, \dots, D_r)$ satisfies the non-eclipse condition. We define

$$M', \Lambda', \Lambda', (\varphi'_t), T'_\pm, t'_\pm, B'_\pm$$

in the same way we defined $M, \Lambda, \Lambda, (\varphi_t), T_\pm, t_\pm, B_\pm$ (see §7.1) by replacing the billiard table $\mathbf{B} = \{D_1, \dots, D_r\}$ by the billiard table $\mathbf{B}' = \{D_0, D_1, \dots, D_r\}$. Let

$$P'_\pm : \Lambda'^{(\pm 1)} \to \pi^{-1}(\partial D'), \quad z \mapsto \varphi_{\pm t'_\pm(z)}(z),$$

where

$$\Lambda'^{(\pm 1)} = \{z \in M' \ : \ t'_\pm(z) < \infty\}.$$

Let $V_0 \subset \pi^{-1}(\partial D_0)$ be a relatively compact neighborhood of $\Lambda' \cap \pi^{-1}(\partial D_0)$ such that $V_0 \cap T\partial D_0 = \emptyset$, and set

$$V_\pm = \{z \in \partial U \cap \Lambda'^{(\pm 1)} \ : \ P'_\pm(z) \in V_0\}.$$

Note that U is a subset of M. However we may see U as a subset of M' since U does not intersect $\pi^{-1}(D_0)$. We also let $W_\pm$ be a neighborhood of $\Lambda_\pm \cap \partial U$ in ∂U such that $W_\pm \cap \mathrm{supp}(Y - X) = \emptyset$ and we set $Y_\pm = W_\pm \cap V_\pm$. We take $\phi_\pm \in C_c^\infty(V_\pm, [0,1])$ (resp. ${\psi^\Lambda}_\pm \in C_c^\infty(W_\pm), [0,1])$ such that $\phi_\pm \equiv 1$ near $(B'_\pm)^{-1}(\Lambda')$ (resp. ${\psi^\Lambda}_\pm \equiv 1$ near $\Lambda_\pm$) ; we define

$$\chi_\pm = \phi_\pm {\psi^\Lambda}_\pm \in C_c^\infty(Y_\pm).$$

Note that $P'_\pm$ realizes a diffeomorphism $V_\pm \to P'_\pm(V_\pm) \subset \pi^{-1}(\partial D_0)$ which we denote by $Q_\pm$. We define $\mathcal{Q}_\pm(s) : \mathcal{D}'^\bullet_c(Y_\pm) \to \mathcal{D}'^\bullet_c(Z_\pm)$, where $Z_\pm = Q_\pm(Y_\pm)$, by

$$\mathcal{Q}_\pm(s)w = \mathrm{e}^{-st'_\pm(\cdot)} \left(Q_\pm^{-1}\right)^* w, \quad w \in \Omega^\bullet_c(Y_\pm)$$

(see Figure 7.2). We finally set, with $Z_\pm = Q_\pm(Y_\pm) \subset \pi^{-1}(\partial D_0)$,

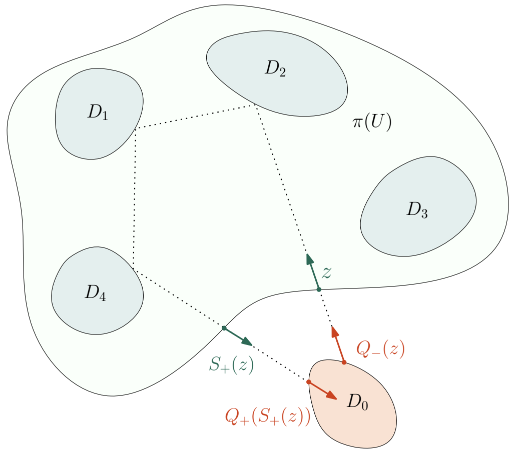

FIGURE 7.2 – The maps $Q_\pm$ and $S_\pm$

$$\mathcal{T}_\pm(s) = \mathcal{Q}_\pm(s)\chi_\pm\mathcal{S}_\pm(s)\chi_\mp\mathcal{Q}_\mp(s)^\top : \Omega^\bullet_c(Z_\mp) \to \mathcal{D}'^\bullet(Z_\pm). \tag{7.2.1}$$

The operator $\mathcal{T}_\pm(s)$ is the transfer operator associated to the first return map to $\pi^{-1}(\partial D_0)$ weigthed by $\mathrm{e}^{-st_{0,\pm}(\cdot)}$, where $t_{0,\pm}(z) = \inf\{t > 0 \ : \ \pi(\varphi'_{\pm t}(z)) \in \partial D_0\}$ are the first (future are past) return times to ∂D_0 of a point $z \in \pi^{-1}(\partial D_0)$.

7.2.2 Composing the scattering maps

Let $Z = Z_+ \cap Z_-$ and

$$\varrho = (\chi_+ \circ Q_+^{-1})(\chi_- \circ Q_-^{-1}) \in C^\infty_c(Z).$$

We have the following result.

Proposition 7.2.1. *For any $n \geqslant 2$, the composition $(\varrho\mathcal{T}_\pm(s))^n : \Omega^\bullet_c(Z) \to \mathcal{D}'^\bullet(Z)$ is well defined.*

Proof. By [Hör90, Theorem 8.2.4] and Proposition 7.1.2 we have

$$\mathrm{WF}'(\mathcal{S}_\pm(s)) \subset \left\{\mathrm{d}\left(\iota_\pm \times \iota_\mp\right)(z,z')^\top \cdot (\xi,\xi') \ : \ (z,z',\xi,\xi') \in \mathrm{WF}'\left(R_\pm(s)\right)\right\}, \tag{7.2.2}$$

where $\iota_\pm \times \iota_\mp : \partial_\pm \times \partial_\mp \hookrightarrow U \times U$ is the inclusion. To prove that $(\varrho\mathcal{T}_\pm(s))^2$ is well defined, it suffices to show by [Hör90, Theorem 8.2.14] that $A \cap B_1 = \emptyset$ where

$$A = \{(z,\xi) \in T^*Z \ : \ \exists z' \in Z, (z',0,z,\xi) \in \mathrm{WF}'\left(\varrho\mathcal{T}_\pm(s)\right)\}$$

and

$$B_1 = \{(z,\xi) \in T^*Z \ : \ \exists z' \in Z, (z,\xi,z',0) \in \mathrm{WF}\left(\varrho\mathcal{T}_\pm(s)\right)\}.$$

Note that $\left(\mathrm{d}\iota_\pm(z)^\top\right)|_{\ker X(z)} : \ker X(z) \to T^*_z\partial_\pm$ is injective for any $z \in \partial_\pm$, since $X(z)$ is transverse to $T_z\partial_\pm$. Moreover $Q_\pm : Y_\pm \to Z_\pm$ is a diffeomorphism and thus $\mathrm{d}Q_\pm^{-1}(z)^\top : T^*_z\partial_\pm \to T^*_z\pi^{-1}(\partial D_0)$ is injective for any $z \in Y_\pm$. Now by (7.2.1) we have

$$\mathrm{WF}'(\varrho\mathcal{T}_\pm(s)) \subset \mathrm{d}(Q_\pm \times Q_\mp)^\top \left(\mathrm{WF}'(\mathcal{S}_\pm(s)) \cap \mathrm{supp}(\chi_\pm \times \chi_\mp)\right),$$

Moreover by (7.1.7) and (7.2.2) we have

$$\mathrm{WF}'(\mathcal{S}_\pm(s)) \subset \mathrm{d}(\iota_\pm \times \iota_\mp)^\top \left(\Upsilon_\pm \cup (E^*_\pm \times E^*_\mp)\right),$$

since $\Delta(T^*U) \cap \pi^{-1}(\partial_\pm \times \partial_\mp) = \emptyset$. By injectivity of $\mathrm{d}Q_\pm^{-1}(z)^\top : T^*_z\partial_\pm \to T^*_z\pi^{-1}(\partial D_0)$ we obtain

$$A \subset \mathrm{d}(Q_\mp^{-1})^\top \mathrm{d}(\iota_\mp)^\top E^*_\mp \quad \text{and} \quad B_1 \subset \mathrm{d}(Q_\pm^{-1})^\top \mathrm{d}(\iota_\pm)^\top E^*_\pm.$$

We claim that this implies $A \cap B_1 = \emptyset$. Indeed, let $(z,\xi) \in T^*Z_\pm$ which lies in $\left(\mathrm{d}(Q_\mp^{-1})^\top \mathrm{d}(\iota_\mp)^\top E^*_\mp\right) \cap \left(\mathrm{d}(Q_\pm^{-1})^\top \mathrm{d}(\iota_\pm)^\top E^*_\pm\right)$. Thus z lies in Λ' and there exists $(z_\pm, \xi_\pm)$ in $E^*_\pm$ such that

$$(z,\xi) = \mathrm{d}(Q_\pm^{-1})^\top \mathrm{d}(\iota_\pm)^\top (z_\pm, \xi_\pm).$$

There are neighborhoods $U_\pm$ of $z_\pm$ in M' and smooth functions $s_\pm : U_\pm \to \mathbb{R}$ such that $Q_\pm(z'_\pm) = \varphi_{s_\pm(z'_\pm)}(z'_\pm)$ for $z'_\pm \in U_\pm$ and $\varphi_{s_\pm(z'_\pm)}(z'_\pm) \in Z$ for $z'_\pm \in U_\pm$. Because $\xi_\pm \in \ker X(z_\pm)$ we see that

$$\mathrm{d}(Q_\pm^{-1})^\top \mathrm{d}(\iota_\pm)^\top (z_\pm, \xi_\pm) = \mathrm{d}\iota^\top \mathrm{d}_z \left(\varphi_{-s_\pm(z_\pm)}\right)^\top \xi_\pm$$

where $\iota : Z \hookrightarrow M'$ is the inclusion. Because $\mathrm{d}\iota^\top : \ker X \to T^*Z$ is injective, we obtain

$$\xi_- = \mathrm{d}\Big[u \mapsto \varphi_{s_-(z_-)-s_+(z_+)}(u)\Big](z_-)^\top \xi_+.$$

Now we have $z_\pm \in \Lambda'$ and since $\xi_\pm \in E^*_\pm$ we obtain $\xi_+ \in E'^*_u(z_+)$ and $\xi_- \in E'^*_s(z_-)$. Thus $\xi \in E'^*_u(z) \cap E'^*_s(z) = \{0\}$. Here we denoted by

$$T_zM' = \mathbb{R}E'_s(z') \oplus \mathbb{R}E'_u(z') \oplus \mathbb{R}X(z'), \quad z' \in \Lambda',$$

the hyperbolic decomposition of TM' over Λ'. We conclude that $A \cap B_1 = \emptyset$, which concludes the case $n = 2$.

By [Hör90, Theorem 8.2.14] we also have the bound

$$\mathrm{WF}((\varrho\mathcal{T}_\pm(s))^2) \subset (\mathrm{WF}'(\varrho\mathcal{T}_\pm(s)) \circ \mathrm{WF}'(\varrho\mathcal{T}_\pm(s))) \cup (B_1 \times \underline{0}) \cup (\underline{0} \times A),$$

where $\underline{0} \subset T^*M'$ denotes the zero section. Therefore, the set B_2, which is defined by

$$B_2 = \left\{(z,\xi) \in T^*Z \;:\; \exists z' \in Z,\ (z,\xi,z',0) \in \mathrm{WF}\left((\varrho\mathcal{T}_\pm(s)^2)\right)\right\},$$

can be written

$$\Big\{(z,\xi) \in T^*Z \;:\; \exists z', z'' \in Z, \exists \eta \in T^*_{z'}Z,\ (z,\xi,z'-\eta) \in \mathrm{WF}(\varrho\mathcal{T}_\pm(s)) \\ \text{and } (z',\eta,z'',0) \in \mathrm{WF}(\varrho\mathcal{T}_\pm(s))\Big\} \cup B_1.$$

As $\mathrm{d}(Q_+^{-1})^\top \mathrm{d}(\iota_+)^\top E_+^* \cap \mathrm{d}(Q_-^{-1})^\top \mathrm{d}(\iota_-)^\top E_-^* = 0$ (as shown above), we obtain

$$B_2 \subset \Big\{(z,\xi) \ : (z,\xi,z',\eta) \in \mathrm{d}(Q_\pm^{-1} \times Q_\mp^{-1})^\top \mathrm{d}(\iota_\pm \times \iota_\mp)^\top \Upsilon_\pm \\ \text{for some } \eta \in \mathrm{d}(Q_\pm^{-1})^\top \mathrm{d}(\iota_\pm)^\top (E_\pm^*)\Big\}.$$

This leads to

$$B_2 \subset \Big\{\mathrm{d}(Q_\pm^{-1})^\top \mathrm{d}(\iota_\pm)^\top \Phi_t(z,\zeta) \ : \ (z,\zeta) \in T^*Y_\mp, \ \langle X(z), \zeta\rangle = 0, \\ \mathrm{d}(\iota_\mp)^\top (z,\zeta) \in \mathrm{d}(Q_\pm \circ Q_\mp^{-1})^\top \mathrm{d}(\iota_\pm)^\top E_\pm^*, \ \varphi_t(z) \in \partial_\pm U, \ t \geqslant 0\Big\}.$$

As before, this set cannot intersect $\mathrm{d}(Q_\mp^{-1})^\top E_\mp^*$ since otherwise we would have $z' \in \Lambda'$ and $\xi' \in T_z^*M'$ contracted in the past and in the future by $\mathrm{d}\varphi_t'^\top$. Thus $B_2 \cap A = \emptyset$ and we obtain that $(\varrho\mathcal{T}_\pm(s))^3$ is well defined. By iterating this process we obtain that $(\varrho\mathcal{T}_\pm(s))^n$ is well defined for every $n \geqslant 2$, which concludes the proof. □

7.2.3 The flat trace of $\mathcal{T}_\pm(s)$

Let $A : \Omega_c^\bullet(\partial) \to \mathcal{D}'^\bullet(Z)$ be an operator such that $\mathrm{WF}'(A) \cap \Delta = \emptyset$, where Δ is the diagonal in $T^*(Z \times Z)$. Then the flat trace of A is defined as

$$\mathrm{tr}_\mathrm{s}^\flat A = \langle \iota_\Delta^* K_A, 1\rangle,$$

where $\iota_\Delta : z \mapsto (z,z)$ is the diagonal inclusion and $A \in \mathcal{D}'^n(Z \times Z)$ is the Schwartz kernel of A, i.e.

$$\int_Z A(u) \wedge v = \int_{Z \times Z} K_A \wedge \pi_1^* u \wedge \pi_2^* v, \quad u, v \in \Omega_c^\bullet(Z),$$

where $\pi_j : Z \times Z \to Z$ is the projection on the j-th factor ($j = 1, 2$). In fact we have

$$\mathrm{tr}_\mathrm{s}^\flat(A) = \sum_{k=0}^{2} (-1)^k \mathrm{tr}^\flat(A_k), \tag{7.2.3}$$

where $\mathrm{tr}^\flat$ is the transversal trace of Attiyah-Bott [AB67] and where we denoted by A_k the operator $C_c^\infty\big(Z, \wedge^k T^*Z\big) \to \mathcal{D}'\big(Z, \wedge^k T^*Z\big)$ induced by A on the space of k-forms (see §B.3.1 for more details). The purpose of this subsection is to prove the following result.

Proposition 7.2.2. *For $n \geqslant 1$, the flat trace of $(\varrho\mathcal{T}_\pm(s))^n$ is well defined and we have*

$$\mathrm{tr}_\mathrm{s}^\flat\left((\varrho\mathcal{T}_\pm(s))^n\right) = n \sum_{r(\gamma)=n} (-1)^{1+m(\gamma)} \frac{\tau^\sharp(\gamma)}{\tau(\gamma)} \mathrm{e}^{-s\tau(\gamma)} \left(\prod_{z \in R(\gamma)} \varrho^2(z)\right)^{\tau(\gamma)/\tau^\sharp(\gamma)} \tag{7.2.4}$$

whenever $\mathrm{Re}(s) \gg 1$, *where the sum runs over all periodic trajectories γ rebounding n times on ∂D_0 and $m(\gamma)$ is the total number of bounds of γ on $\partial D_0, \dots, \partial D_r$. Here $\tau^\sharp(\gamma)$ is the primitive length of γ and*

$$R(\gamma) = \{(\gamma(\tau), \dot\gamma(\tau)) \ : \ \tau \in \mathbb{R}\} \cap \pi^{-1}(\partial D_0)$$

is the set of incidence vectors of γ along D_0.

Corollary 7.2.3. *As $s \mapsto (\varrho \mathcal{T}_\pm(s))^n$ extends meromorphically to the whole complex plane, so does the right hand side of (7.2.4).*

Proof. For $z \in Z$ we define the first (future and past) return times to $\pi^{-1}(\partial D_0)$ by

$$t_{\pm,0}(z) = \inf\{t > 0 \ : \ \varphi'_{\pm t}(z) \in \pi^{-1}(\partial D_0)\}.$$

We set $\Lambda_{\pm,0} = \{z \in Z \ : \ t_{\pm,0}(z) < \infty\}$, and we define by $B_{\pm,0} : Z \to \pi^{-1}(\partial D_0)$ the first (future and past) return maps to $\pi^{-1}(\partial D_0)$ by

$$B_{\pm,0}(z) = \varphi'_{t_{\pm,0}}(z), \quad z \in \Lambda_{\pm,0}.$$

For $n \geqslant 1$ we define the sets $\Lambda^{(n)}_{\pm,0} \subset Z$ by induction as follows. We set $\Lambda^{(1)}_{\pm,0} = \Lambda_{\pm,0}$ and

$$\Lambda^{(n+1)}_{\pm,0} = \left\{ z \in \Lambda_{\pm,0} \ : \ B_{\pm,0}(z) \in \Lambda^{(n)}_{\pm,0} \right\}, \quad n \geqslant 1.$$

In particular $(B_{\pm,0})^n(z)$ is well defined for $z \in \Lambda^{(n)}_{\pm,0}$. We finally set

$$t^{(n)}_{\pm,0}(z) = \sum_{k=0}^{n-1} t_{\pm,0}\left((B_{\pm,0})^k(z)\right), \quad z \in \Lambda^{(n)}_{\pm,0},$$

and $t^{(n)}_{\pm,0}(z) = +\infty$ for $z \in Z \setminus \Lambda^{(n)}_{\pm,0}$. We now fix $n \geqslant 1$. Let $g \in C^\infty(\mathbb{R}, [0,1])$ such that $g \equiv 1$ on $]-\infty, 1]$ and $g \equiv 0$ on $[2, +\infty[$. For $L > 0$ we define

$$g_L(z) = g\left(t^{(n)}_{\pm,0}(z) - L\right), \quad z \in Z.$$

Then by definition of $\mathcal{T}_\pm(s)$, the operator $g_L\left(\varrho \mathcal{T}_\pm(s)\right)^n : \Omega^\bullet_c(Z) \to \mathcal{D}'^\bullet(Z)$ coincides with the operator

$$w \longmapsto g_L(\cdot)\left(\prod_{k=0}^{n} \varrho^2\left((B_{\mp,0})^k(\cdot)\right)\right) \mathrm{e}^{-s t^{(n)}_{\pm,0}(\cdot)}\left((B_{\mp,0})^n\right)^* w.$$

It now follows from the Atiyah-Bott trace formula [AB67, Corollary 5.4] that [5]

$$\langle \iota^*_\Delta K_{\varrho,\pm}(s), g_L\rangle = \sum_{\substack{z \in Z \\ B^n_{\mp,0}(z) = z}} \operatorname{sgn}\det(1 - \mathrm{d}(B_{\mp,0})^n(z))\mathrm{e}^{-t^{(n)}_{\pm,0}(z)} g_L(z)\left(\prod_{k=0}^{n-1} \varrho^2\left(B^k_{\mp}(z)\right)\right). \tag{7.2.5}$$

Now it is not hard to see that

$$\operatorname{sgn}\det(1 - \mathrm{d}(B_{\mp,0})^n(z)) = \operatorname{sgn}\det(1 - P_\gamma) = \begin{cases} 1, & \text{if} \quad m(\gamma) \text{ is odd}, \\ -1, & \text{if} \quad m(\gamma) \text{ is even}, \end{cases}$$

where γ is the closed orbit generated by z. This yields $\operatorname{sgn}\det(1 - \mathrm{d}(B_{\mp,0})^n(z)) = (-1)^{1+m(\gamma)}$. Next, it is a classical fact that for every $k, n \geqslant 1$, there is $C_k > 0$ such that

$$\|\mathrm{d}^k\left((B_{\pm,0})^n\right)(z)\| \leqslant C_k \exp\left(C_k t^{(n)}_{\pm,0}(z)\right), \quad z \in \Lambda^{(n)}_{\pm,0}.$$

Thus we may proceed exactly as in the proof of [Chab, Proposition 3.6] to take the limit in (7.2.5) when $L \to +\infty$ to obtain (7.2.4). □

5. See the proof of [Chab, Proposition 3.6] for more details.

7.3 A Tauberian argument

In this section we use a Tauberian theorem of Delange [Del54] to derive an asymptotic growth of a weighted sum of periodic trajectories rebounding a fixed number of times on ∂D_0.To that aim we wish to work with series having positive coefficients, and we first explain how Proposition 7.2.2 can be adapted to remove the sign $(-1)^{1+m(\gamma)}$.

7.3.1 Doubling manifold

Let

$$\widehat{M'} = (N' \times \{-1, 1\})/\sim, \quad N' = S\mathbb{R}^2 \setminus \left(\pi^{-1}(D'^{\circ}) \cup G'\right),$$

where $G' = T\partial D'$ and $D' = \bigcup_{j=0}^{r} D_j$, and where $(x, v, a) \sim (y, w, b)$ if and only if, for some $j \in \{0, \dots, r\}$, it holds

$$x = y \in \partial D_j, \quad w = v - 2\langle v, n_j(x)\rangle n_j(x) \quad \text{and} \quad a = -b.$$

Let $\hat{\pi} : \widehat{M'} \to M'$ be the natural projection, which is a 2-fold covering, and denote by $J : \widehat{M'} \to \widehat{M'}$ the involution induced by $(x, v, a) \mapsto (x, v, -a)$. Then there is a unique continuous flow $(\widehat{\varphi}'_t)$ acting on $\widehat{M'}$ such that $\hat{\pi} \circ \widehat{\varphi}'_t = \varphi'_t \circ \hat{\pi}$. Clearly, the flow $(\widehat{\varphi}'_t)$ is hyperbolic on $\widehat{\Lambda}' = \hat{\pi}^{-1}(\Lambda')$. Moreover, each periodic orbit $\gamma : [0, \tau(\gamma)] \to \Lambda'$ of the flow (φ'_t) with an *even* number of bounds on $\partial D'$ gives rise to two periodic orbits of $(\widehat{\varphi}'_t)$ which are generated by the two points lying in $\hat{\pi}^{-1}(\gamma(0))$; every periodic orbit of $(\widehat{\varphi}'_t)$ is obtained in this way.

Next, we define $\widehat{\widetilde{R}}_\pm(s), \widehat{\widetilde{\mathcal{S}}}_\pm(s), \widehat{\mathcal{Q}}_\pm(s), \widehat{\chi}_\pm, \widehat{\mathcal{T}}_\pm(s)$ and $\hat{\varrho}$ in the same way we defined $\widetilde{R}_\pm(s), \widetilde{\mathcal{S}}_\pm(s), \mathcal{Q}_\pm(s), \chi_\pm, \mathcal{T}_\pm(s)$ and ϱ, by using the flow $(\widehat{\varphi}'_t)$ instead of (φ'_t). Clearly, Propositions 7.1.2 and 7.2.1 extend for those operators, if we replace Z by $\widehat{Z} = \hat{\pi}^{-1}(Z)$ and $\widetilde{\partial}_\pm$ by $\hat{\pi}^{-1}(\widetilde{\partial}_\pm)$. Moreover, thanks to the description of the periodic orbits of $(\widehat{\varphi}'_t)$ given above, we may redo the proof of Proposition 7.2.2 to obtain the formula

$$\frac{1}{2}\operatorname{tr}^\flat_{\mathrm{s}}\left((1 - J^*)(\hat{\varrho}\widehat{\mathcal{T}}_\pm(s))^n\right) = -n \sum_{r(\gamma)=n} \frac{\tau^\sharp(\gamma)}{\tau(\gamma)} \mathrm{e}^{-s\tau(\gamma)} \left(\prod_{z \in R(\gamma)} \varrho^2(z)\right)^{\tau(\gamma)/\tau^\sharp(\gamma)} \tag{7.3.1}$$

which is valid for $\mathrm{Re}(s) \gg 1$, as it follows from the fact that there is a $2:1$ correspondance between fixed points of $J \circ \widehat{\varphi}'_t$ and fixed points of φ'_t with an *odd* number of bounds on $\partial D'$.

7.3.2 Zeta functions

Let $\mathcal{P}_{\mathbf{B}}$ be the set of primitive periodic orbits of (φ_t), for the billiard table $\mathbf{B}$. We define the Ruelle zeta function $\zeta_{\mathbf{B}}$ associated to the billiard flow $\mathbf{B}$ by

$$\zeta_{\mathbf{B}}(s) = \prod_{\gamma \in \mathcal{P}_{\mathbf{B}}} \left(1 - \mathrm{e}^{-s\tau(\gamma)}\right)^{-1}, \quad s \in \mathbb{C},$$

where the product converges whenever $\mathrm{Re}(s)$ is large enough. By [Mor07, Theorem 1.3], there is $h_{\mathbf{B}} > 0$ and $c_{\mathbf{B}} > 0$ such that $\zeta_{\mathbf{B}}$ admits a meromorphic extension to the half plane $\{\mathrm{Re}(s) > -c_{\mathbf{B}}\}$; moreover, $\zeta_{\mathbf{B}}$ is analytic and nonvanishing on the line $\{\mathrm{Re}(s) = h_{\mathbf{B}}\}$ except for a simple pole at $s = h_{\mathbf{B}}$ (as it follows from [Mor91, Remark 3.1] and [PP83, Proposition 9]) ; hence $\zeta'_{\mathbf{B}}/\zeta_{\mathbf{B}}$ is analytic on $\{\mathrm{Re}(s) = h_{\mathbf{B}}\}$, except for a simple pole with residue -1 at $s = h_{\mathbf{B}}$.

In what follows, we set $\widehat{U} = \hat{\pi}^{-1}(U)$

$$\Omega_0^k = \{w \in \Omega_c^k(\widehat{U}) \ : \ \iota_{\widehat{X}} w = 0\},$$

where $\hat{X}$ is the generator of $(\widehat{\varphi}'_t)$ on $\widehat{U}$. Then it follows from [DG16] (see also [BSW21, §4]) that we may write, for $\mathrm{Re}(s)$ large enough, any $\varepsilon > 0$ small and $\hat{\chi} \in C_c^\infty(\widehat{U}, [0,1])$ satisfying $\hat{\chi} \equiv 1$ on Λ,[6]

$$\zeta'_{\mathbf{B}}(s)/\zeta_{\mathbf{B}}(s) = -\frac{1}{2}\sum_{k=0}^{2}(-1)^k \mathrm{e}^{\mp\varepsilon s}\mathrm{tr}^\flat\left((1-J^*)\,\hat{\chi}\widehat{\varphi}^*_{\mp\varepsilon}\widehat{\widehat{R}}_\pm(s)\hat{\chi}|_{\Omega_0^k}\right); \tag{7.3.2}$$

moreover, the residue of $\zeta'_{\mathbf{B}}(s)/\zeta_{\mathbf{B}}(s)$ at $s = s_0$ is given by

$$-\frac{1}{2}\sum_{k=0}^{2}(-1)^k \mathrm{tr}^\flat\left((1-J^*)\,\hat{\chi}\widehat{\Pi}_\pm(h_{\mathbf{B}})\hat{\chi}|_{\Omega_0^k}\right). \tag{7.3.3}$$

where $\widehat{\Pi}_\pm(s_0)$ is the residue of $\widehat{\widehat{R}}_\pm(s)$ at $s = s_0$ (see §7.1.6). Next, we know that $s \mapsto \widehat{\widehat{R}}_\pm(s)|_{\Omega_0^0}$ is holomorphic on $\{\mathrm{Re}(s) > 0\}$, simply because the integral defining $\widehat{\widehat{R}}_\pm(s)|_{\Omega_0^0}$ converges absolutely in this region. This implies that

$$\widehat{\Pi}_\pm(s)|_{\Omega_0^2} = 0, \quad \mathrm{Re}(s) > 0,$$

since the map $u \mapsto u \wedge \mathrm{d}\hat{\alpha}$ realizes an isomorphism $\mathrm{ran}\,\widehat{\Pi}_\pm(s_0)|_{\Omega_0^0} \to \mathrm{ran}\,\widehat{\Pi}_\pm(s_0)|_{\Omega_0^2}$, where we put $\hat{\alpha} = \hat{\pi}^*\alpha$.

Finally, let $\eta(s) = \sum_{\gamma\in\widetilde{\mathcal{P}}_{\mathbf{B}}} \tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)} = \zeta'_{\mathbf{B}}(s)/\zeta_{\mathbf{B}}(s)$, where $\widetilde{\mathcal{P}}_{\mathbf{B}}$ is the set of periodic trajectories of (φ_t). Also, let $\eta_{\mathrm{even}}(s)$ (resp. $\eta_{\mathrm{odd}}(s)$) be the series defined similarly by summing over periodic γ's with an even (resp. odd) number of bounces $m(\gamma)$. Let $\mathcal{P}_{\mathbf{B}}^{\mathrm{even}}$ (resp. $\mathcal{P}_{\mathbf{B}}^{\mathrm{odd}}$) be the set of primitive $\gamma \in \mathcal{P}_{\mathbf{B}}$ such that $m(\gamma)$ is even (resp. odd). Using the symbolic coding and similar arguments to that used in the proof of Lemma 7.4.1 below, it is not hard to construct injective maps $F_\pm : \mathcal{P}_{\mathbf{B}}^{\mathrm{even/odd}} \to \mathcal{P}_{\mathbf{B}}^{\mathrm{odd/even}}$ such that for some $C > 0$ it holds

$$\tau(\gamma) - C \leqslant \tau(F_\pm(\gamma)) \leqslant \tau(\gamma) + C, \quad \gamma \in \mathcal{P}_{\mathbf{B}}^{\mathrm{even/odd}}.$$

This estimate implies that both $\eta_{\mathrm{even}}(s)$ and $\eta_{\mathrm{odd}}(s)$ have a simple pole at $s = h_{\mathbf{B}}$, since $\eta(s)$ does and $\eta(s) = \eta_{\mathrm{even}}(s) + \eta_{\mathrm{odd}}(s)$. Moreover, the residues of $\eta_{\mathrm{even}}(s)$ and $\eta_{\mathrm{odd}}(s)$ at $s = h_{\mathbf{B}}$ are given respectively by

$$\frac{1}{2}\,\mathrm{tr}^\flat(\hat{\chi}\widehat{\Pi}_\pm(h_{\mathbf{B}})\hat{\chi}) \quad \text{and} \quad -\frac{1}{2}\,\mathrm{tr}^\flat(\hat{\chi}J^*\widehat{\Pi}_\pm(h_{\mathbf{B}})\hat{\chi}). \tag{7.3.4}$$

6. Again, we use that the periodic orbits of $(\widehat{\varphi}'_t)$ in $\hat{\Lambda} = \pi^{-1}(\Lambda)$ are in 2 :1 correspondance with the periodic orbits of (φ_t) bouncing an even number of times on ∂D, while the fixed points of $J\widehat{\varphi}_t$ are in 2 :1 correspondance with fixed points of φ_t bouncing an odd number of times on ∂D.

The first one coincides with $\operatorname{rank}\Pi_\pm(h_{\mathbf{B}})$ (see for example [DG16, §4]). Moreoever, since $\eta_{\mathrm{even}}(s) \leqslant \eta(s)$, it follows that this number is equal to 1 or 2 ; however it cannot be equal to 2, because otherwise $\eta_{\mathrm{odd}}(s)$ would not have a pole at $s = h_{\mathbf{B}}$, since the residue of $\eta(s)$ at $s = h_{\mathbf{B}}$ is equal to 1. Therefore

$$\operatorname{rank}\widehat{\Pi}_\pm(h_{\mathbf{B}}) = 1, \tag{7.3.5}$$

and hence both residues in (7.3.4) are equal to 1/2. Thus it follows that

$$J^*\widehat{\Pi}_\pm(h_{\mathbf{B}}) = -\widehat{\Pi}_\pm(h_{\mathbf{B}}), \tag{7.3.6}$$

since J^* preserves $\operatorname{ran}\widehat{\Pi}_\pm(h_{\mathbf{B}})$ and $J^2 = \mathrm{Id}$.

7.3.3 A Tauberian argument

Taking the notations of §7.3.1, we set

$$\hat{A}_\pm = \widehat{\mathcal{Q}}_\pm(h_{\mathbf{B}})\hat{\chi}_\pm\hat{\iota}_\pm^*\iota_{\hat{X}}\widehat{\Pi}_\pm(h_{\mathbf{B}})(\hat{\iota}_\mp)_*\hat{\chi}_\mp\widehat{\mathcal{Q}}_\mp(h_{\mathbf{B}})^\top,$$

where $\hat{\iota}_\pm : \widehat{\widetilde{\partial}}_\pm = \hat{\pi}^{-1}(\widetilde{\partial}_\pm) \hookrightarrow \hat{U}$ is the inclusion. Then we have, as operators $\Omega_c^\bullet(\hat{Z}) \to \mathcal{D}'^\bullet(\hat{Z})$,

$$\frac{1-J^*}{2}(\hat{\varrho}\widehat{\mathcal{T}}_\pm(s))^n = \frac{(1-J^*)(\hat{A}_\pm)^n}{2(s-h_{\mathbf{B}})^n} + \mathcal{O}((s-h_{\mathbf{B}})^{-n+1}), \quad s \to h_{\mathbf{B}}.$$

Now note that

$$\frac{1-J^*}{2}\hat{A}_\pm = \frac{1}{2}\widehat{\mathcal{Q}}_\pm(h_{\mathbf{B}})\hat{\chi}_\pm\hat{\iota}_\pm^*\hat{\iota}_{\hat{X}}(1-J^*)\widehat{\Pi}_\pm(h_{\mathbf{B}})(\hat{\iota}_\mp)_*\hat{\chi}_\mp\widehat{\mathcal{Q}}_\mp(h_{\mathbf{B}})^\top = \hat{A}_\pm$$

where we used (7.3.6). As $\hat{A}_\pm$ is of rank one by (7.3.5), we have

$$\operatorname{tr}^\flat\Big((\hat{A}_\pm)^n|_{\Omega_0^1}\Big) = \operatorname{tr}^\flat\big(\hat{A}_\pm|_{\Omega_0^1}\big)^n$$

Thus, letting $c_\pm = \operatorname{tr}^\flat\big(A_\pm|_{\Omega_0^1}\big)$, we have

$$\frac{1}{2}\operatorname{tr}_{\mathrm{s}}^\flat\Big((1-J^*)(\hat{\varrho}\widehat{\mathcal{T}}_\pm(s))^n\Big) = -\frac{(c_\pm)^n}{(s-h_{\mathbf{B}})^n} + \mathcal{O}((s-h_{\mathbf{B}})^{-n+1}), \quad s \to h_{\mathbf{B}}. \tag{7.3.7}$$

Now we define

$$N_\varrho(t,n) = \sum_{\substack{\gamma\in\mathcal{P} \\ r(\gamma)=n \\ \tau(\gamma)\leqslant t}} I_\varrho(\gamma), \quad t \geqslant 0,$$

where we set, for a closed trajectory $\gamma : [0, \tau(\gamma)] \to M'$,

$$I_\varrho(\gamma) = \prod_{z\in R(\gamma)} \rho^2(\gamma) \quad \text{where} \quad R(\gamma) = \pi^{-1}(D_0) \cap \{(\gamma(\tau), \dot{\gamma}(\tau)) \ : \ \tau \in [0, \tau(\gamma)]\}.$$

Note that if $r(\gamma) = n$ one has $\sharp R(\gamma) = n\tau(\gamma)/\tau^\sharp(\gamma)$.

Proposition 7.3.1. *Assume that* $c_\pm > 0$. *Then*

$$N_\varrho(t,n) \sim \frac{(c_\pm t)^n}{n!}\frac{\mathrm{e}^{h_{\mathbf{B}}t}}{h_{\mathbf{B}}t}, \quad t \to +\infty.$$

Proof. Here we follow the argument of §4.5.1. Define

$$g_{n,\varrho}(t) = \sum_{\substack{\gamma\in\mathcal{P}\\ r(\gamma)=n}} \tau(\gamma) \sum_{\substack{k\geqslant 1\\ k\tau(\gamma)\leqslant t}} I_\varrho(\gamma)^k, \quad t \geqslant 0.$$

For $\mathrm{Re}(s)$ large enough we set $G_{n,\varrho}(s) = \displaystyle\int_0^\infty g_{n,\varrho}(t)\mathrm{e}^{-ts}\mathrm{d}t$. Then a simple computation starting from (7.3.1) shows that

$$G_{n,\varrho}(s) = \frac{1}{s}\sum_{r(\gamma)=n} \tau^\sharp(\gamma) I_\varrho(\gamma)^{\tau(\gamma)/\tau^\sharp(\gamma)}\mathrm{e}^{-s\tau(\gamma)} = \frac{\partial_s \operatorname{tr}_{\mathrm{s}}^\flat\Big(((1-J^*)\hat{\varrho}\widehat{\mathcal{T}}_\pm(s))^n\Big)}{2ns},$$

where the sum runs over all periodic orbits (not necessarily primitive) γ such that $r(\gamma) = n$. By (7.3.7) we have

$$G_{n,\varrho}(h_{\mathbf{B}}s) = \frac{(c_\pm)^n}{h_{\mathbf{B}}^{n+2}(s-1)^{n+1}} + \mathcal{O}((s-1)^{-n}), \quad s \to 1.$$

Then applying a Tauberian theorem from Delange [Del54, Théorème III] we obtain

$$\frac{1}{h_{\mathbf{B}}} g_{n,\varrho}(t/h_{\mathbf{B}}) \sim \frac{(c_\pm)^n}{h_{\mathbf{B}}^{n+2}}\frac{\mathrm{e}^t}{n!}t^n, \quad t \to +\infty,$$

which reads $g_{n,\chi}(t) \sim \dfrac{(c_\pm t)^n}{n! h_{\mathbf{B}}}\exp(h_{\mathbf{B}}t)$ as $t \to +\infty$. Now note that

$$g_{n,\varrho}(t) \leqslant \sum_{\substack{\gamma\in\mathcal{P}\\ r(\gamma)=n\\ \tau(\gamma)\leqslant t}} \tau(\gamma)\lfloor t/\tau(\gamma)\rfloor I_\varrho(\gamma) \leqslant t N_\varrho(t)$$

which gives $\liminf_{t\to+\infty}\dfrac{N_\varrho(t)}{g_{n,\varrho}(t)/t} \geqslant 1$. On the other hand, let

$$\zeta_{n,\varrho}(s) = \prod_{\substack{\gamma\in\mathcal{P}\\ r(\gamma)=n}} \left(1 - I_\varrho(\gamma)\mathrm{e}^{-s\tau(\gamma)}\right)^{-1}, \quad \mathrm{Re}(s) \gg 1.$$

Then we have

$$\zeta_{n,\varrho}(s) \geqslant \prod_{\substack{\gamma\in\mathcal{P}\\ r(\gamma)=n}} \left(1 + I_\varrho(\gamma)\mathrm{e}^{-s\tau(\gamma)}\right) \geqslant \prod_{\substack{\gamma\in\mathcal{P}\\ r(\gamma)=n\\ \tau(\gamma)\leqslant t}} \left(1 + I_\varrho(\gamma)\mathrm{e}^{-st}\right) \geqslant \mathrm{e}^{-st}N_\varrho(t). \tag{7.3.8}$$

As $\partial_s \log\zeta_{n,\varrho}(s) = -sG_{n\varrho}(s)$, it follows that $\zeta_{n,\varrho}$ extends holomorphically on $\{\mathrm{Re}(s) > h_{\mathbf{B}}\}$ (as $G_{n,\rho}$ does). Let $\sigma > 1$, and $\varepsilon > 0$ such that $(h_{\mathbf{B}}+\varepsilon)/\sigma < h_{\mathbf{B}}$. Then by (7.3.8) applied with $s = h_{\mathbf{B}} + \varepsilon$ we have

$$N_\varrho(t/\sigma) \leqslant \zeta_{n,\varrho}(h_{\mathbf{B}}+\varepsilon)\exp\left(\frac{(h_{\mathbf{B}}+\varepsilon)t}{\sigma}\right).$$

This implies that $N_\varrho(t/\sigma)/N_\varrho(t) \to 0$ as $t \to +\infty$. Now we write

$$g_{n,\varrho}(t) \geqslant \sum_{\substack{\gamma\in\mathcal{P}\\ r(\gamma)=n\\ \tau(\gamma)\leqslant t}} \tau(\gamma) I_\varrho(\gamma) \geqslant \sum_{\substack{\gamma\in\mathcal{P}\\ r(\gamma)=n\\ t/\sigma\leqslant\tau(\gamma)\leqslant t}} \frac{t}{\sigma} I_\varrho(\gamma) = \frac{t}{\sigma}\left(N_\varrho(t) - N_\varrho(t/\sigma)\right).$$

This leads to

$$\limsup_{t\to+\infty} \frac{N_\varrho(t)}{g_{n,\varrho}(t)/t} \leqslant \sigma \limsup_{t\to+\infty}\left(1-\frac{N_\varrho(t/\sigma)}{N_\varrho(t)}\right)^{-1} = \sigma.$$

As $\sigma > 1$ is arbitrary, the proof of the lemma is complete, since we have

$$g_{n,\varrho}(t)/t \sim \frac{(c_\pm t)^n}{n!}\frac{\mathrm{e}^{h_{\mathbf{B}}t}}{h_{\mathbf{B}}t}$$

as t goes to infinity. □

7.4 A priori bounds

In this section we derive some *a priori* bounds on $N(n,t)$ (the number of primitive periodic orbits bouncing n times on ∂D_0 and of length not greater than t) by using the fact that the billiard flow is conjugated to a subshift of finite type. This will allow us to convert the asymptotics obtained in §7.3 into an asymptotics on $N(n,t)$.

7.4.1 Coding

Let Σ'_N be the set of finite sequences $u = u_1 \cdots u_N$ with $u_j \in \{0,1,\ldots,r\}$ and $u_j \neq u_{j+1}$ (with $j \in \mathbb{Z}/N\mathbb{Z}$), and such that u is distinct from its cyclic permutations. We also define Σ_N as above by replacing $\{0,1,\ldots,r\}$ by $\{1,\ldots,r\}$. By §7.1.3 we have a one-to-one correspondance

$$\mathcal{P}_{\mathbf{B}'} \longleftrightarrow \left(\bigcup_{N=2}^{\infty} \Sigma'_N\right) / \sim \tag{7.4.1}$$

where $u \sim v$ if and only if u is a cyclic permutation of v. For any $\gamma \in \mathcal{P}_{\mathbf{B}'}$ we will denote by $\mathrm{wl}(\gamma)$ its word length, that is, the length of (any) word which is associated to γ via the above correspondance.

For any sequence $u \in \Sigma'_N$, we will denote by $\gamma_u : \mathbb{R} \to \Lambda'$ the closed billiard trajectory (parameterized by arc length) starting from the point $z_u \in \Lambda'$ which is associated to the sequence

$$\cdots uuu \cdots \in \Sigma'.$$

Its period is then defined by

$$\tau(\gamma_u) = \sum_{k=0}^{N-1} t'_+(B'^k(z_u)),$$

where t_+ is defined in §7.1.3. We have the following result.

Lemma 7.4.1. *There is $C>0$ such that the following holds. Let $\gamma:[0,T]\to\Lambda'$ be a billiard trajectory (parameterized by arc length) such that $\gamma(0),\gamma(T)\in\pi^{-1}(\partial D_0)$ and denote by $0=t_0<\cdots<t_N=T$ the times for which γ hits ∂D and assume that $N>2$. Let $u=u_1\cdots u_{N-1}\in\{0,\dots,r\}^{N-1}$ be the finite sequence such that it holds $\pi(\gamma(t_k))\in\partial D_{u_k}$ for $k=1,\dots,N-1$, and assume that $u_1\neq u_{N-1}$ so that γ_u is well defined. Then*

$$\tau(\gamma_u)-C\leqslant T\leqslant\tau(\gamma_u)+C.$$

Proof. By Lemma 7.1.1, it holds, for some $C>0$ and $\beta>1$ which are independent of γ,

$$\operatorname{dist}(B'^k(z_u),\gamma(t_k))\leqslant C\beta^{-N/2+|k-N/2|},\quad k=1,\dots,N-1.$$

Now note that $t'_+:\{z\in\pi^{-1}(\partial D)\ :\ t'_+(z)<+\infty\}\to\mathbb{R}_+$ is locally Lipschitz continuous. As Λ' is compact, it follows that for some $C'>0$ we have

$$\left|t'_+(B'^k(z_u))-t'_+(\gamma(t_k))\right|\leqslant C'\beta^{-N/2+|k-N/2|}$$

and thus

$$|\tau(\gamma_u)-T|\leqslant 2L_m+C'\sum_{k=1}^{N-1}\beta^{-N/2+|k-N/2|}\leqslant 2L_m+\frac{C'}{\beta-1},$$

where $L_m=\sup\{\operatorname{dist}(x_i,x_j)\ :\ x_i\in D_i,\ x_j\in D_j,\ i\neq j\}$. This concludes the proof. □

7.4.2 The bounds

Let $\mathcal{P}_{\mathbf{B}}$ be the set of oriented primitive periodic orbits of the flow associated to the billiard $\mathbf{B}$, and set $\mathcal{P}_{\mathbf{B}}(t)=\{\gamma\in\mathcal{P}_{\mathbf{B}}\ :\ \tau(\gamma)\leqslant t\}$. Then by [Mor91] we have

$$\sharp\{\gamma\in\mathcal{P}_{\mathbf{B}}\ :\ \tau(\gamma)\leqslant t\}\sim\frac{e^{h_{\mathbf{B}}t}}{h_{\mathbf{B}}t},\quad t\to+\infty. \tag{7.4.2}$$

In what follows, we will denote by $\mathcal{P}_{\mathbf{B}'}(n,t)$ the set of primitive periodic trajectories of the billiard $\mathbf{B}'$ of period less than t which make exactly n rebounds on ∂D_0, and $N(n,t)=\sharp\mathcal{P}_{\mathbf{B}'}(n,t)$. Finally we denote by $\widetilde{\mathcal{P}}_{\mathbf{B}}(t)$ (resp. $\widetilde{\mathcal{P}}_{\mathbf{B}'}(n,t)$) the set of (not necessarily primitive) periodic orbits for the billiard $\mathbf{B}$ (resp. for the billiard $\mathbf{B}'$) of period less or equal than t (resp. and making n rebounds on ∂D_0); we denote $\widetilde{N}(t)=\sharp\widetilde{\mathcal{P}}_{\mathbf{B}}(t)$ and $\widetilde{N}(n,t)=\sharp\widetilde{\mathcal{P}}_{\mathbf{B}'}(n,t)$. It is a classical fact that we have

$$\widetilde{N}(t)\sim N(t),\quad t\to+\infty, \tag{7.4.3}$$

as it can be seen from the equalities

$$\widetilde{N}(t)=\sum_{\tau(\gamma)\leqslant t}1=\sum_{\gamma\in\mathcal{P}}\sum_{k\tau(\gamma)\leqslant t}1=\sum_{\substack{\gamma\in\mathcal{P}\\ t/2<\tau(\gamma)\leqslant t}}1+\sum_{\substack{\gamma\in\mathcal{P}\\ \tau(\gamma)\leqslant t/2}}\lfloor t/\tau(\gamma)\rfloor,$$

and the fact that $\sum_{\substack{\gamma\in\mathcal{P}\\ \tau(\gamma)\leqslant t/2}}\lfloor t/\tau(\gamma)\rfloor\ll N(t)$ as $t\to+\infty$ by (7.4.2).

Proposition 7.4.2. *For each $n \geqslant 1$, there is $C_n > 0$ such that if t is large enough we have*

$$C_n^{-1} t^{n-1} \exp(h_{\mathbf{B}} t) \leqslant N(n,t) \leqslant C_n t^{n-1} \exp(h_{\mathbf{B}} t). \tag{7.4.4}$$

Proof. We start with the case $n = 1$. Consider the map $F : \Sigma_N \to \Sigma'_{N+1}$ defined by $F(u_1 \cdots u_N) = 0u_1 \cdots u_N$ (note that for any word $u \in \Sigma_N$, $F(u)$ is still a primitive word as it contains exactly one zero in its letters). By Lemma 7.4.1, we have

$$\tau(\gamma_u) - C \leqslant \tau(\gamma_{F(u)}) \leq \tau(\gamma_u) + C, \quad u \in \Sigma_N.$$

The map F is obviously injective. Recalling the correspondance (7.4.1) (for both billiards $\mathbf{B}$ and $\mathbf{B}'$), we thus have

$$N(1,t) \geqslant \sum_{N=2}^{\infty} \sum_{\substack{u \in \Sigma_N \\ \tau(\gamma_u) \leqslant t - C}} 1 \;=\; \sum_{\gamma \in \mathcal{P}_{\mathbf{B}}(t-C)} \mathrm{wl}(\gamma),$$

where the last equality comes from the fact that each $\gamma \in \mathcal{P}_{\mathbf{B}}$ corresponds to exactly $\mathrm{wl}(\gamma)$ words in $\Sigma_{\mathcal{A}}$. Note that for some $C > 0$ it holds

$$C^{-1}\tau(\gamma) \leqslant \mathrm{wl}(\gamma) \leqslant C\tau(\gamma), \quad \gamma \in \mathcal{P}_{\mathbf{B}}. \tag{7.4.5}$$

In particular we obtain

$$N(1,t) \geqslant \frac{t}{2C} \sharp\left(\mathcal{P}_{\mathbf{B}}(t) \setminus \mathcal{P}_{\mathbf{B}}(t/2)\right).$$

By (7.4.2), we obtain that the first inequality of (7.4.4) holds for $n = 1$. For the second one, consider the set $\widetilde{\Sigma}_N$ of finite words $u_1 \cdots u_N$ with $u_j \neq u_{j+1}$ for $j \in \mathbb{Z}/N\mathbb{Z}$ (note that $\Sigma_N \subset \widetilde{\Sigma}_N$ is the set of primitive words within $\widetilde{\Sigma}_N$). Consider the map $G : \widetilde{\Sigma}_N \to \Sigma'_{N+2}$ defined by

$$G(u_1 \cdots u_N) = 0u_1 \cdots u_N u_1, \quad u_1 \cdots u_N \in \widetilde{\Sigma}_N.$$

Every primitive periodic orbit bouncing exactly one time on ∂D_0 can be encoded by a finite word of the form $F(u)$ or $G(u)$ for some $u \in \widetilde{\Sigma}_N$ where $N \geqslant 2$ (note that F extends to a map $F : \widetilde{\Sigma}_N \to \Sigma'_{N+1}$). In particular, by Lemma 7.4.1, we have for some $C > 0$

$$\mathcal{P}(1,t) \subset \bigcup_N \Big\{ F\left(\left\{u \in \widetilde{\Sigma}_N \;:\; \tau(\gamma_u) \leqslant t + C\right\}\right) \cup G\left(\left\{u \in \widetilde{\Sigma}_N \;:\; \tau(\gamma_u) \leqslant t + C\right\}\right)\Big\}.$$

With (7.4.5) in mind, this leads to

$$N(1,t) \leqslant 2 \sum_{N=2}^{\infty} \sum_{\substack{u \in \widetilde{\Sigma}_N \\ \tau(\gamma_u) \leqslant t + C}} 1 \leqslant 2 \sum_{\substack{\gamma \in \widetilde{\mathcal{P}}_{\mathbf{B}} \\ \tau(\gamma) \leqslant t + C}} \mathrm{wl}(\gamma) \leqslant 2(t + C)\widetilde{N}(t + C) \leqslant C \exp(h_{\mathbf{B}} t),$$

where the last inequality holds for t large enough and comes from (7.4.3). The case $n = 1$ is proven.

We now proceed by induction and assume that (7.4.4) holds for every $n = 1, \dots, m$, for some $m \geqslant 1$. Similarly to (7.4.3), the estimate (7.4.4) also holds if we replace $N(n,t)$ by $\widetilde{N}(n,t)$. Every element of $\mathcal{P}_{\mathbf{B}'}(m+1,t)$ can be represented by the concatenation of a word (starting from 0) representing an element of $\widetilde{\mathcal{P}}_{\mathbf{B}'}(m,t_1)$ and a word (starting from 0) representing an element of $\widetilde{\mathcal{P}}_{\mathbf{B}'}(1,t_2)$, where $t_1+t_2 \leqslant t+2C$ (for some constant C). More precisely, for $N,k \geqslant 1$, set

$$A(k) = \Big\{u_1 \cdots u_N \in \widetilde{\Sigma}'_N \ : \ N \geqslant 2, \quad u_1 = 0, \quad u_N \neq 0, \quad \sharp\{j : u_j = 0\} = k\Big\}.$$

Then every element γ of $\mathcal{P}_{\mathbf{B}'}(m+1,t)$ can be represented by a word uv (i.e. $\gamma = \gamma_{uv}$) where $u \in A(m)$ and $v \in A(1)$. Moreover, by Lemma 7.4.1, we must have

$$\tau(\gamma) - 2C \leqslant \tau(\gamma_u) + \tau(\gamma_v) \leqslant \tau(\gamma) + 2C \tag{7.4.6}$$

for some C which does not depend of γ. Note also that for each periodic trajectory making k rebounds on ∂D_0, there are at most k words in $A(k)$ representing it (since the words have to start by the letter 0). Summarizing the above facts, we have for t large enough (in what follows C is a constant depending only on m that may change at each line)

$$\begin{aligned}
\widetilde{N}(m+1,t) &\leqslant \sum_{\substack{u\in A(m)\\ \tau(\gamma_u)\leqslant t+C}} \ \sum_{\substack{v\in A(1)\\ \tau(\gamma_v)\leqslant t-\tau(\gamma_u)+C}} 1\\
&\leqslant \sum_{\substack{u\in A(m)\\ \tau(\gamma_u)\leqslant t+C}} \widetilde{N}(1, t-\tau(\gamma_u)+C)\\
&\leqslant \sum_{\substack{u\in A(m)\\ \tau(\gamma_u)\leqslant t+C}} C\exp(h_{\mathbf{B}}(t-\tau(\gamma_u)+C))\\
&\leqslant \sum_{k=1}^{t+C} m\widetilde{N}(m,k)C\exp(h_{\mathbf{B}}(t-k+C))\\
&\leqslant C\sum_{k=1}^{t+C} k^{m-1}\exp(h_{\mathbf{B}}k)\exp(h_{\mathbf{B}}(t-k+C))\\
&\leqslant Ct^m\exp(h_{\mathbf{B}}t),
\end{aligned}$$

where we used $\widetilde{N}(m,t) \leqslant Ct^{m-1}\exp(h_{\mathbf{B}})$ as it follows from the induction hypothesis. For the lower bound, we proceed as follows. The map $A(m)\times A(1) \to A(m+1)$ defined by $(u,v) \mapsto uv$ is injective; moreover, every element of $\widetilde{\mathcal{P}}_{\mathbf{B}'}(m+1,t)$ is represented by exactly $m+1$ elements of $A(m+1)$. By (7.4.6), we have

$$\widetilde{N}(m+1,t) \geqslant \frac{1}{m+1} \sum_{\substack{u\in A(m)\\ \tau(\gamma_u)\leqslant t-C}} \ \sum_{\substack{v\in A(1)\\ \tau(\gamma_v)\leqslant t-\tau(\gamma_u)-C}} 1.$$

Let $T > 0$ large enough (it will be chosen later). By similar computations as above, we have

$$\widetilde{N}(m+1,t) \geqslant C \sum_{k=1}^{(t-C)/T} \left(\widetilde{N}(m,(k+1)T) - \widetilde{N}(m,kT)\right) \times \exp(h_{\mathbf{B}}(t-(k+1)T-C)). \tag{7.4.7}$$

If k is large enough, we have by the induction hypothesis

$$\begin{aligned}\widetilde{N}(m,(k+1)T) - \widetilde{N}(m,kT) &\geqslant C_m^{-1}[(k+1)T]^{m-1}\mathrm{e}^{h_{\mathbf{B}}(k+1)T} - C_m[kT]^{m-1}\mathrm{e}^{h_{\mathbf{B}}kT} \\ &\geqslant (kT)^{m-1}\mathrm{e}^{h_{\mathbf{B}}kT}\left(C_m^{-1}\left(1+\frac{1}{k}\right)^{m-1}\mathrm{e}^{h_{\mathbf{B}}T} - C_m\right).\end{aligned}$$

If T is large enough the last term of the above equation is bounded from below by $C(kT)^{m-1}\mathrm{e}^{h_{\mathbf{B}}kT}$ for some $C > 0$ independent of k. Injecting this in (7.4.7), we obtain

$$\begin{aligned}\widetilde{N}(m+1,t) &\geqslant C \sum_{k=1}^{(t-C)/T} (kT)^{m-1}\exp(h_{\mathbf{B}}kT)\exp(h_{\mathbf{B}}(t-(k+1)T-C)) \\ &\geqslant Ct^m \exp(h_{\mathbf{B}}t).\end{aligned}$$

Thus we proved that (7.4.4) holds for $\widetilde{N}(m+1,t)$. We now show that this also holds for $N(m+1,t)$, as follows. Because of Lemma 7.4.1 and the fact that any nonprimitive word in $A(m+1)$ can be written as the concatenation of $(m+1)/d$ identical words (where $d < m+1$ is a divisor of $m+1$) we have, for t large enough,

$$\begin{aligned}\widetilde{N}(m+1,t) - N(m+1,t) &\leqslant \sum_{d \mid m+1} \widetilde{N}\left(d, \frac{td}{m+1} + C\right) \\ &\leqslant C \sum_{d \mid m+1} \left(\frac{td}{m+1}\right)^{d-1} \exp\left(h_{\mathbf{B}}\left(\frac{td}{m+1} + C\right)\right),\end{aligned}$$

where the sums run over the divisors of $m+1$ which are stricty less than $m+1$. In particular, we have $\widetilde{N}(m+1,t) - N(m+1,t) \leqslant t^{(m+1)/2}\exp(h_{\mathbf{B}}t/2)$ for t large, and thus $N(m+1,t)$ also satisfies (7.4.4). This concludes the proof. □

7.5 Proof of the main result

In this section we prove the estimate annouced in the introduction. In fact, we will prove that $N_\varrho(n,t) \sim N(n,t)$ as $t \to +\infty$, which will imply the sought result.

7.5.1 First considerations

If $\gamma : \mathbb{R}/\tau(\gamma)\mathbb{Z} \to \Lambda'$ is a periodic orbit rebounding exactly n times on ∂D_0, we denote $I_1(\gamma), \dots, I_n(\gamma) \subset \mathbb{R}/\tau(\gamma)\mathbb{Z}$ the cyclically ordered sequence of intervals satisfying $\gamma(I_j^\circ) \notin \partial D_0$ for each j, where I_j° denotes the interior of I_j (this sequence is unique modulo cyclic permutations). We start by the following easy result.

Lemma 7.5.1. *There is $t_0 > 0$ such that the following holds. For every $\gamma \in \widetilde{\mathcal{P}}_{\mathbf{B}'}$ such that*

$$\ell(I_j(\gamma)) \geqslant t_0, \quad j = 1, \ldots, n,$$

we have $I_\varrho(\gamma) = 1$.

Proof. Let γ as above (for some large $t_0 > 0$ which will be chosen later) and $z \in R(\gamma)$ (see §B.3.1). Let $z_\pm = B'_\pm(z)$. Then $z_\pm \in \Lambda_\pm^{(m)}$, where $m = m(t_0) \to +\infty$ as $t_0 \to +\infty$. Here we set

$$\Lambda_\pm^{(m)} = \{z \in M \ : \ \sharp T_\pm(z) \geqslant m\}.$$

In particular, by the proof of Lemma 7.4.1 we have $\mathrm{dist}(z_\pm, \Gamma_\pm) \leqslant C\beta^{-m}$. Thus if $t_0 > 0$ is big enough, we have $\chi_\pm(z_\pm) = 1$ since $\chi_\pm \equiv 1$ on $\Gamma_\pm$. As a consequence $\varrho(z) = 1$, also by definition of ϱ. Thus, we get

$$I_\varrho(\gamma) = \prod_{z \in R(\gamma)} \varrho(z)^2 = 1,$$

which concludes the proof. □

For any $t_0 > 0$ we will denote $\widetilde{N}(n, t_0, t) = \sharp \widetilde{\mathcal{P}}_{\mathbf{B}'}(n, t_0, t)$ where

$$\widetilde{\mathcal{P}}_{\mathbf{B}'}(n, t_0, t) = \{\gamma \in \widetilde{\mathcal{P}}_{\mathbf{B}'} \ : \ r(\gamma) = n, \ \ell(I_j(\gamma)) \leqslant t_0 \text{ for some } 1 \leqslant j \leqslant n\}.$$

Lemma 7.5.2. *Let $t_0 > 0$ and $n \geqslant 2$. Then for some $C > 0$ we have for t large enough*

$$\widetilde{N}(n, t_0, t) \leqslant C t^{n-2} \exp(h_{\mathbf{B}} t).$$

Proof. By Lemma 7.4.1, there is $C > 0$ such that the following holds. Every trajectory $\gamma \in \widetilde{\mathcal{P}}_{\mathbf{B}'}(n, t_0, t)$ can be represented by a word in $\widetilde{\Sigma}'_N$ obtained by the concatenation of two words $u \in A(n-1)$ and $v \in A(1)$ satisfying

$$\tau(\gamma_u) \leqslant t + C, \quad \tau(\gamma_v) \leqslant t_0 + C.$$

Now for t large enough one has

$$\sharp\{u \in A(n-1) \ : \ \tau(\gamma_u) \leqslant t + C\} \leqslant (n-1)(t+C)^{n-2} \exp(h_{\mathbf{B}} t)$$

by Proposition 7.4.2. As $\{v \in A(1) \ : \ \tau(\gamma_v) \leqslant t_0 + C\}$ is finite, the lemma is proven. □

7.5.2 Proof of Theorem 7.0.1

First, we note that the constants $c_\pm$ given in §?? is positive. Indeed, if $c_\pm = 0$, then $s \mapsto \mathrm{tr}^\flat_{\mathrm{s}}(\varrho \mathcal{T}_\pm(s))$ would be regular at $s = h_{\mathbf{B}}$ by the proof of Proposition 7.3.1. In particular, we would have

$$N_\varrho(1, t) \ll \exp(h_{\mathbf{B}} t), \quad t \to \infty.$$

However, by Lemma 7.5.1, we have $I_\varrho(\gamma) = 1$ whenever $\tau(\gamma)$ is large enough and $r(\gamma) = 1$, which gives $N_\varrho(1, t) \sim N(1, t)$ as $t \to \infty$. Now $N(1, t) \geqslant C \exp(h_{\mathbf{B}} t)$ for large t by Proposition 7.4.2, which contradicts the fact that $N_\varrho(1,t) \ll \exp(h_{\mathbf{B}}t)$. Thus $c_\pm > 0$. By Lemmas 7.5.1 and 7.5.2 we have

$$N(n,t) - N_\varrho(n,t) \leqslant N(n,t,t_0) \leqslant Ct^{n-2}\exp(h_{\mathbf{B}}t).$$

Thus, by Propositions 7.3.1 and 7.4.2, we obtain $N_\varrho(n,t) \sim N(n,t)$ as $t \to \infty$, which reads

$$N(n,t) \sim \frac{(c_\pm t)^n}{n!}\frac{\mathrm{e}^{h_{\mathbf{B}}t}}{h_{\mathbf{B}}t}, \quad t \to \infty.$$

This concludes the proof of Theorem 7.0.1.

Chapitre 8

Obstacles et fonctions zêta dynamiques

Comme le précédent, ce chapitre concerne un système de billard constitué d'obstacles convexes dans l'espace euclidien. Nous obtenons un prolongement méromorphe pour certaines séries dynamiques liées aux résonances quantiques associées au problème de Dirichlet. En utilisant des résultats d'Ikawa et de Fried, nous montrons en outre qu'il y a une bande avec une infinité de ces résonances. Ce chapitre contient l'article *Dynamical zeta functions for billiards* [CP22] écrit en collaboration avec Vesselin Petkov.

Sommaire

8.1 Introduction

Let $D_1, \dots, D_r \subset \mathbb{R}^d$, $d \geqslant 2$, be compact strictly convex disjoint obstacles with smooth boundary and let $D = \bigcup_{j=1}^r D_j$. Throughout this paper we suppose the following non-eclipse condition

$$D_k \cap \text{convex hull}\,(D_i \cup D_j) = \emptyset, \tag{8.1.1}$$

for any $1 \leqslant i, j, k \leqslant r$ such that $i \neq k$ and $j \neq k$. Under this condition all period rays for the billiard flow in $\mathbb{R}^d \setminus D^\circ$ are ordinary reflecting ones without tangent segments to the boundary of D. Notice that if (8.1.1) is not satisfied, for generic perturbations of ∂D all periodic reflecting rays in $\mathbb{R}^d \setminus \mathring{D}$ have no segments tangent to ∂D (see Theorem 6.3.1 in [PS17]). We consider the (non grazing) billiard flow $(\varphi_t)_{t\in\mathbb{R}}$ (see §7.1.1 for a precise definition). For any periodic γ, denote by P_γ its associated linearized Poincaré map and by $\tau(\gamma)$ its period. Let $\mathcal{P}$ be the set of all periodic rays. The counting function of the lengths of periodic rays satisfies the bound

$$\sharp\{\gamma \in \mathcal{P} : \tau(\gamma) \leqslant \tau\} \leqslant e^{a\tau}, \quad \tau > 0,$$

for some $a > 0$. Moreover, for some constants $C, b_1, b_2 > 0$ we have (see for instance [Pet99])

$$Ce^{b_1\tau(\gamma)} \leqslant |\det(I - P_\gamma)| \leqslant e^{b_2\tau(\gamma)}, \quad \gamma \in \mathcal{P}.$$

By using these estimates, for $\mathrm{Re}(s) \gg 1$ we define two Dirichlet series

$$\eta_{\mathrm{N}}(s) = \sum_{\gamma\in\mathcal{P}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(1-P_\gamma)|^{1/2}}, \quad \eta_{\mathrm{D}}(s) = \sum_{\gamma\in\mathcal{P}} (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(1-P_\gamma)|^{1/2}},$$

where for any periodic γ, we denoted by $\tau^\sharp(\gamma)$ its primitive period, and by $m(\gamma)$ the number of reflexions of γ on the obstacles.

The series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$ are related to the resonances of the self-adjoint operators $-\Delta_{\mathrm{b}}$, $\mathrm{b} = \mathrm{N}, \mathrm{D}$, on $\mathcal{H} = L^2(\mathbb{R}^d \backslash D)$, with Neumann and Dirichlet boundary conditions on ∂D, respectively, and domains $\mathcal{D}_{\mathrm{b}} \subset \mathcal{H}$. To explain this relation, consider the resolvents

$$\mathcal{R}_{\mathrm{b}}(\mu) = \left(-\Delta_{\mathrm{b}} - \mu^2\right)^{-1},$$

which are analytic in $\{\mu \in \mathbb{C} : \mathrm{Im}\,\mu < 0\}$. Then $\mathcal{R}_{\mathrm{b}}(\mu) : \mathcal{H}_{\mathrm{comp}} \longrightarrow \mathcal{D}_{\mathrm{b,loc}}$ has a meromorphic continuation to $\mu \in \mathbb{C}$ if d is odd, and in the logarithmic covering of $\mathbb{C}\backslash\{0\}$ if d is even (see [LP89] for d odd and [DZ19] for $d \geqslant 2$). The poles μ_j, $\mathrm{Im}\,\mu_j > 0$, of these continuations are called *resonances*. Introduce the distribution $u \in \mathcal{D}'(\mathbb{R})$ given by the trace

$$u(t) = 2\mathrm{tr}_{L^2(\mathbb{R}^d)}\left(\cos(t\sqrt{-\Delta_{\mathrm{b}}}) \oplus 0 - \cos(t\sqrt{-\Delta_0})\right),$$

where Δ_0 is the free Laplacian in $\mathbb{R}^d$ and $\cos(t\sqrt{-\Delta_{\mathrm{b}}}) \oplus 0$ acts as 0 on $L^2(D)$. Then for d odd, [Mel82] (see also [BGR82] for a slightly weaker result) proved that in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ we have

$$u(t) = \sum_j m(\mu_j) e^{i|t|\mu_j},$$

where $m(\mu_j)$ is the multiplicity of μ_j. Here in the notations we omitted the dependence on the boundary conditions. The series above converge in the sense of distributions since we have the bound $\sharp\{\mu_j : |\mu_j| \leqslant r\} \leqslant Cr^d$ for all $r > 0$ (see [DZ19]). The reader may see also [Zwo97] and [DZ19] for a proof treating the singularity of $u(t)$ at $t = 0$. For d even, the situation is more complicated since the resonances are defined in a logarithmic covering of $\mathbb{C} \setminus \{0\}$. Let $\Lambda = \mathbb{C} \setminus \mathrm{e}^{i\frac{\pi}{2}\mathbb{R}^+}$ and for $\rho > 0$ let

$$\Lambda_\rho = \{\mu \in \Lambda : |\operatorname{Im}\mu| \leqslant \rho|\operatorname{Re}\mu|\}$$

be a conic neighborhood of $\mathbb{R}$. Choose a function ψ in $C_c^\infty(\mathbb{R}; [0,1])$ equal to 1 in a neighborhood of 0 and denote by $\sigma_{\mathrm{b}}(\lambda)$ the scattering phase related to $-\Delta_{\mathrm{b}}$ (see [Zwo98] for the notation). Following the work of Zworski [Zwo98], there exists a function $v_{\rho,\psi} \in C^\infty(\mathbb{R} \setminus \{0\})$ such that in the sense of distributions $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ one has

$$\begin{aligned} u(t) = \sum_{\mu_j \in \Lambda_\rho} m(\mu_j) e^{i\mu_j|t|} + m(0) \\ + 2\int_0^\infty \psi(\lambda) \frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}(\lambda)\cos(t\lambda)\mathrm{d}\lambda + v_{\rho,\psi}(t), \end{aligned} \tag{8.1.2}$$

where $m(0)$ is a constant. The reader may consult [Sjö97] for a local trace formula involving the resonances. Concerning the singularities of the distribution u, it follows from [BGR82] that we have

$$\operatorname{sing\,supp} u \subset \{\tau(\gamma) \in \mathbb{R}^+ : \gamma \in \mathcal{P}\}.$$

Under the condition (8.1.1), every periodic trajectory γ is an ordinary reflecting ray and the leading singularity of $u(t)$ related to $t = \tau(\gamma)$ was described by Guillemin and Melrose [GM79]. More precisely, the singularity related to γ has the form

$$(-1)^{m(\gamma)}\tau^\sharp(\gamma)|\det(I - P_\gamma)|^{-1/2}\delta(t - \tau(\gamma)) + L^1_{\mathrm{loc}}(\mathbb{R})$$

(see for instance, Corollary 4.3.4 in [PS17]), where for the Neumann problem the factor $(-1)^{m(\gamma)}$ must be omitted. Taking the sum of the Laplace transforms of the leading singularities of all $\gamma \in \mathcal{P}$, we obtain the dynamical zeta functions $\eta_{\mathrm{N}}(s), \eta_{\mathrm{D}}(s)$.

The analytic singularities of $\eta_{\mathrm{N}}(s)$ and $\eta_{\mathrm{D}}(s)$ are important for the analysis of the distribution of the resonances (see [Ika88b, Ika90a, Ika90b, Ika92, Sto09, Pet08] and the papers cited there). By using the Ruelle transfer operator and symbolic dynamics (see [Ika90a, Pet99, Sto09, Mor91]), a meromorphic continuation of $\eta_{\mathrm{N}}(s), \eta_{\mathrm{D}}(s)$ has been proved in a domain $s_0 - \alpha \leqslant \operatorname{Re} s$ with a suitable $\alpha > 0$, where s_0 is the abscissa of absolute convergence of the Dirichlet series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$. Recently, a meromorphic continuation on $\mathbb{C}$ of the series

$$\sum_{\gamma \in \mathcal{P}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(1 - P_\gamma)|}, \quad \operatorname{Re}(s) \gg 1, \tag{8.1.3}$$

has been proved by Küster–Schütte–Weich [KSW21] (see also [BSW21] for results concerning weighted zeta functions). On the other hand, a meromorphic continuation in the whole complex plan of the semi-classical zeta function for contact Anosov flows was established by Faure–Tsujii [FT17]. Their zeta function is similar to the function ζ_{N} defined in (8.1.4) below. The meromorphic continuation of the Ruelle zeta function for general Anosov flows was established by Giulietti–Liverani–Pollicott [GLP13] (see also the work of Dyatlov–Zworski [DZ16] for another microlocal proof). In this paper the series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$ are simply called dynamical zeta functions following previous works [Pet99, Pet08] and we refer to the book of Baladi [Bal18] for more references concerning zeta functions for hyperbolic dynamical systems.

Our main result is the following

Theorem 8.1.1. *The functions η_{N} and η_{D} admit a meromorphic continuation to the whole complex plane with simple poles and integer residues.*

One may consider also the zeta functions $\zeta_{\mathrm{b}}(s)$ associated to the boundary conditions $\mathrm{b} = \mathrm{D}, \mathrm{N}$, defined for $\operatorname{Re} s$ large enough by

$$\zeta_{\mathrm{b}}(s) = \exp\left(-\sum_{\gamma\in\mathcal{P}} (-1)^{m(\gamma)\varepsilon(\mathrm{b})} \frac{\mathrm{e}^{-s\tau(\gamma)}}{\mu(\gamma)|\det(1-P_\gamma)|^{1/2}}\right), \tag{8.1.4}$$

where $\varepsilon(\mathrm{D}) = 1$, $\varepsilon(\mathrm{N}) = 0$ and $\tau(\gamma) = \mu(\gamma)\tau^\sharp(\gamma)$. Notice that we have

$$\frac{\zeta_{\mathrm{b}}'(s)}{\zeta_{\mathrm{b}}(s)} = \eta_{\mathrm{b}}(s), \quad \mathrm{b} = \mathrm{D}, \mathrm{N}, \quad \operatorname{Re} s \gg 1. \tag{8.1.5}$$

In particular, since by the above theorem $\eta_{\mathrm{b}}(s)$ has simple poles with integer residues, it follows by a classical argument of complex analysis that we have the following

Corollary 8.1.2. *For $\mathrm{b} = \mathrm{D}, \mathrm{N}$, the function $s \mapsto \zeta_{\mathrm{b}}(s)$ extends meromorphically to the whole complex plane.*

In fact, we will prove a slightly more general result. For $q \in \mathbb{N}_{\geqslant 2}$, consider the Dirichlet series

$$\eta_q(s) = \sum_{m(\gamma)\in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|1-P_\gamma|^{1/2}}, \quad \operatorname{Re}(s) \gg 1,$$

where the sum runs over all periodic rays γ with $m(\gamma) \in q\mathbb{N}$. We will show that η_q admits a meromorphic continuation to the whole complex plane, with simple poles and residues valued in $\mathbb{Z}/q$ (see Theorem 8.4.1). In particular, considering the function $\zeta_q(s)$ defined by

$$\zeta_q(s) = \exp\left(-\sum_{\gamma\in\mathcal{P},\, m(\gamma)\in q\mathbb{N}} \frac{\mathrm{e}^{-s\tau(\gamma)}}{\mu(\gamma)|\det(1-P_\gamma)|^{1/2}}\right), \quad \operatorname{Re} s \gg 1,$$

one gets $q\zeta_q'/\zeta_q = q\eta_q$. Thus the function $s \mapsto \zeta_q(s)^q$ extends meromorphically to the whole complex plane since its logarithmic derivative is $q\eta_q$ and by Theorem 4 the function $q\eta_q$ has simple poles with integer residues. One reason for which it is interesting to study those functions is the relation

$$\eta_{\mathrm{D}}(s) = -\frac{\mathrm{d}}{\mathrm{d}s} \log \frac{\zeta_2(s)^2}{\zeta_{\mathrm{N}}(s)} = 2\eta_2(s) - \eta_{\mathrm{N}}(s), \tag{8.1.6}$$

which allows to express $\eta_{\mathrm{D}}(s)$ for $\operatorname{Re} s \gg 1$ as the difference of two Dirichlet series with positive coefficients. In particular, to show that $\eta_{\mathrm{D}}(s)$ has a meromorphic extension to $\mathbb{C}$, it is sufficient to prove that both series $\eta_{\mathrm{N}}(s)$ and $2\eta_2(s)$ have this property.

The distribution of the resonances μ_j depends on the geometry of the obstacles and for trapping obstacles and d odd it was conjectured that there exists $\delta > 0$ such that

$$N_{0,\delta} = \sharp\{\mu_j \in \mathbb{C} \;:\; 0 < \operatorname{Im} \mu_j \leqslant \delta\} = \infty. \tag{8.1.7}$$

For d even we must count

$$N_{0,\delta} = \sharp\{\mu_j \in \mathbb{C} \;:\; 0 < \operatorname{Im} \mu_j \leqslant \delta,\ 0 < \arg z < \pi\} \tag{8.1.8}$$

since a meromorphic extension of $\mathcal{R}_{\mathrm{D}}(\mu)$ is possible on the Riemann logarithmic surface $\Lambda = \{-\infty < \arg z < +\infty\}$. This conjecture for d odd was introduced by Ikawa [Ika90a] and it is known as the modified Lax-Phillips conjecture (MLPC). In this direction, for d odd, Ikawa [Ika88b, Ika90a] proved that for strictly convex disjoint obstacles satisfying (8.1.1) the existence of at least one singularity of $\eta_{\mathrm{N}}(s)$ or $\eta_{\mathrm{D}}(s)$ implies the existence of $\delta > 0$ for which (8.1.7) holds for the Neumann or Dirichlet boundary problem. The proof in [Ika90a] can be modified to cover also the case d even, applying the trace formula of Zworski (8.1.2). The existence of a singularity of the dynamical zeta function trivially holds for the Neumann problem since $\eta_{\mathrm{N}}(s)$ is a Dirichlet series with positive coefficients, and by a classical result, $\eta_{\mathrm{N}}(s)$ must have a singularity at $s_0 \in \mathbb{R}$, where $\operatorname{Re} s = s_0$ is the line of absolute convergence of $\eta_{\mathrm{N}}(s)$. Moreover, for d odd (see [Pet02]) there are constants $c_0, \varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$ it holds

$$\sharp\left\{\mu_j \in \mathbb{C} :\ 0 < \operatorname{Im} \mu_j \leqslant \frac{c_0}{\varepsilon},\ |\mu_j| \leqslant r\right\} \geqslant C_\varepsilon r^{1-\varepsilon}.$$

The situation for the Dirichlet problem is more complicated since $\eta_{\mathrm{D}}(s)$ is analytic for $\operatorname{Re} s \geqslant s_0$, s_0 being the abscissa of absolute convergence [Pet99]. Moreover, for $d = 2$ [Sto01] and for $d \geqslant 3$ under some conditions [Sto12] Stoyanov proved that there exists $\varepsilon > 0$ such that $\eta_{\mathrm{D}}(s)$ is analytic for $\operatorname{Re} s \geqslant s_0 - \varepsilon$. The reason of this cancellation of singularities is related to the change of signs in the Dirichlet series defining $\eta_{\mathrm{D}}(s)$, as it is emphasized by the relation (8.1.6). Despite many works in the physical literature concerning the n-disk problem (see for example [CVW97, Wir99, LZ02, PWB+12, BWP+13] and the references cited there), a rigorous proof of the (MLPC) was established only for sufficiently small balls [Ika90b] and for obstacles with sufficiently small diameters [Sto09]. In this direction we prove the following

Theorem 8.1.3. *Assume that the boundary ∂D is real analytic. Then the function $\eta_{\mathrm{D}}(s)$ has at least one pole and the* (MLPC) *holds.*

For convenience of the reader we explain briefly the ideas of the proofs of Theorems 8.1.1 and 8.1.3. First, in §8.2 we make some geometric preparations. The non-grazing billiard flow φ_t is defined in $M = B/\sim$, where

$$B = S\mathbb{R}^d \setminus (\pi^{-1}(\mathring{D}) \cup \mathcal{D}_g),$$

$\pi : S\mathbb{R}^d \to \mathbb{R}^d$ is the natural projection, $\mathcal{D}_{\rm g}$ is the grazing part (see §7.1.1) and $(x, v) \sim (y, w)$ if and only if $(x, v) = (y, w)$ or $x = y \in \partial D$ and w is equal to the reflected direction of v at $x \in \partial D$ (see §7.1.1). By using this factorization, the flow φ_t becomes continuous in M. However, to apply the Dyatlov–Guillarmou theory [DG16] in order to study the spectral properties of φ_t — which are intimately related to the dynamical zeta functions — we need to work with a smooth flow. For this reason we use a special smooth structure near the set ∂D with smooth charts introduced in the recent work of Küster–Schütte–Weich [KSW21]. In this smooth model one obtains a smooth flow φ_t which is uniformly hyperbolic when restricted to the trapped set K of φ_t, which is compact. The periodic points are dense in K and for any $z \in K$ the tangent space T_zM has the decomposition $T_zM = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z)$ with unstable and stable spaces $E_u(z)$, $E_s(z)$, where X is the generator of φ_t. A meromorphic continuation of the resolvent $(X + s)^{-1}$ has been established in [DG16] in a general setting, and as in [DZ16] and [DG16], estimates on the wavefront set of the resolvent $(X + s)^{-1}$ allow to define its flat trace which is linked to the series (8.1.3). This implies a meromorphic continuation of this series in $\mathbb{C}$ (see [KSW21]).

To prove a meromorphic continuation of the zeta function $\eta_{\rm N}(s)$ which is defined with factors $|\det(I - P_\gamma)|^{-1/2}$ instead of $|\det(1 - P_\gamma)|^{-1}$, a natural approach would consist in studying the Lie derivative $\mathcal{L}_X$ acting on sections of the unstable bundle $E_u(z)$ (see for example [FT17, pp. 6–8]). However, $E_u(z)$ in general is not smooth with respect to z, but only Hölder continuous. Thus we are led to change the geometrical setting as in the work of Faure–Tsujii [FT17] (notice that the Grassmann bundle introduced below also appears in [BR75] and [GL08]). One introduces the Grasmannian bundle $\pi_G : G \to V$ over a neighborhood V of K ; for every $z \in V$ the fiber $\pi_G^{-1}(z)$ is formed by all $(d-1)$-dimensional planes of T_zN. We define $\widetilde{K}_u = \{E_u(z) : z \in K\} \subset G$ and we introduce the natural lifted smooth flow $\widetilde{\varphi}_t$ on G. Then by [BR75, Lemma A.3], the set $\widetilde{K}_u$ is hyperbolic for the flow $\widetilde{\varphi}_t$. We introduce the tautological bundle $\mathcal{E} \to G$ by setting

$$\mathcal{E} = \{(\omega, v) \in \pi_G^*(TV) \ : \ v \in [\omega]\},$$

where $[\omega]$ denotes the subspace of $T_{\pi_G(z)}V$ that $\omega \in G$ represents, and $\pi_G^*(TV)$ is the pull-back of the tangent bundle $TV \to V$ by π_G. Next, we define the vector bundle $\mathcal{F} \to G$ by

$$\mathcal{F} = \{(\omega, W) \in TG \ : \ {\rm d}\pi_G(w) \cdot W = 0\}$$

which is a subbundle of the bundle $TG \to G$. Finally, we set

$$\mathcal{E}_{k,\ell} = \wedge^k \mathcal{E}^* \otimes \wedge^\ell \mathcal{F}, \quad 0 \leqslant k \leqslant d-1, \quad 0 \leqslant \ell \leqslant d^2 - d,$$

and define a suitable flow $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ as well as a transfer operator (see §8.2)

$$\Phi_{-t}^{k,\ell,*} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell}).$$

For a periodic orbit γ of φ_t, this geometrical setting allows to express the term $|\det(I - P_\gamma)|^{-1/2}$ as a finite sum involving the traces of $\Phi_{\tau(\gamma)}^{k,\ell}$ along the periodic orbit $\widetilde{\gamma} = \{(\gamma(t), E_u(\gamma(t)) \ : \ t \in [0, \tau(\gamma)]\}$ of the flow $(\widetilde{\varphi}_t)$ (see Lemma 8.3.1). In this context we may apply the Dyatlov–Guillarmou theory for the generators of the transfer operators $\Phi_{-t}^{k,\ell,*}$ and by using the Guillemin flat trace formula [Gui77] (see also [DZ16, Appendix B] or [BSW21]), we obtain the meromorphic continuation of $\eta_{\rm N}(s)$. Finally, the meromorphic continuation of $\eta_q(s)$ is obtained in a similar way, by considering in addition a certain *q-reflexion bundle* $\mathcal{R}_q \to G$ on which the flow $\widetilde{\varphi}_t$ can be lifted (see §8.4.1).

The strategy to prove Theorem 8.1.3 goes as follows. First, the representation (8.1.6) tells us that, if $\eta_{\rm D}(s)$ can be extended to an entire function, then the function $\zeta_2(s)^2/\zeta_{\rm N}(s)$ has neither zeros nor poles on the whole complex plane. For obstacles with real analytic boundary we may use real analytic charts near ∂D to define a real analytic structure on M which makes φ_t a real analytic flow. In this context we may apply a result of Fried [Fri95] to the billiard flow lifted to the Grassmannian bundle, and we show that the meromorphic functions ζ_2 and $\zeta_{\rm N}$ have finite order. This crucial point implies that $\zeta_2(s)^2/\zeta_{\rm N}(s)$ has also finite order. Finally, by using Hadamard's factorisation theorem, one concludes that we may write $\zeta_2(s)^2/\zeta_{\rm N}(s) = e^{Q(s)}$ for some polynomial $Q(s)$. This leads to $\eta_{\rm D}(s) = -Q'(s)$ and we obtain a contradiction. Notice that this argument works if the functions $\zeta_2(s)$ and $\zeta_{\rm N}(s)$ have finite order. The recent work of Bonthonneau–Jézéquel [BJ20] about Anosov flows suggests that this should be satisfied for obstacles with *Gevrey* regular boundary ∂D. In particular, the (MLPC) should be true for such obstacles. However in this paper we are not going to study this generalization.

This chapter is organized as follows. In §2 one introduces the geometric setting of the billiard flow φ_t and its smooth model. We define the Grasmannian extension G and the bundles $\mathcal{E}, \mathcal{F}$, $\mathcal{E}_{k,l} = \Lambda^k\mathcal{E}^\star \otimes \Lambda^\ell\mathcal{F}$ over G. Next, we discuss the setting, where we apply the Dyatlov-Guillarmou theory [DG16] for the first order operator $\mathbf{Q}_{k,\ell} = \nabla_{\tilde{Y}} + \mathbf{A}_{k,\ell}$ leading to a meromorphic continuation of the cut-off resolvent $\mathbf{R}_{k,\ell}(s) = \tilde{\chi}(\mathbf{Q}_{k,\ell} + s)^{-1}\tilde{\chi}$. In §3 we treat the flat trace of the resolvent $\mathbf{R}_\varepsilon^{k,\ell}(s) = \mathrm{e}^{-\varepsilon(\mathbf{Q}_{k,\ell}+s)}\mathbf{R}_{k,\ell}(s)$, $\varepsilon > 0$ and we obtain a meromoprhic continuation of $\eta_N(s)$. In §4 we study the dynamical zeta functions $\eta_q(s)$ for particular rays γ having number of reflections $m(\gamma) \in q\mathbb{N}$, $q \geqslant 2$. Applying the result for $\eta_2(s)$, we deduce the meromorphic continuation of $\eta_D(s)$. Finally, in §5 we treat the modified Lax-Philips conjecture for obstacles with real analytic boundary. In Appendix we present a proof for $d \geqslant 2$ of the uniform hyperbolicity of the flow φ_t in the Euclidean metric in $\mathbb{R}^d$.

8.2 Geometrical setting

In this section, we consider $D_1, \dots, D_r \subset \mathbb{R}^d$ some pairwise disjoint, smooth, strictly convex obstacles, satisfying the Ikawa non-eclipse condition (8.1.1). In particular we are in the same setting as that of the beginning of Chapter 7 and we will take the notations of §§7.1.1,7.1.2 and 7.1.3.

8.2.1 The Grassmann extension

We consider a neighborhood V of K in M, with smooth boundary. We embed V into a compact manifold without boundary N (for example by taking the doubling manifold of the closure of V), and we arbitrarily extend X to obtain a smooth vector field on N, which we still denote by X. The associated flow is still denoted by (φ_t) (note however that this new flow (φ_t) is now complete).

We consider the $(d-1)$-Grassmann bundle

$$\pi_G : G \to N$$

over N, that is, for every $z \in N$, the set $\pi_G^{-1}(z)$ consists of all $(d-1)$-dimensional planes of T_zN. Moreover, $\pi_G^{-1}(z)$ can be identified with the Grasmannian $G_{d-1}(\mathbb{R}^{2d-1})$ which is isomorphic to $O(2d-1)/(O(d-1)\times O(d))$, $O(k)$ being the space of $k\times k$ orthogonal matrices with elements in $\mathbb{R}$. The dimension of $O(k)$ is $k(k-1)/2$, hence the dimension of $\pi_G^{-1}(z)$ is $d(d-1)$. Note that G is a smooth compact manifold. We may lift the flow φ_t to a flow $\widetilde{\varphi}_t : G \to G$ which is simply defined by

$$\widetilde{\varphi}_t(E) = \mathrm{d}\varphi_t(z)(E) \subset T_{\varphi_t(z)}N, \quad z \in N, \quad E \subset T_zN, \quad t \in \mathbb{R}. \tag{8.2.1}$$

Introduce the set

$$\widetilde{K}_u := \{(z, E_u(z)) \ : \ z \in K\} \subset G.$$

Clearly, $\widetilde{K}_u$ is invariant under the action of $\widetilde{\varphi}_t$, since $\mathrm{d}\varphi_t(z)(E_u(z)) = E_u(\varphi_t(z))$. As K is a hyperbolic set, it follows from [BR75, Lemma A.3] that the set $\widetilde{K}_u$ is hyperbolic for $\widetilde{\varphi}_t$ and we have a decomposition

$$T_\omega G = \mathbb{R}\widetilde{X}(\omega) \oplus \widetilde{E}_u(\omega) \oplus \widetilde{E}_s(\omega), \quad \omega \in \widetilde{K}_u.$$

Here $\widetilde{X}$ is the generator of the flow $(\widetilde{\varphi}_t)$ and the spaces $\widetilde{E}_s(\omega)$ and $\widetilde{E}_u(\omega)$ are defined as follows. For small $\varepsilon > 0$, let

$$W_s(z,\varepsilon) = \{z' \in M : \mathrm{dist}\,(\varphi_t(z), \varphi_t(z')) \leqslant \varepsilon \text{ for every } t \geqslant 0\}$$

and

$$W_u(z,\varepsilon) = \{z' \in M : \mathrm{dist}\,(\varphi_{-t}(z), \varphi_{-t}(z')) \leqslant \varepsilon \text{ for every } t \geqslant 0\}$$

be the local stable and unstable manifolds at z, where dist is any smooth distance on M. For $b = s, u$, we define

$$\widetilde{W}_b(z) = TW_b(z,\varepsilon) = \{(z', E_b(z')) \ : \ z' \in W_b(z,\varepsilon)\} \subset G.$$

We finally set, for $\omega = (z, E_u(z)) \in \widetilde{K}_u$,

$$\widetilde{E}_u(\omega) = T_\omega(\widetilde{W}_u(z)),$$

and also we define $\widetilde{E}_s(\omega)$ as the tangent space at ω of the manifold

$$\widetilde{W}_{s,\mathrm{tot}}(z) = \big\{E \in \pi_G^{-1}(W_s(z,\varepsilon)) : \mathrm{dist}(E_u(z), E) < \varepsilon\big\},$$

where dist is any smooth distance on TN.

Lemma 8.2.1. *For any $\omega = (z, E) \in G$ we have natural isomorphisms*

$$\widetilde{E}_u(\omega) \simeq E_u(z), \quad \widetilde{E}_s(\omega) \simeq E_s(z) \oplus \ker \mathrm{d}\pi_G(\omega).$$

Under these identifications, we have

$$\mathrm{d}\widetilde{\varphi}_t|_{\widetilde{E}_u(\omega)} \simeq \mathrm{d}\varphi_t|_{E_u(z)}, \quad \mathrm{d}\widetilde{\varphi}_t|_{\widetilde{E}_s(\omega)} \simeq \mathrm{d}\varphi_t|_{E_s(z)} \oplus \mathrm{d}\widetilde{\varphi}_t|_{\ker \mathrm{d}\pi_G(\omega)}.$$

Proof. Note that if $\omega = (z, E) \in G$, by (8.2.1) one has

$$d\pi_G(\omega) \circ d\widetilde{\varphi}_t(\omega) = d(\pi_G \circ \widetilde{\varphi}_t)(\omega) = d(\varphi_t \circ \pi_G)(\omega) = d\varphi_t(z) \circ d\pi_G(\omega). \qquad (8.2.2)$$

This equality shows that $d\widetilde{\varphi}_t$ preserves $\ker d\pi_G$. Looking at the definitions of $\widetilde{W}_u(z)$ and $W_u(z, \varepsilon)$, we see that

$$d\pi_G(\omega)|_{\widetilde{E}_u(z)} : \widetilde{E}_u(z) \to E_u(z)$$

realizes an isomorphism. Then by (8.2.2), it is clear that $d\pi_G(\omega)|_{T_\omega \widetilde{W}_u(z)}$ realizes a conjugation between $d\widetilde{\varphi}_t(\omega)|_{\widetilde{E}_u(\omega)}$ and $d\varphi_t(z)|_{E_u(z)}$. Similarly, $d\pi_G|_{T_\omega \widetilde{W}_s(\omega)}$ realizes an isomorphism $T_\omega \widetilde{W}_s(\omega) \simeq E_s(z)$, which conjugates $d\widetilde{\varphi}_t|_{\widetilde{E}_s(\omega)}$ and $d\varphi_t|_{E_s(z)}$. Thus the lemma will be proven if we show that we have the direct sum

$$\widetilde{E}_s(z) = T_\omega \widetilde{W}_{s,\mathrm{tot}}(z) = T_\omega \widetilde{W}_s(z) \oplus \ker d\pi_G(\omega).$$

To see this, take a local trivialization $\widetilde{W}_{s,\mathrm{tot}}(z) \to W_s(z, \varepsilon) \times G_{d-1}(\mathbb{R}^{2d-1})$ sending ω on (z, E_0) for some $E_0 \in G_{d-1}(\mathbb{R}^{2d-1})$ and such that $\widetilde{W}_s(z)$ is sent to $W_s(z, \varepsilon) \times \{E_0\}$. In these coordinates one has the identifications

$$T_\omega \widetilde{W}_s(z) \simeq E_s(z) \oplus \{0\} \quad \text{and} \quad \ker d\pi_G(\omega) \simeq \{0\} \oplus T_{E_0} G_{d-1}(\mathbb{R}^{2d-1}).$$

As $T_\omega \widetilde{W}_{s,\mathrm{tot}}(z)$ is identified with $E_s(z) \oplus T_{E_0} G_{d-1}(\mathbb{R}^{2d-1})$, the proof is complete. □

We conclude this paragraph by noting that for any $\omega = (z, E) \in \widetilde{K}_u$ we have

$$\begin{aligned} \dim \widetilde{E}_u(\omega) + \dim \widetilde{E}_s(\omega) &= \dim E_u(z) + \dim E_s(z) + \dim \ker d\pi_G(\omega) \\ &= \dim M - 1 + \dim \pi_G^{-1}(z) \\ &= \dim G - 1, \end{aligned}$$

since $\dim G = \dim M + \dim \pi_G^{-1}(z)$.

8.2.2 Vector bundles

We define the tautological vector bundle $\mathcal{E} \to G$ by

$$\mathcal{E} = \{(\omega, u) \in \pi_G^*(TN) \ : \ \omega \in G, \ u \in [\omega]\},$$

where $[\omega] = E$ denotes the $(d-1)$ dimensional subspace of $T_{\pi_G(z)}N$ represented by $\omega = (z, E)$ and $\pi_G^*(TN)$ is the pullback bundle of TN. Also, we define the vector bundle $\mathcal{F} \to G$ by

$$\mathcal{F} = \{(\omega, W) \in TG \ : \ d\pi_G(\omega) \cdot W = 0\}.$$

It is a subbundle of the bundle $TG \to G$. The dimensions of the fibres $\mathcal{E}_\omega$ and $\mathcal{F}_\omega$ of $\mathcal{E}$ and $\mathcal{F}$ over ω are given by

$$\dim \mathcal{E}_\omega = d - 1, \quad \dim \mathcal{F}_\omega = \dim \ker d\pi_G(\omega) = \dim \pi_G^{-1}(z) = d^2 - d$$

for any $\omega \in G$ with $\pi_G(\omega) = z$. Finally, we set

$$\mathcal{E}_{k,\ell} = \wedge^k \mathcal{E}^* \otimes \wedge^\ell \mathcal{F}, \quad 0 \leqslant k \leqslant d-1, \quad 0 \leqslant \ell \leqslant d^2 - d,$$

where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$, that is, we repalce the fibre $\mathcal{E}_\omega$ by its dual space $\mathcal{E}_\omega^*$. We consider $\mathcal{E}^*$ and not $\mathcal{E}$ since the map $\mathrm{d}\varphi_t(\pi_G(z)) : \mathcal{E}_\omega \to \mathcal{E}_{\widetilde{\varphi}_t(\omega)}$ is expanding for $\omega \in \widetilde{K}_u$ and $t \to +\infty$, whereas $\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top} : \mathcal{E}_\omega^* \to \mathcal{E}^*_{\widetilde{\varphi}_t(\omega)}$ is contracting. Indeed, for $\omega = (z, E_u(z)) \in \widetilde{K}_u$ and $u \in E_u(z)^*$ (here $E_u(z)^*$ is the dual vector space of $E_u(z)$ and it does not coincide with $E_u^*(z)$) one has

$$\langle \mathrm{d}\varphi_t(z)^{-\top} u, v\rangle = \langle u, \mathrm{d}\varphi_{-t}(\varphi_t(z))v\rangle, \quad v \in \mathrm{d}\varphi_t(z)E_u(z) = E_u(\varphi_t(z)).$$

Consequently, $\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}$ is contracting on $\mathcal{E}_\omega^*$ when $\omega \in \widetilde{K}_u$ since $\mathrm{d}\varphi_{-t}(\varphi_t(z))$ is contracting on $E_u(\varphi_t(z))$. This fact will be convenient later for the proof of Lemma 8.3.1 below.

In what follows we use the notation $\omega = (z, \eta) \in G$ and $u \otimes v \in \mathcal{E}_{k,\ell}|_\omega$. By using the flow $\widetilde{\varphi}_t$, we introduce a flow $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ by setting

$$\Phi_t^{k,\ell}(\omega, u\otimes v) = \Big(\widetilde{\varphi}_t(\omega),\ b_t(\omega)\cdot \Big[\big(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\big)^{\wedge k}(u) \otimes \mathrm{d}\widetilde{\varphi}_t(\omega)^{\wedge \ell}(v)\Big]\Big), \quad (8.2.3)$$

where ${}^{-\top}$ denotes the inverse transpose and

$$b_t(\omega) = |\det \mathrm{d}\varphi_t(\pi_G(\omega))|_{[\omega]}|^{1/2} \cdot |\det(\mathrm{d}\widetilde{\varphi}_t(\omega)|_{\ker \mathrm{d}\pi_G})|^{-1}.$$

Here the determinants are taken with respect to any choice of smooth metrics g_N and g_G on N and G, as follows. If $\omega = (z, E) \in G$ and $t \in \mathbb{R}$, then the number $|\det \mathrm{d}\varphi_t(z)|_{[\omega]}|$ is defined as the absolute value of the ratio

$$\frac{(\mathrm{d}\varphi_t(z)|_{[\omega]})^{\wedge^{d-1}} \cdot \mu_\omega}{\mu_{\widetilde{\varphi}_t(\omega)}}$$

where $\mu_\omega = e_{1,\omega} \wedge \ \cdots \wedge e_{d-1,\omega} \in \wedge^{d-1}[\omega]$ (resp. $\mu_{\widetilde{\varphi}_t(\omega))} \in \wedge^{d-1}[\widetilde{\varphi}_t(\omega)]$) is a volume element given by any basis $e_{1,\omega}, \ldots, e_{d-1,\omega}$ of $[\omega]$ (resp. $[\widetilde{\varphi}_t(\omega)]$) which is orthonormal with respect to the scalar product induced by $g_N|_{[\omega]}$ (resp. $g_N|_{[\widetilde{\varphi}_t(\omega)]}$). The number $|\det(\mathrm{d}\widetilde{\varphi}_t(\omega)|_{\ker \mathrm{d}\pi_G})|$ is defined similarly. Taking local trivializations of $\mathcal{E}^*$ and $\mathcal{F}$, we see that the action of $\Phi_t^{k,\ell}$ is smooth. Thus we have a diagram

$$\begin{array}{ccc}
\mathcal{E}_{k,\ell} & \xrightarrow{\Phi_t^{k,\ell}} & \mathcal{E}_{k,\ell} \\
\downarrow & & \downarrow \\
G & \xrightarrow{\tilde{\varphi}_t} & G \\
\downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_G} \\
N & \xrightarrow{\varphi_t} & N
\end{array}$$

Now we consider the transfer operator

$$\Phi_{-t}^{k,\ell,*} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$$

defined by

$$\Phi_{-t}^{k,\ell,*}\mathbf{u}(\omega) = \Phi_t^{k,\ell}\big[\mathbf{u}(\widetilde{\varphi}_{-t}(\omega))\big], \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}). \tag{8.2.4}$$

Let $\mathbf{P}_{k,\ell} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$ be the generator of $\Phi_{-t}^{k,\ell,*}$, that is,

$$\mathbf{P}_{k,\ell}\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\Phi_{-t}^{k,\ell,*}\mathbf{u}\right), \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

Then we have the equality

$$\mathbf{P}_{k,\ell}(f\mathbf{u}) = (\widetilde{X}f)\mathbf{u} + f(\mathbf{P}_{k,\ell}\mathbf{u}), \quad f \in C^\infty(G), \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}). \tag{8.2.5}$$

Next, we want to study the spectral properties of the operator $\mathbf{P}_{k,\ell}$ applying the work of Dyatlov–Guillarmou [DG16]. For this purpose, one needs to find a neighborhood of $\widetilde{K}_u$ which has convexity properties with respect to $\widetilde{X}$. However, it is not clear that a such neighborhood exists, and one needs to modify slightly $\widetilde{X}$ outside a neighborhood of $\widetilde{K}_u$ to obtain the desired properties. This is down in §8.2.3 below.

8.2.3 Isolating blocks

By [CE71, Theorem 1.5], there exists an arbitrarily small neighborhood $\widetilde{V}_u$ of $\widetilde{K}_u$ in G such that the following holds.

(i) The boundary $\partial\widetilde{V}_u$ of $\widetilde{V}_u$ is smooth;

(ii) The set $\partial_0\widetilde{V}_u = \{z \in \partial\widetilde{V}_u : \widetilde{X}(z) \in T_z\partial\widetilde{V}_u\}$ is a smooth submanifold of codimension 1 of $\partial\widetilde{V}_u$;

(iii) There is $\varepsilon > 0$ such that for any $z \in \partial\widetilde{V}_u$ one has

$$\widetilde{X}(z) \in T_z\partial\widetilde{V}_u \quad \Longrightarrow \quad \widetilde{\varphi}_t(z) \notin \operatorname{clos}\widetilde{V}_u, \quad |t| \in \,]0,\varepsilon[\,,$$

where clos A denotes the closure of a set A.

In what follows we denote

$$\Gamma_\pm(\widetilde{X}) = \{z \in \widetilde{V}_u : \widetilde{\varphi}_t(z) \in \widetilde{V}_u, \ \mp t > 0\}.$$

A function $\tilde{\rho} \in C^\infty(\operatorname{clos}\widetilde{V}_u, \mathbb{R}_{\geqslant 0})$ will be called a boundary defining function for $\widetilde{V}_u$ if we have $\{z \in \operatorname{clos}\widetilde{V}_u : \tilde{\rho}(z) = 0\} = \partial\widetilde{V}_u$ and $\mathrm{d}\tilde{\rho}(z) \neq 0$ for any $z \in \partial\widetilde{V}_u$.

By [GMT21, Lemma 2.3] (see also [KSW21, Lemma 5.2]), we have the following result.

Lemma 8.2.2. *For any small neighborhood $\widetilde{W}_0$ of $\partial_0\widetilde{V}_u$ in* clos $\widetilde{V}_u$*, we may find a vector field $\widetilde{Y}$ on* clos $\widetilde{V}_u$ *which is arbitrarily close to $\widetilde{X}$ in the C^∞-topology, such that the following holds.*

(1) $\operatorname{supp}(\widetilde{Y} - \widetilde{X}) \subset \widetilde{W}_0$*;*

(2) $\Gamma_\pm(\widetilde{X}) = \Gamma_\pm(\widetilde{Y})$ *where $\Gamma_\pm(\widetilde{Y})$ is defined as $\Gamma_\pm(\widetilde{X})$ by replacing the flow $(\widetilde{\varphi}_t)$ by the flow generated by $\widetilde{Y}$.*

(3) *For any defining function $\tilde{\rho}$ of $\widetilde{V}_u$ and any $\omega \in \partial\widetilde{V}_u$ we have*

$$\widetilde{Y}\tilde{\rho}(\omega) = 0 \quad \Longrightarrow \quad \widetilde{Y}^2\tilde{\rho}(\omega) < 0. \tag{8.2.6}$$

From now on, we will fix $\widetilde{V}_u, \widetilde{W}_0$ and $\widetilde{Y}$ as above. By [DG16, Lemma 2.1] we may find a smooth extension of $\widetilde{Y}$ on G (still denoted by $\widetilde{Y}$) so that for every $\omega \in G$ and $t \geqslant 0$, we have

$$\omega, \widetilde{\varphi}_t(\omega) \in \operatorname{clos} \widetilde{V}_u \quad \Longrightarrow \quad \widetilde{\varphi}_\tau(\omega) \in \operatorname{clos} \widetilde{V}_u \text{ for every } \tau \in [0,t]. \tag{8.2.7}$$

Let $(\psi_t)_{t\in\mathbb{R}}$ be the flow generated by $\widetilde{Y}$ and set $\widetilde{\Gamma}_\pm = \Gamma_\pm(\widetilde{Y})$ for simplicity. The extended unstable/stable bundles $\widetilde{E}_\pm^* \subset T^*\widetilde{V}_u$ over $\widetilde{\Gamma}_\pm$ are defined by

$$\widetilde{E}_\pm^*(\omega) = \{\Omega \in T_\omega^*\widetilde{V}_u \ : \ \Psi_t(\Omega) \to_{t\to\pm\infty} 0\},$$

where Ψ_t is the symplectic lift of $\widetilde{\psi}_t$, that is

$$\Psi_t(\Omega) = \left(\mathrm{d}\widetilde{\psi}_t(\omega)^{-\top} \cdot \Omega\right), \quad (\omega, \Omega) \in T^*G, \quad t \in \mathbb{R},$$

where ${}^{-\top}$ denotes the inverse transpose. Then by [DG16, Lemma 2.10], the bundles $\widetilde{E}_\pm^*(\omega)$ depend continuously on $\omega \in \widetilde{\Gamma}_\pm$, and for any smooth norm $|\cdot|$ on T^*G, and moreover for some constants $C, \beta > 0$ we have

$$|\Psi_{\pm t}(\Omega)| \leqslant C\mathrm{e}^{-\beta t}|\Omega|, \quad t \geqslant 0, \quad \Omega \in E_\mp^*(\omega).$$

8.2.4 Dyatlov–Guillarmou theory

Let $\nabla^{k,\ell}$ be any smooth connexion on $\mathcal{E}_{k,\ell}$. Then by (8.2.5) we have

$$\mathbf{P}_{k,\ell} = \nabla^{k,\ell}_{\widetilde{X}} + \mathbf{A}_{k,\ell}$$

for some $\mathbf{A}_{k,\ell} \in C^\infty(G, \mathrm{End}(\mathcal{E}_{k,\ell}))$. We define a new operator $\mathbf{Q}_{k,\ell}$ by setting

$$\mathbf{Q}_{k,\ell} = \nabla^{k,\ell}_{\widetilde{Y}} + \mathbf{A}_{k,\ell} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell}).$$

Note that $\mathbf{Q}_{k,\ell}$ coincides with $\mathbf{P}_{k,\ell}$ near $\widetilde{K}_u$ since $\widetilde{Y}$ coincides with $\widetilde{X}$ near $\widetilde{K}_u$. Clearly, we have

$$\mathbf{Q}_{k,\ell}(f\mathbf{u}) = (\widetilde{Y}f)\mathbf{u} + f(\mathbf{Q}_{k,\ell}\mathbf{u}), \quad f \in C^\infty(G), \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}). \tag{8.2.8}$$

Consider the transfer operator $\mathrm{e}^{-t\mathbf{Q}_{k,\ell}} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$, which is characterized by

$$\partial_t \mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u} = -\mathbf{Q}_{k,\ell}\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u}, \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}), \quad t \in \mathbb{R}.$$

Fix any norm on $\mathcal{E}_{k,\ell}$; this fixes a scalar product on $L^2(G, \mathcal{E}_{k,\ell})$. Then for some $C > 0$ we have

$$\|\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\|_{L^2(G,\mathcal{E}_{k,\ell})\to L^2(G,\mathcal{E}_{k,\ell})} \leqslant C\mathrm{e}^{C|t|}, \quad t \in \mathbb{R}.$$

For $\mathrm{Re}(s) \gg 1$, the resolvent $(\mathbf{Q}_{k,\ell} + s)^{-1}$ on $L^2(G, \mathcal{E}_{k,\ell})$ is given by

$$(\mathbf{Q}_{k,\ell} + s)^{-1} = \int_0^\infty \mathrm{e}^{-t(\mathbf{Q}_{k,\ell}+s)}\mathrm{d}t : L^2(G, \mathcal{E}_{k,\ell}) \to L^2(G, \mathcal{E}_{k,\ell}). \tag{8.2.9}$$

Let $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ be such that $\widetilde{\chi} \equiv 1$ on $\widetilde{K}_u$. Define the operator

$$\mathbf{R}_{k,\ell}(s) = \widetilde{\chi}(\mathbf{Q}_{k,\ell} + s)^{-1}\widetilde{\chi}, \quad \mathrm{Re}(s) \gg 1,$$

from $C_c^\infty(\widetilde{V}_u, \mathcal{E}_{k,\ell})$ to $\mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell})$, where $\mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell})$ denotes the space of distributions valued in $\mathcal{E}_{k,\ell}$. Thanks to (8.2.6), (8.2.7) and (8.2.8), we are in position to apply [DG16, Theorem 1] in order to obtain a meromorphic extension of $\mathbf{R}_{k,\ell}(s)$ to the whole plane $\mathbb{C}$. Moreover, according to [DG16, Theorem 2], for every $s_0 \in \mathbb{C}$ in a small neighborhood of s_0 one has the representation

$$\mathbf{R}_{k,\ell}(s) = \mathbf{R}_{H,k,\ell}(s) + \sum_{j=1}^{J(s_0)} \frac{(-1)^{j-1}(\mathbf{Q}_{k,\ell} + s_0)^{j-1}\Pi_{s_0}^{k,\ell}}{(s - s_0)^j} \tag{8.2.10}$$

where $\mathbf{R}_{H,k,\ell}(s) : C_c^\infty(\widetilde{V}_u, \mathcal{E}_{k,\ell}) \to \mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell})$ is a holomorphic family of operators near $s = s_0$. Let us denote by $K_{\mathbf{R}_{H,k,\ell}(s)}$ and $K_{\Pi_{s_0}^{k,\ell}}$ the Schwartz kernels of the operators $\mathbf{R}_{H,k,\ell}(s)$ and $\Pi_{s_0}^{k,\ell}$, respectively. By [DG16, Lemma 4.5], we have

$$\mathrm{WF}'(K_{\mathbf{R}_{H,k,\ell}}(s)) \subset \Delta(T^*\widetilde{V}_u) \cup \Upsilon_+ \cup (\widetilde{E}_+^* \times \widetilde{E}_-^*). \tag{8.2.11}$$

Here $\Delta(T^*\widetilde{V}_u)$ is the diagonal in $T^*(\widetilde{V}_u \times \widetilde{V}_u)$,

$$\Upsilon_+ = \{(\Psi_t(\omega, \Omega), \omega, \Omega) \ : \ (\omega, \Omega) \in T^*\widetilde{V}_u, \ t \geqslant 0, \ \langle \widetilde{Y}(\omega), \Omega\rangle = 0\},$$

while the bundles $\widetilde{E}_\pm^*$ and Ψ_t are defined in §8.2.3. Finally, we have

$$\mathrm{supp}(K_{\Pi_{s_0}^{k,\ell}}) \subset \Gamma_+ \times \Gamma_- \quad \text{and} \quad \mathrm{WF}'(K_{\Pi_{s_0}^{k,\ell}}(s)) \subset \widetilde{E}_+^* \times \widetilde{E}_-^*. \tag{8.2.12}$$

8.3 The dynamical zeta function for the Neumann problem

8.3.1 The Guillemin trace formula

Consider a smooth flow Φ_t acting on $C^\infty(V, \mathcal{V})$ which has the form

$$\Phi_t(x, u) = (\varphi_t(x), F(x, t) \cdot u), \quad (x, u) \in \mathcal{V},$$

where φ_t is a hyperbolic flow on V and $F(x, t)$ is an isomorphism $\mathcal{V}_x \to \mathcal{V}_{\varphi_t(x)}$ for every x, t. For $\varrho \in C_c^\infty(\mathbb{R}^+)$ define $\mathbf{T}_\varrho : C^\infty(V, \mathcal{V}) \to C^\infty(V, \mathcal{V})$ by

$$\mathbf{T}_\varrho \mathbf{u} = \int_0^\infty \varrho(t)(\Phi_{-t}^* \mathbf{u})\mathrm{d}t, \quad \mathbf{u} \in C^\infty(V, \mathcal{V}),$$

where

$$(\Phi_{-t}^* \mathbf{u})(x) = \Phi_t(\mathbf{u}(\varphi_{-t}x)), \quad \mathbf{u} \in C^\infty(V, \mathcal{V}), \quad x \in V.$$

Assume that all periodic trajectories γ of φ_t are nondegenerate, that is $1 \notin \mathrm{spect}\, P_\gamma$, where

$$P_\gamma = \mathrm{d}\varphi_{-\tau(\gamma)}(z)|_{E_u(z) \oplus E_s(z)}$$

is the linearized Poincaré map of the orbit γ (here z is any point of γ).

Then following [DZ16, Appendix B] (see also [BSW21]), the flat trace of $\mathbf{T}_\varrho$ (see Appendix B.3) is well defined and given by

$$\operatorname{tr}^\flat \mathbf{T}_\varrho = \sum_\gamma \frac{\varrho(\tau(\gamma))}{|\det(1-P_\gamma)|} \int_0^{\tau^\sharp(\gamma)} \operatorname{tr} F(\gamma(t), \tau(\gamma)) \mathrm{d}t,$$

where the sum runs over all periodic orbits γ of the flow φ_t, and $\tau^\sharp(\gamma)$ is the primitive period of γ.

8.3.2 The flat trace of cut-off resolvent

For $\varrho \in C_c^\infty(\mathbb{R}^+)$ define

$$\mathbf{T}_\varrho^{k,\ell}\mathbf{u} = \widetilde{\chi}\left(\int_0^\infty \varrho(t)(\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u})\mathrm{d}t\right)\widetilde{\chi}, \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}),$$

where the cut-off function $\widetilde{\chi}$ was defined in the previous section. By the above statement for the flat trace we have

$$\operatorname{tr}^\flat(\mathbf{T}_\varrho^{k,\ell}) = \sum_{\widetilde{\gamma}} \frac{\varrho(\tau_\gamma)}{|\det(1-\widetilde{P}_\gamma)|} \int_0^{\tau^\sharp(\gamma)} \operatorname{tr} \Phi_{\tau(\gamma)}^{k,\ell}(\widetilde{\gamma}(t))\mathrm{d}t,$$

where the sum runs over all periodic orbits $\tilde{\gamma}$ of $\tilde{\varphi}_t$ and $\widetilde{P}_\gamma$ is the linearized Poincaré map of the closed orbit $t \mapsto \widetilde{\gamma}(t) = (\gamma(t), E_u(\gamma(t)) \in G$ of the flow $\widetilde{\varphi}_t$. Note that $\Phi_{\tau(\gamma)}^{k,\ell}(\widetilde{\gamma}(t))$ is conjugated to $\Phi_{\tau(\gamma)}^{k,\ell}(\widetilde{\gamma}(t'))$ for any t, t'. Hence

$$\int_0^{\tau^\sharp(\gamma)} \operatorname{tr} \Phi_{\tau(\gamma)}^{k,\ell}(\widetilde{\gamma}(t))\mathrm{d}t = \tau^\sharp(\gamma) \operatorname{tr} \Phi_{\tau(\gamma)}^{k,\ell}(\omega_{\widetilde{\gamma}})$$

for any $\omega_{\widetilde{\gamma}} \in \widetilde{\gamma}$.

By proceeding exactly as in [DZ16, §4], we obtain, with (8.2.9) in mind

$$\operatorname{tr}^\flat \mathbf{R}_\varepsilon^{k,\ell}(s) = \sum_{\widetilde{\gamma}} \frac{\mathrm{e}^{-s\tau(\gamma)}\tau^\sharp(\gamma) \operatorname{tr} \Phi_{\tau(\gamma)}^{k,\ell}(\omega_{\widetilde{\gamma}})}{|\det(1-\widetilde{P}_\gamma)|}, \quad \operatorname{Re}(s) \gg 1, \tag{8.3.1}$$

where for $\operatorname{Re}(s)$ large enough and $\varepsilon > 0$ small, we set

$$\mathbf{R}_\varepsilon^{k,\ell}(s) = \widetilde{\chi}\mathrm{e}^{-\varepsilon(\mathbf{Q}_{k,\ell}+s)}(\mathbf{Q}_{k,\ell}+s)^{-1}\widetilde{\chi}.$$

More precisely, we apply the above formula with the functions $\varrho_{s,T}(t) = \mathrm{e}^{-st}\varrho_T(t)$, where $\varrho_T \in C_c^\infty(\mathbb{R}^+)$ satisfies $\operatorname{supp}\varrho_T \subset [\delta, T+1]$ for δ small and $\varrho_T \equiv 1$ on $[2\delta, T]$. Then by taking the limit $T \to \infty$ which can be justified as in [DZ16, §4], we obtain (8.3.1) exploiting (8.2.9). The fact that $\operatorname{tr}^\flat \mathbf{R}_\varepsilon^{k,\ell}(s)$ is well defined follows from the information for the wave front set given in (8.2.11) and from multiplicativity of wavefront sets [Hör90, Theorem 8.2.14].

Next one obtains the following formula similar to that in [FT17] which we will prove for the sake of completeness.

Lemma 8.3.1. *For any periodic orbit $\tilde{\gamma}$ related to a periodic orbit γ, we have*

$$\frac{1}{|\det(I-\widetilde{P}_\gamma)|}\sum_{k=0}^{d-1}\sum_{\ell=0}^{d^2-d}(-1)^{k+\ell}\operatorname{tr}\Phi^{k,\ell}_{\tau(\gamma)}(\omega_{\tilde{\gamma}}) = |\det(1-P_\gamma)|^{-1/2}.$$

Proof. Let $\gamma(t)$ be a periodic orbit and let $\tilde{\gamma}(t) = (\gamma(t), E_u(\gamma(t)),\ \omega_{\tilde{\gamma}} \in \tilde{\gamma}$. Set

$$P_{\gamma,u} = \mathrm{d}\varphi_{-\tau(\gamma)}(z)|_{E_u(z)}, \quad P_{\gamma,s} = \mathrm{d}\varphi_{-\tau(\gamma)}(z)|_{E_s(z)},$$
$$P_{\gamma,\perp} = \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})|_{\ker \mathrm{d}\pi_G(\omega)}, \quad P^{-1}_{\gamma,\perp} = \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})^{-1}|_{\ker \mathrm{d}\pi_G(\omega)}.$$

The linearized Poincaré map $\widetilde{P}_\gamma$ of the closed orbit $\widetilde{\gamma}$ satisfies

$$\begin{aligned}\det(I-\widetilde{P}_\gamma) &= \det\Big(I - \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}|_{\widetilde{E}_s(\omega)\oplus\widetilde{E}_u(\omega)}\Big)\\ &= \det(I-P_\gamma)\det(I-P_{\gamma,\perp})\end{aligned} \tag{8.3.2}$$

since $\widetilde{E}_s(\omega) \simeq E_s(z) \oplus \ker \mathrm{d}\pi_G(\omega)$ and $\widetilde{E}_u(\omega) \simeq E_u(z)$ by Lemma 8.2.1. Recall the well known formula

$$\det(I-A) = \sum_{j=0}^{k}(-1)^j \operatorname{tr}\wedge^j A$$

which is valid for any endomorphism A of a k-dimensional vector space. By (8.2.3), we get

$$\begin{aligned}&\sum_{k,\ell}(-1)^{k+\ell}\operatorname{tr}\Phi^{k,\ell}_{\tau(\gamma)}(\omega_{\tilde{\gamma}})\\ &\qquad= b_{\tau(\gamma)}(\omega_{\tilde{\gamma}})\left(\sum_{k=0}^{d-1}(-1)^k\operatorname{tr}\wedge^k P_{\gamma,u}\right)\left(\sum_{\ell=0}^{d^2-d}(-1)^\ell\operatorname{tr}\wedge^\ell P^{-1}_{\gamma,\perp}\right)\\ &\qquad= |\det(P_{\gamma,u})|^{-1/2}|\det(P_{\gamma,\perp})|\det(I-P_{\gamma,u})\det(I-P^{-1}_{\gamma,\perp}).\end{aligned} \tag{8.3.3}$$

Here we have used the equality

$$\begin{aligned}b_{\tau(\gamma)}(\omega_{\tilde{\gamma}}) &= |\det \mathrm{d}\varphi_{\tau(\gamma)}(\pi_G(\omega_{\tilde{\gamma}}))|_{[\omega_{\tilde{\gamma}}]}|^{1/2}\cdot\left|\det\left(\mathrm{d}\widetilde{\varphi}_{\tau(\gamma)}(\omega_{\tilde{\gamma}})|_{\ker \mathrm{d}\pi_G}\right)\right|^{-1}\\ &= |\det(P_{\gamma,u})|^{-1/2}|\det(P_{\gamma,\perp})|\end{aligned}$$

because $P_{\gamma,u}$ and $P_{\gamma,\perp}$ are defined with $d\varphi_{-t}$ and $d\tilde{\varphi}_{-t}$, respectively. Therefore (8.3.2) yields

$$\sum_{k,\ell}(-1)^{k+\ell}\frac{\operatorname{tr}\Phi^{k,\ell}_{\tau(\gamma)}(\omega_{\tilde{\gamma}})}{|\det(I-\widetilde{P}_\gamma)|} = \frac{\det(I-P_{\gamma,u})\det(I-P^{-1}_{\gamma,\perp})|\det(P_{\gamma,u})|^{-1/2}}{|\det(I-P_\gamma)||\det(I-P_{\gamma,\perp})||\det(P_{\gamma,\perp})|^{-1}}. \tag{8.3.4}$$

Since P_γ is a linear symplectic map, we have

$$\det(I-P^{-1}_{\gamma,s}) = \det(I-P_{\gamma,u}),\ \det(P_{\gamma,s}) = \det(P^{-1}_{\gamma,u}),$$

and one deduces

$$\begin{aligned}|\det(I-P_\gamma)| &= |\det(I-P_{\gamma,u})||\det(I-P_{\gamma,s})|\\ &= |\det(P_{\gamma,s})||\det(I-P_{\gamma,u})||\det(I-P^{-1}_{\gamma,s})|\\ &= |\det(P_{\gamma,u})|^{-1}|\det(I-P_{\gamma,u})|^2\end{aligned}$$

For $t > 0$ the map $\mathrm{d}\widetilde{\varphi}_t = (\mathrm{d}\widetilde{\varphi}_{-t})^{-1}$ is contracting on $\ker \mathrm{d}\pi_G \subset \tilde{E}_s(\omega_{\tilde{\gamma}})$ (resp. $\mathrm{d}\varphi_{-t}$ is contracting on $E_u(z)$) and these contractions yield $\det(I - P_{\gamma,\perp}^{-1}) > 0$ (resp. $\det(I - P_{\gamma,u}) > 0$). Thus the terms involving $P_{\gamma,\perp}$ in (8.3.4) cancel and since

$$|\det(I - P_\gamma)|^{-1/2} = |\det(P_{\gamma,u})|^{1/2} \det(I - P_{\gamma,u})^{-1},$$

the right hand side of (8.3.4) is equal to $|\det(I - P_\gamma)|^{-1/2}$. □

8.3.3 Meromorphic continuation of η_{N}

From Lemma 8.3.1 and (8.3.1), we deduce that for $\mathrm{Re}(s) \gg 1$, we have

$$\eta_{\mathrm{N}}(s) = \sum_{k=0}^{d-1} \sum_{\ell=0}^{d^2-d} (-1)^{k+\ell} \operatorname{tr}^\flat \mathbf{R}_\varepsilon^{k,\ell}(s),$$

where $\eta_{\mathrm{N}}(s)$ is defined by

$$\eta_{\mathrm{N}}(s) = \sum_\gamma \frac{\tau^\sharp(\gamma) \mathrm{e}^{-\tau(\gamma)s}}{|\det(1 - P_\gamma)|^{1/2}}.$$

Since for every k, ℓ the family $s \mapsto \mathbf{R}_\varepsilon^{k,\ell}(s)$ extends to a meromorphic family on the whole complex plane, so does $s \mapsto \eta_{\mathrm{N}}(s)$. Indeed, it follows from the proof of [DG16, Lemma 4.2] that $s \mapsto K_{\mathbf{R}_\varepsilon^{k,\ell}(s)}$ is continuous as a map [1]

$$\mathbb{C} \setminus \mathrm{Res}(\mathbf{R}_\varepsilon^{k,\ell}) \to \mathcal{D}'_\Gamma(G \times G, \mathcal{E}_{k,\ell} \boxtimes \mathcal{E}_{k,\ell}^*).$$

Here $K_{\mathbf{R}_\varepsilon^{k,\ell}(s)}$ is the Schwartz kernel of $\mathbf{R}_\varepsilon^{k,\ell}(s)$ and

$$\Gamma = \Delta_\varepsilon \cup \Upsilon_{+,\varepsilon} \cup E_+^* \times E_-^*,$$

where $\Delta_\varepsilon = \{(\Psi_\varepsilon(\Omega), \Omega) : (\Omega, \Omega) \in \Delta\}$ and

$$\Upsilon_{+,\varepsilon} = \{(\Psi_t(\Omega), \Omega) : t \geqslant \varepsilon,\ \langle \Omega, X \rangle = 0\},$$

while $\mathcal{D}'_\Gamma(G \times G, \mathcal{E}_{k,\ell} \boxtimes \mathcal{E}_{k,\ell}^*)$ is the space of distributions valued in $\mathcal{E}_{k,\ell} \boxtimes \mathcal{E}_{k,\ell}^*$ whose wavefront set is contained in Γ. This space is endowed with its usual topology (see [Hör90, §8.2]). In particular, $s \mapsto \operatorname{tr}^\flat \mathbf{R}_\varepsilon^{k,\ell}(s)$ is continuous on $\mathbb{C} \setminus \mathrm{Res}(\mathbf{R}_\varepsilon^{k,\ell})$ by [Hör90, Theorem 8.2.4]. Finally, Cauchy's formula implies that this map is meromorphic on $\mathbb{C}$ and this completes the proof that the Dirichlet series $s \mapsto \eta_{\mathrm{N}}(s)$ admits a meromorphic continuation in $\mathbb{C}$. Finally, by proceeding exactly as in [DG16, §5], one is able to show that η_{N} has integer residues.

1. This follows from the fact that the estimates on the wavefront set of $\mathbf{R}_\varepsilon^{k,\ell}(s)$ given in [DG16] are locally uniform with respect to $s \in \mathbb{C}$.

8.4 The dynamical zeta function for particular rays

In this section we adapt the above construction to prove the following result.

Theorem 8.4.1. *Let $q \in \mathbb{N}_{\geqslant 1}$. The function $\eta_q(s)$ defined by*

$$\eta_q(s) = \sum_{m(\gamma)\in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|1-P_\gamma|^{1/2}}, \quad \mathrm{Re}(s) \gg 1,$$

where the sum runs over all periodic rays γ with $m(\gamma) \in q\mathbb{N}$, admits a meromorphic continuation to the whole complex plane with simple poles and residues valued in $\mathbb{Z}/q$.

Note that for large $\mathrm{Re}(s)$ we have the formula

$$\eta_\mathrm{D}(s) = 2\eta_2(s) - \eta_\mathrm{N}(s). \tag{8.4.1}$$

In particular, Theorem 8.4.1 implies that $\eta_\mathrm{D}(s)$ also extends meromorphically to the whole complex plane, since $\eta_\mathrm{N}(s)$ does by the preceding section. In particular, we obtain Theorem 8.1.1 since $2\eta_2(s)$ has simple poles with residues in $\mathbb{Z}$.

8.4.1 The q-reflection bundle

For $q \geqslant 2$ we define the *q-reflection bundle* $\mathcal{R}_q \to M$ by

$$\mathcal{R}_q = \left(\left[S\mathbb{R}^d \setminus \left(\pi^{-1}(D^\circ) \cup \mathcal{D}_\mathrm{g}\right)\right] \times \mathbb{R}^q\right) / \approx, \tag{8.4.2}$$

where the equivalence classes of the relation $\approx$ are defined as follows. For $(x, v) \in S\mathbb{R}^d \setminus (\pi^{-1}(D^\circ) \cup \mathcal{D}_\mathrm{g})$ and $\xi \in \mathbb{R}^q$, we set $[(x, v, \xi)] = \{(x, v, \xi)\}$ if $x \in \mathbb{R}^d \setminus D$ and

$$[(x, v, \xi)] = \{(x, v, \xi), (x, v', A(q)\cdot\xi)\} \qquad \text{if } (x, v) \in \mathcal{D}_\mathrm{in},\ (x, v') \in \mathcal{D}_\mathrm{out},$$

where $A(q)$ is the $q \times q$ matrix with entries in $\{0, 1\}$ given by

$$A(q) = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

This indeed defines an equivalence relation since $(x, v') \in \mathcal{D}_\mathrm{out}$ whenever $(x, v) \in \mathcal{D}_\mathrm{in}$. Note that

$$A(q)^q = \mathrm{Id}, \quad \mathrm{tr}\, A(q)^j = 0, \quad j = 1, \dots, q-1. \tag{8.4.3}$$

Let us describe the smooth structure of $\mathcal{R}_q$, using the charts of M and the notations of §7.1.2. For $z_\star \in \mathcal{D}_\mathrm{in}$, we denote by $U_{z_\star}$ the image of $\psi_{z_\star}$. Then the bundle $\mathcal{R}_q \to M$ can be defined by defining its transition maps, as follows. Let $U = \psi(B \setminus \pi^{-1}(\partial D))$ be the chart domain of ψ. In the coordinates (7.1.2), we have $U_{z_\star} \cap U = U_+ \sqcup U_-$ where

$$U_+ =]0, \varepsilon[\times B^{2d-2}(0, \delta) \quad \text{and} \quad U_- =]-\varepsilon, 0[\times B^{2d-2}(0, \delta).$$

Then we define the transition map $\alpha_{z_\star} : U_{z_\star} \cap U \to \mathrm{GL}(\mathbb{R}^q)$ of the bundle $\mathcal{R}_q$ with respect to the pair of charts $(\psi_{z_\star}, \psi)$ to be the locally constant map defined by

$$\alpha_{z_\star}(z) = \begin{cases} \mathrm{Id} & \text{if } z \in U_-, \\ A(q) & \text{if } z \in U_+. \end{cases}$$

For $z_\star, z'_\star \in \mathcal{D}_{\mathrm{in}}$, the transition map of $\mathcal{R}_q$ for the pair of charts $(\psi_{z_\star}, \psi_{z'_\star})$ is declared to be constant and equal to Id on $U_{z_\star} \cap U_{z'_\star}$. In this way we obtain a smooth bundle $\mathcal{R}_q$ over M, which is clearly homeomorphic to the quotient space (8.4.2). Since the transition maps of $\mathcal{R}_q$ are locally constant, there is a natural flat connexion d^q on $\mathcal{R}_q$ which is given in the charts by the trivial connexion on $\mathbb{R}^q$.

Consider a small smooth neighborhood V of K. As in §7.1.3, we embed V into a smooth compact manifold without boundary N, and we fix an extension of $\mathcal{R}_q$ to N (this is always possible if we choose N to be the doubling manifold of V). Consider any connexion ∇^q on the extension of $\mathcal{R}_q$ which coincides with d^q near K, and denote by

$$P_{q,t}(z) : \mathcal{R}_q(z) \to \mathcal{R}_q(\varphi_t(z))$$

the parallel transport of ∇^q along the curve $\{\varphi_\tau(z) \ : \ 0 \leqslant \tau \leqslant t\}$. We have a smooth action of φ_t on $\mathcal{R}_q$ which is given by

$$\varphi_t^q(z,\xi) = (\varphi_t(z), P_{q,t}(z)\cdot\xi), \quad (z,\xi) \in \mathcal{R}_q.$$

From (8.4.3), and the fact that ∇^q coincides with d^q near K, we easily deduce that for any periodic orbit $\gamma = (\varphi_\tau(z))_{\tau\in[0,\tau(\gamma)]}$, we have

$$\operatorname{tr} \varphi^q_{\tau(\gamma)}(z) = \begin{cases} q & \text{if} \quad m(\gamma) = 0 \mod q, \\ 0 & \text{if} \quad m(\gamma) \neq 0 \mod q. \end{cases} \tag{8.4.4}$$

8.4.2 Transfer operators acting on G

Now consider the bundle

$$\mathcal{E}^q_{k,\ell} = \mathcal{E}_{k,\ell} \otimes \pi_G^* \mathcal{R}_q,$$

where $\pi_G^*\mathcal{R}_q$ is the pullback of $\mathcal{R}_q$ by π_G and $\mathcal{E}_{k,\ell}$ is defined in §8.2.2, so that $\pi_G^*\mathcal{R}_q \to G$ is a vector bundle over G. We may lift the flow φ_t^q to a flow $\Phi_t^{k,\ell,q}$ on $\mathcal{E}^q_{k,\ell}$ which is defined locally near $\widetilde{K}_u$ by

$$\begin{aligned}\Phi_t^{k,\ell,q}&(\omega, u\otimes v\otimes\xi)\\ &= \left(\widetilde{\varphi}_t(\omega),\ b_t(\omega)\cdot\left[\left(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\right)^{\wedge k}(u)\otimes(\mathrm{d}\widetilde{\varphi}_t(\omega))^{\wedge\ell}(v)\otimes P_{q,t}(z)\cdot\xi\right]\right)\end{aligned}$$

for any $\omega = (z,E) \in G$, $u\otimes v\otimes\xi \in \mathcal{E}^q_{k,\ell}(\omega)$ and $t \in \mathbb{R}$. Here $b_t(\omega)$ is defined in 8.2.2. As in §8.2.4, we consider a smooth connexion $\nabla^{k,\ell,q} = \nabla^{k,\ell} \otimes \pi_G^*\nabla^q$ on $\mathcal{E}^q_{k,\ell}$. Define the transfer operator

$$\Phi_{-t}^{k,\ell,q,*} : C^\infty(G, \mathcal{E}^q_{k,l}) \to C^\infty(G, \mathcal{E}^q_{k,\ell})$$

by

$$\Phi_{-t}^{k,\ell,q,*}\mathbf{u}(\omega) = \Phi_{-t}^{k,\ell,q}[\mathbf{u}(\tilde{\varphi}_t(\omega))], \ \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}^q).$$

Then the operator

$$\mathbf{P}_{k,\ell,q} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\Phi_{-t}^{k,\ell,q,*}\right),$$

which is defined near $\widetilde{K}_u$, can be written locally as $\nabla_{\widetilde{X}}^{k,\ell,q} + \mathbf{A}_{k,\ell,q}$ for some $\mathbf{A}_{k,\ell,q} \in C^\infty(\widetilde{U}_u, \mathrm{End}\,\mathcal{E}_{k,\ell}^q)$ which is defined in some small neighborhood $\widetilde{U}_u$ of $\widetilde{K}_u$. Next, we choose some $\mathrm{B}_{k,\ell,q} \in C^\infty(G, \mathrm{End}\,\mathcal{E}_{k,\ell}^q)$ which coincides $\mathbf{A}_{k,\ell,q}$ near $\widetilde{K}_u$. We consider $\widetilde{V}_u$ and $\widetilde{Y}$ as in §8.2.3, and set

$$\mathbf{Q}_{k,\ell,q} = \nabla_{\widetilde{Y}}^{k,\ell,q} + \mathrm{B}_{k,\ell,q} : C^\infty(G, \mathcal{E}_{k,\ell}^q) \to C^\infty(G, \mathcal{E}_{k,\ell}^q).$$

8.4.3 Meromorphic continuation of $\eta_q(s)$

For $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ such that $\widetilde{\chi} \equiv 1$ near $\widetilde{K}_u$, we define

$$\mathbf{R}_\varepsilon^{k,\ell,q}(s) = \widetilde{\chi}\mathrm{e}^{-\varepsilon(\mathbf{Q}_{k,\ell,q}+s)}(\mathbf{Q}_{k,\ell,q} + s)^{-1}\widetilde{\chi}$$

and by the argument of the preceding section one obtains a meromorphic continuation of $\mathbf{R}_\varepsilon^{k,\ell,q}(s)$. Now notice that, with the notations of §8.3.2, for any periodic orbit γ of (φ_t) we have

$$\mathrm{tr}\,\Phi_{\tau(\gamma)}^{k,\ell,q}(\omega_{\widetilde{\gamma}}) = \mathrm{tr}\left(\Phi_{\tau(\gamma)}^{k,\ell}(\omega_{\widetilde{\gamma}})\right)\mathrm{tr}\left(\varphi_{\tau(\gamma)}^q(z_\gamma)\right),$$

where z_γ is any point of γ. However, by (8.4.4) one gets $\mathrm{tr}\,\varphi_{\tau(\gamma)}^q(z_\gamma) = \mathbf{1}_{q\mathbb{N}}(m(\gamma))$. In particular, proceeding exactly as in the the preceding section, we obtain that for $\mathrm{Re}(s)$ large enough,

$$\sum_{k,\ell}(-1)^{k+\ell}\,\mathrm{tr}^\flat\,\mathbf{R}_\varepsilon^{k,\ell,q}(s) = q\sum_{m(\gamma)\in q\mathbb{N}}\frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(1-P_\gamma)|^{1/2}}. \tag{8.4.5}$$

Thus, repeating the argument of §8.3, we establish a meromorphic continuation of the function $s \mapsto \eta_q(s)$. Finally, by using (8.4.5), we may proceed exactly as in §8.3.3 to show that $q\eta_q$ has integer residues. This completes the proof of Theorem 8.4.1.

8.5 The modified Lax–Phillips conjecture for real analytic obstacles

In this section, we assume that the obstacles $D_1, \ldots, D_r$ have real analytic boundary. Then the smooth structure on M defined in §7.1.2 induces an analytic structure on M. Indeed, with notations of §7.1.2, the local parameterizations $F_{z_\star}$ of $\mathcal{D}_{\mathrm{in}}$ can be chosen to be real analytic, as $\mathcal{D}_{\mathrm{in}}$ is a real analytic submanifold of $S\mathbb{R}^2$. This makes the transition maps (7.1.3) real analytic, and thus we obtain a real analytic structure on M. In the charts $\psi_{z_\star}$ and ψ, the billiard flow is a translation and it defines a real analytic flow. Of course, the Grassmannian bundle $G \to M$ also becomes real analytic. Consequently, the lifted flow $\widetilde{\varphi}_t$ on G, which is defined by (8.2.1), is real analytic as well.

Consider the bundles $\mathcal{E}^q_{k,\ell} \to G$ defined in §8.4.2 for $q \in \mathbb{N}_{\geqslant 2}$, $1 \leqslant k \leqslant d-1$ and $1 \leqslant \ell \leqslant d^2 - d$. If $q = 1$ the bundles $\mathcal{E}^1_{k,\ell} \to G$ are isomorphic to $\mathcal{E}_{k,\ell}$, $\mathcal{E}_{k,\ell}$ being the bundles defined in §8.3. As before we naturally extend the flow $\widetilde{\varphi}_t$ to a flow $\Phi_t^{k,\ell,q}$ (which is non complete) on $\mathcal{E}^q_{k,\ell}$. We set

$$\mathcal{E}^+_q = \bigoplus_{k+\ell \text{ even}} \mathcal{E}^q_{k,\ell} \quad \text{and} \quad \mathcal{E}^-_q = \bigoplus_{k+\ell \text{ odd}} \mathcal{E}^q_{k,\ell}.$$

Define the flows $\Phi^+_{t,q}$ and $\Phi^-_{t,q}$, acting respectively on the bundles $\mathcal{E}^+_q$ and $\mathcal{E}^-_q$, by

$$\Phi^+_{t,q} = \bigoplus_{k+\ell \text{ even}} \Phi_t^{k,\ell,q} \quad \text{and} \quad \Phi^-_{t,q} = \bigoplus_{k+\ell \text{ odd}} \Phi_t^{k,\ell,q}.$$

Then $\Phi^\pm_{t,q}$ is a virtual lift of $\widetilde{\varphi}_t$ to the virtual bundles $\mathcal{E}^\pm_q$, in the sense of [Fri95, p. 176]. Also, following [Fri95, p. 176], given a periodic ray γ, one defines $\chi_\gamma(\mathcal{E}^\pm_q) = \chi_\gamma(\mathcal{E}^+_q) - \chi_\gamma(\mathcal{E}^-_q)$. More precisely, given a point $\omega = (z, E) \in G$, $z \in \gamma$, and a bundle $\xi \to G$ over G, one considers the transformation $\Phi_{\tau(\gamma)} : \xi_\omega \to \xi_\omega$, where ξ_ω is the fibre over ω and Φ_t is the lift of the flow $\tilde{\varphi}_t$ to ξ. Then we set $\chi_\gamma(\xi) = \operatorname{tr} \Phi_{\tau(\gamma)}$. For a period ray γ related to a primitive periodic ray $\gamma^\sharp$ one defines $\mu(\gamma) \in \mathbb{N}$ determined by the equality $\tau(\gamma) = \mu(\gamma)\tau(\gamma^\sharp)$.

After this preparation one introduces the zeta function

$$\zeta_q(s) = \exp\Big(-\frac{1}{q}\sum_{\tilde{\gamma}} \frac{\chi_\gamma(\mathcal{E}^\pm_q)}{\mu(\gamma)|\det(1 - \tilde{P}_\gamma)|} \mathrm{e}^{-s\tau(\gamma)}\Big), \quad \mathrm{Re}(s) \gg 1.$$

This function corresponds exactly to the *flat-trace function* $T^\flat(s)$ introduced by Fried [Fri95, p. 177]. On the other hand, one has

$$\chi_\gamma(\mathcal{E}^\pm_q) = \sum_{k,\ell} (-1)^{k+\ell} \operatorname{tr} \Phi^{k,\ell,q}_{\tau(\gamma)}(\omega_{\tilde{\gamma}}).$$

According to the analysis of §8.3 for the function $\zeta_{\mathrm{N}}(s)$, one deduces that

$$\frac{\mathrm{d}}{\mathrm{d}s} \log \zeta_1(s) = \sum_\gamma \frac{\tau(\gamma^\sharp) e^{-s\tau(\gamma)}}{|\det(I - P_\gamma)|^{1/2}} = \eta_{\mathrm{N}}(s), \quad \mathrm{Re}\, s \gg 1.$$

Similarly, the argument of §8.4 implies

$$\frac{\mathrm{d}}{\mathrm{d}s} \log \zeta_2(s)^2 = 2 \sum_{\substack{\gamma \\ m(\gamma) \in 2\mathbb{N}}} \frac{\tau(\gamma^\sharp) e^{-s\tau(\gamma)}}{|\det(I - P_\gamma)|^{1/2}} = 2\eta_2(s), \quad \mathrm{Re}\, s \gg 1.$$

Consequently, the representation (8.4.1) yields

$$\eta_{\mathrm{D}}(s) = -\frac{\mathrm{d}}{\mathrm{d}s} \log\Big(\frac{\zeta_2(s)^2}{\zeta_1(s)}\Big), \quad \mathrm{Re}\, s \gg 1. \tag{8.5.1}$$

For obstacles with real analytic boundary the flow $\widetilde{\varphi}_t$ is real analytic and the bundles $\mathcal{E}_q^{\pm}$ are real analytic, too. Thus we are in position to apply the principal result of Fried [Fri95, Theorem p. 180] (see also pp. 177–178) saying that the zeta functions $s \mapsto \zeta_k(s)$, $k = 1, 2$, are meromorphic in $\mathbb{C}$ with finite order $\rho(\zeta_k)$. Thus ζ_2^2/ζ_1 is a meromorphic function with order $\max\{\rho(\zeta_1), \rho(\zeta_2^2)\}$.

Proof of Theorem 8.1.3. Denote by $\{\mu_j\} \subset \mathbb{C}$ the set of resonances for the wave equation in the domain $\mathbb{R}^d \setminus D$, with Dirichlet boundary conditions. Our purpose is to to prove that there is $\delta > 0$ such that

$$\sharp\{\mu_j \ : \ |\operatorname{Im}\mu_j| \leqslant \delta\} = \infty.$$

By the work of Ikawa [Ika88b, Ika90a] and a slight modification of its proof to cover the case d even [2], it is sufficient to show that the Dirichlet series $\eta_{\mathrm{D}}(s)$ cannot be continued as an entire function on $\mathbb{C}$, that is, $\eta_{\mathrm{D}}(s)$ has at least one pole. We proceed by contradiction and assume that $\eta_{\mathrm{D}}(s)$ is an entire function. Applying the representation (8.5.1), this means that $\zeta_2(s)^2/\zeta_1(s)$ has neither poles nor zeros. As we have mentioned above, this function has finite order, so by the Hadamard factorisation theorem we deduce that $\zeta_2(s)^2/\zeta_1(s) = \exp(Q(s))$ for some polynomial $Q(s)$. This implies that $\eta_{\mathrm{D}}(s) = -Q'(s)$ is a polynomial, which is impossible. Indeed, since $\eta_{\mathrm{D}}(s) \to 0$ as $\operatorname{Re}(s) \to +\infty$, this implies that $Q'(s)$ must be the zero polynomial. By uniqueness of the development of an absolutely convergent Dirichlet series of the form $\sum_n a_n e^{-\lambda_n s}$ [Per08], this leads to a contradiction. □

8.6 Hyperbolicity of the billiard flow

In this section we show that the non-grazing flow (ϕ_t) defined in §7.1.1 is uniformly hyperbolic on the trapped set K_e. As it was mentioned in §7.1.3, we can obtain the uniform hyperbolicity of the flow (φ_t) on K in the smooth model from that for (ϕ_t) on K_e. The flow (ϕ_t) is hyperbolic on K_e if for every $z = (x, v) \in \mathring{B} \cap K_e$ we have a splitting

$$T_z\mathbb{R}^d = \mathbb{R}X(z) \oplus E_s(z) \oplus E_u(z),$$

where $X(z) = v$ and $E_s(z)/E_u(z)$ are stable/unstable spaces such that $\mathrm{d}\phi_t(z)$ maps $E_{s/u}(z)$ onto $E_{s/u}(\phi_t(z))$ whenever $\phi_t(z) \in \mathring{B} \cap K_e$, and if for some constants $C, \nu > 0$ independent of $z \in K_e$, we have

$$\|\mathrm{d}\phi_t(z) \cdot v\| \leqslant \begin{cases} Ce^{-\nu t}\|v\|, & v \in E_s(z),\ t \geqslant 0, \\ Ce^{-\nu|t|}\|v\|, & v \in E_u(z),\ t \leqslant 0. \end{cases} \tag{8.6.1}$$

First, we consider the case of periodic points. Our purpose is to define the unstable and stable manifolds $E_u(z)$ and $E_s(z)$ at a periodic point z, and to estimate the norm of $\mathrm{d}\phi_t(z)|_{E_b(z)}$ for $b = u, s$. Consider a periodic ray γ with reflection points $z_i = (q_i, \omega_i)$, $q_i \in \partial D$, $\omega_i \in S^{d-1}$, $i = 0, \ldots, m(\gamma) = m$. We will apply the representation of the Poincaré map established in Theorem 2.3.1 and Proposition 2.3.2 in [PS17].

2. for d even one applies the trace formula provided by Zworski [Zwo98] and one repeats the argument of [Ika90a].

To do this, we recall some notations given in Section 2 of [PS17]. Let $\Pi_i \subset \mathbb{R}^d$ be the plane passing thought q_i and orthogonal to the line q_iq_{i+1} and let Π'_i be the plan passing thought q_i and orthogonal to ω_{i-1}. For $j = i \,(\mathrm{mod}\, m)$ we set $\Pi_j = \Pi_i$, $q_j = q_i$. Set $\lambda_i = \|q_{i-1} - q_i\|$ and let σ_i be the symmetry with respect to the tangent plane $\alpha_i = T_{q_i}\partial D$. Clearly,

$$\sigma_i(\omega_i) = \omega_{i+1}, \quad \sigma_i(\Pi'_i) = \Pi_i, \quad \Pi_0 = \Pi_m.$$

We identify the plans Π_{i-1} and Π'_i by using a translation along the line determined by the segment $[q_{i-1}, q_i]$ and we will write $\sigma_i(\Pi_{i-1}) = \Pi_i$.

We may identify $\Pi_i \times \Pi_i$ with $\Sigma_{z_i} = T_{z_i}(T\mathbb{R}^d)/E_{z_i}$, where E_{z_i} is the two-dimensional space spanned by ω_i and the cone axis at z_i. We will denote $\overline{\mathcal{D}}_{\mathrm{in}} = \{(x, v) \,:\, x \in \partial D, |v| = 1, \langle v, n(x)\rangle \geqslant 0\}$. Then define the *billiard ball map*

$$\mathrm{B} : \overline{\mathcal{D}}_{\mathrm{in}} \ni (x, v) \longmapsto (y, R_y w) \in \overline{\mathcal{D}}_{\mathrm{in}},$$

where $R_y : S_y\mathbb{R}^d \to S_y\mathbb{R}^d$ is the reflexion with respect to $T_y\partial D$ and

$$(y, w) = \phi_{\tau_+(x,v)}(x, v)$$

where

$$\tau_+(x, v) = \inf\{t > 0 \;:\; \pi(\phi_t(x, v)) \in \partial D\}.$$

This map is well defined near $K_e \cap \mathcal{D}_{\mathrm{in}}$. Given $(u, v) \in \Pi_{i-1} \times \Pi_{i-1}$ sufficiently close to $(0, 0)$, consider the line $\ell(u, v)$ passing through u and having direction $\omega_{i-1} + v$ (the point v is identified with the vector v). Then $\ell(u, v)$ intersects ∂D at a point $p = p(u, v)$ close to q_i. Let $\ell'(u, v)$ be the line symmetric to $\ell(u, v)$ with respect to the tangent plane to ∂D at p and let $u' \in \Pi_i$ be the intersection point of $\ell'(u, v)$ with Π_i. There exists a unique $v' \in \Pi_i$ for which $\omega_i + v'$ has the direction of $\ell'(u, v)$. Thus we get a map

$$\Psi_i : \Pi_{i-1} \times \Pi_{i-1} \ni (u, v) \longmapsto (u', v') \in \Pi_i \times \Pi_i$$

defined for (u, v) in a small neighborhood of $(0, 0)$ (see Figure 8.6). The smoothness of the billiard ball map implies the smoothness of Ψ_i. Next consider the second fundamental form $S(\xi, \eta) = \langle G_i(\xi), \eta\rangle$ for D at q_i, where

$$G_i = \mathrm{d}n_j(q_i) : \alpha_i \longrightarrow \alpha_i$$

is the Gauss map. Introduce a symmetric linear map $\widetilde{\xi}_i$ on Π_i defined by for $\xi, \eta \in \Pi'_i$ by

$$\langle \widetilde{\xi}_i \sigma_i(\xi), \sigma_i(\eta)\rangle = -2\langle \omega_{i-1}, n_j(q_i)\rangle \langle G_i(\pi_i(\xi)), \pi_i(\eta)\rangle,$$

where $\langle ., .\rangle$ denotes the scalar product in $\mathbb{R}^d$ and $\pi_i : \Pi'_i \longrightarrow \alpha_i$ be the projection on α_i along $\mathbb{R}\omega_{i-1}$.

Notice that the non-eclipse condition (8.1.1) implies that there exists $\beta_0 \in \,]0, \pi/2[$ depending only of D such that for all incoming directions ω_{i-1} and all reflexion points $q_i \in \partial D_j$, it holds

$$-\langle \omega_{i-1}, n_j(q_i)\rangle = \langle \omega_i, n_j(q_i)\rangle \geqslant \cos\beta_0 > 0.$$

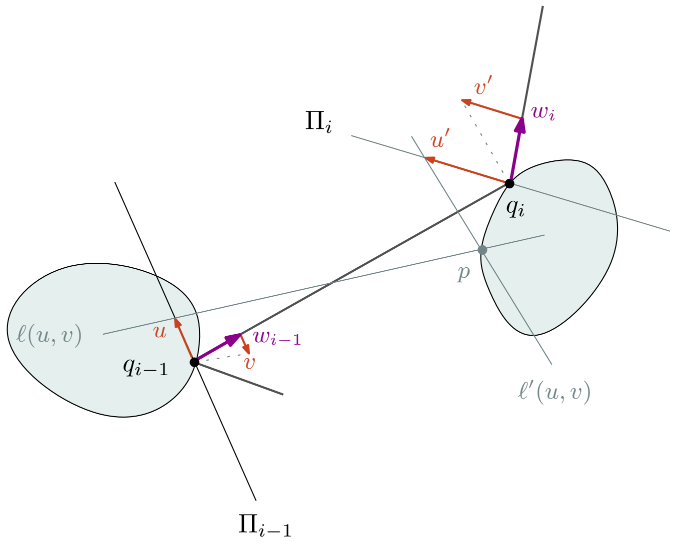

FIGURE 8.1 – The map $\Psi_i : (u, v) \mapsto (u', v')$

Consequently, the symmetric map $\widetilde{\xi}_i$ has spectrum included in $[\mu_1, \mu_2]$ with $0 < \mu_1 < \mu_2$ depending only of $\kappa = \cos\beta_0$ and the sectional curvatures of ∂D. Finally, define the symmetric map

$$\xi_i = s_i^{-1}\widetilde{\xi}_i s_i : \Pi_m \longrightarrow \Pi_m$$

with $s_i = \sigma_i \circ \sigma_{i-1} \circ \cdots \circ \sigma_1$. By Theorem 2.3.1 in [PS17], the map $\mathrm{d}\Psi_i(0,0)$ has the form

$$\mathrm{d}\Psi_i(0,0) = \begin{pmatrix} I & \lambda_i I \\ \tilde{\xi}_i & I + \lambda_i \tilde{\xi}_i \end{pmatrix} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix},$$

and the linearized Poincaré map P_γ related to γ is given by

$$P_\gamma = \mathrm{d}(\Psi_m \circ \cdots \circ \Psi_1)(0,0) : \Pi_0 \times \Pi_0 \longrightarrow \Pi_0 \times \Pi_0,$$

which reads

$$P_\gamma = \begin{pmatrix} s_m & 0 \\ 0 & s_m \end{pmatrix} \begin{pmatrix} I & \lambda_m I \\ \xi_m & I + \lambda_m \xi_m \end{pmatrix} \cdots \begin{pmatrix} I & \lambda_1 I \\ \xi_1 & I + \lambda_1 \xi_1 \end{pmatrix}.$$

Next, we repeat without changes the argument of Proposition 2.3.2 in [PS17]. For $k = 0, 1, \ldots, m$, consider the space S_k^+ of linear symmetric non-negative definite maps $M : \Pi_k \longrightarrow \Pi_k$. Next, let $\mathrm{S}_k^+(\varepsilon) \subset \mathrm{S}_k^+$ be the space of maps such that $M \geqslant \varepsilon I$ with $\varepsilon > 0$. To study the spectrum of P_γ, consider the subspace

$$L_0 = \{(u, M_0 u) : u \in \Pi_0\}, \quad M_0 \in \mathrm{S}_0^+,$$

which is Lagrangian with respect to the natural symplectic structure on $\Pi_0 \times \Pi_0$. By the action of the map $\mathrm{d}\Psi_1(0,0)$, the space L_0 is transformed into

$$L_1 = \left\{ \big(\sigma_1(I + \lambda_1 M_0)u,\ \sigma_1((I + \lambda_1 \xi_1)M_0 + \xi_1)u\big) \ :\ u \in \Pi_0 \right\} \subset \Pi_1 \times \Pi_1.$$

Introduce the operator

$$\mathrm{A}_i : \mathrm{S}_{i-1}^+ \longrightarrow \mathrm{S}_i^+$$

defined by
$$\mathrm{A}_i(M) = \sigma_i M(I + \lambda_i M)^{-1}\sigma_i^{-1} + \tilde{\xi}_i.$$
Therefore we may write $L_1 = \{(u, M_1 u) : u \in \Pi_1\}$ with $M_1 = \mathrm{A}_1(M_0)$. By recurrence, one defines

$$L_k = \{(u, M_k u) : \ u \in \Pi_k\}, \quad M_k = \mathrm{A}_k(M_{k-1}), \quad k = 1, 2, \ldots, m.$$

The maps A_k are contractions from $\mathrm{S}_{k-1}^+(\varepsilon)$ to $\mathrm{S}_k^+(\varepsilon)$, and hence

$$\mathrm{A} = \mathrm{A}_m \circ \cdots \circ \mathrm{A}_1$$

is also a contraction from $\mathrm{S}_0^+(\varepsilon)$ to $\mathrm{S}_0^+(\varepsilon)$. We choose $M_0 \in \mathrm{S}_0^+(\varepsilon)$ as a fixed point of A and notice that $\varepsilon > 0$ can be chosen uniformly for all periodic rays. Thus we deduce
$$P_\gamma \begin{pmatrix} u \\ M_0 u \end{pmatrix} = \begin{pmatrix} Su \\ M_0 Su \end{pmatrix}$$
with a map $S : \Pi_0 \longrightarrow \Pi_0$ having the form

$$S = \sigma_m(I + \lambda_m \mathrm{A}'_{m-1}(M_0)) \circ \sigma_{m-1}(I + \lambda_{m-1}\mathrm{A}'_{m-2}(M_0)) \circ \cdots \circ \sigma_1(I + \lambda_1 M_0),$$

where $\mathrm{A}'_k = \mathrm{A}_k \circ \mathrm{A}_{k-1} \circ \cdots \circ \mathrm{A}_1$. Setting

$$d_0 = \min_{i \neq j} \mathrm{dist}\,(D_i, D_j) > 0, \quad d_1 = \max_{i \neq j} \mathrm{dist}\,(D_i, D_j),$$

and $\beta = \log(1 + \varepsilon d_0)$, one obtains

$$\|Su\| \geqslant (1 + d_0\varepsilon)^m \|u\| = e^{\beta m}\|u\|.$$

Obviously, the eigenvalues of S are eigenvalues of P_γ and we conclude that P_γ has $(d-1)$ eigenvalues $\nu_1, \ldots, \nu_{d-1}$ satisfying

$$|\nu_j| \geqslant e^{\beta m}, \quad j = 1, \ldots, d-1.$$

For $0 < \tau < \lambda_1$, consider a point $\rho = \phi_\tau(z) \in \mathring{B} \cap \gamma$, where $z = (x, v) \in \mathcal{D}_{\mathrm{in}}$. The map $\phi_\tau : \mathcal{D}_{\mathrm{in}} \to \mathring{B}$ is smooth near z and moreover $\mathrm{d}\phi_\tau(z) : \Sigma_z \to \Sigma_{\phi_\tau(z)}$. We identify $\Pi_0 \times \Pi_0$ with Σ_z and $\Sigma_{\phi_\tau(z)}$ with the image

$$\mathrm{d}\phi_\tau(z)\Sigma_z = \begin{pmatrix} I & \tau I \\ 0 & I \end{pmatrix} (\Pi_0 \times \Pi_0)$$

Next we define the unstable subspace of $\Sigma_{\phi_\tau(z)}$ as

$$E_u(\phi_\tau(z)) = \mathrm{d}\phi_\tau(z)(L_0) = \begin{pmatrix} I & \tau I \\ 0 & I \end{pmatrix} (L_0).$$

Let $0 < \tau < \lambda_1$, $0 < \sigma < \lambda_{p+1}$ and $p \geqslant 1$, and set $t = -\tau + \sum_{j=1}^{p} \lambda_j + \sigma$. Then ϕ_t is smooth near ρ, and we have

$$\mathrm{d}\phi_t(\rho)|_{\Sigma_\rho} = \mathrm{d}\phi_\sigma(\mathrm{B}^p(z)) \circ \mathrm{dB}^p(z) \circ \mathrm{d}\phi_\tau(z)^{-1} : \Sigma_\rho \to \Sigma_{\phi_t(\rho)}.$$

Thus we have the diagram

$$\begin{array}{ccccccc} & & E_u(\rho) & \xrightarrow{\mathrm{d}\phi_t(\rho)} & E_u(\phi_t(\rho)) & & \\ & & \Big\downarrow {\scriptstyle \mathrm{d}\phi_{-\tau}(\rho)} & & \Big\uparrow {\scriptstyle \mathrm{d}\phi_\sigma(\mathrm{B}^p(z))} & & \\ \Pi_0 & \xrightarrow{\chi_0} & L_0 & \xrightarrow{\mathrm{dB}^p(z)} & L_p & \xleftarrow{\chi_p} & \Pi_p, \end{array}$$

where $\chi_0 : \Pi_0 \ni u \mapsto (u, M_0u) \in L_0 \subset \Pi_0 \times \Pi_0$ and $\chi_p : \Pi_p \ni u \mapsto (u, M_pu) \in L_p \subset \Pi_p \times \Pi_p$. It is easy to obtain an estimate of the action of $\mathrm{d}\phi_t(\rho)|_{E_u(\rho)}$ for $\rho = \phi_\tau(z),\ v = \mathrm{d}\phi_\tau(z)(u, M_0u) \in E_u(\rho)$. Clearly,

$$\mathrm{d}\phi_t(\rho) \cdot v = (\mathrm{d}\phi_\sigma(\mathrm{B}^p(z)) \circ \mathrm{dB}^p(z))(u, M_0u).$$

By the above argument we deduce

$$\mathrm{dB}^p(z)(u, M_0u) = (S_pu, M_pS_pu) \in L_p$$

with

$$S_p = \sigma_p(I + \lambda_p\mathrm{A}'_{p-1}(M_0)) \circ \sigma_{p-1}(I + \lambda_{p-1}\mathrm{A}'_{p-2}(M_0)) \circ \cdots \circ \sigma_1(I + \lambda_1 M_0).$$

Setting $\beta_0 = \beta/d_1$ and $w = (u, M_0u) = \mathrm{d}\phi_{-\tau}(\rho) \cdot v$, we have

$$\|\mathrm{dB}^p(z) \cdot w\| = \|(S_pu, M_pS_pu)\| \geqslant \|S_pu\| \geqslant e^{\frac{\beta}{d_1}pd_1}\|u\| \geqslant e^{\beta_0(t+\tau-\sigma)}\|u\|,$$

and hence we get

$$\|\mathrm{dB}^p(z) \cdot w\| \leqslant C_0e^{-\beta_0 d_1}e^{\beta_0 t}\|w\| = C_0e^{-\beta_0 d_1}e^{\beta_0 t}\|\mathrm{d}\phi_{-\tau}(\rho)v\|. \tag{8.6.2}$$

Here we used the estimate

$$\|w\| = \Big(\|u\|^2 + \|M_0u\|^2\Big)^{1/2} \leqslant (1 + B_0^2)^{1/2}\|u\|$$

with $\|M_0\|_{\Pi_0\to\Pi_0} \leqslant B_0$ and we set $C_0 = (1 + B_0^2)^{-1/2}$. The constant B_0 can be chosen uniformly for all M_k and all periodic points since for every non-negative symmetric map M one has

$$\|M(I + \lambda_k M)^{-1}\| \leqslant \frac{1}{\lambda_k} \leqslant \frac{1}{d_0},$$

and the norms $\|\widetilde{\xi}_k\|$ are uniformly bounded by a constant depending on the sectional curvatures and $\kappa > 0$. Hence

$$\|\mathrm{A}_k(M)\| \leqslant B_0 \tag{8.6.3}$$

and the same is true for the fixed point $M_0 = \mathrm{A}_m(M_{m-1})$. Consequently, the estimate (8.6.3) is uniform for all periodic points. Finally, estimating the norm of $\mathrm{d}\phi_{-\sigma}(\mathrm{B}^p(z)) = \begin{pmatrix} I & -\sigma I \\ 0 & I \end{pmatrix}$, we obtain $\|\mathrm{d}\phi_\sigma(\mathrm{B}^p(z))\zeta\| \geqslant (1 + d_1)^{-1}\|\zeta\|$ and

$$\begin{aligned} \|\mathrm{d}\phi_t(\rho)v\| &\geqslant (1 + d_1)^{-1}C_0e^{-\beta_0 d_1}e^{\beta_0 t}\|\mathrm{d}\phi_{-\tau}(\rho)v\| \\ &\geqslant (1 + d_1)^{-2}C_0e^{-\beta_0 d_1}e^{\beta_0 t}\|v\|. \end{aligned}$$

Our case is a partial one of a more general setting (see [LW94]) concerning Lagrangian spaces $\{(u, Mu)\}$ with positive definite linear maps M. Such spaces are called positive Lagrangian. A linear symplectic map L is called monotone if it maps positive Lagrangian onto positive Lagrangian. In [LW94] it is proved that any monotone symplectic map is a contraction on the manifold of positive Lagrangian spaces. After a suitable conjugation the map L has the representation (see Proposition 3 in [LW94])

$$L = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} I & R \\ P & I + PR \end{pmatrix}$$

with positive definite matrices P, R. In our situation we have $A = I$, $R = \lambda_i I$, $P = \xi_i$.

To determine the stable space $E_s(z)$ at z, we will study the flow ϕ_t for $t < 0$ and repeat the above argument leading to a fixed point. The linear map P_γ^{-1} for a periodic ray γ with m reflexions has the representation

$$P_\gamma^{-1} = (\mathrm{d}\Psi_1)^{-1} \circ \cdots \circ (\mathrm{d}\Psi_m)^{-1} : \Pi_0 \times \Pi_0 \longrightarrow \Pi_0 \times \Pi_0,$$

where

$$(\mathrm{d}\Psi_k)^{-1} = \begin{pmatrix} \sigma_k^{-1} & 0 \\ 0 & \sigma_k^{-1} \end{pmatrix} \begin{pmatrix} I + \lambda_k \xi_k & -\lambda_k I \\ -\xi_k & I \end{pmatrix}.$$

Recall that $\Pi_0 = \Pi_m$. Consider a Lagrangian $Q_0 = Q_m = \{(u, -N_m u) : u \in \Pi_0\}$ with a symmetric non-negative definite map $N_m \in \mathrm{S}_0^+$. Then

$$\begin{aligned}(\mathrm{d}\Psi_m)^{-1} Q_m &= \Big\{\Big(\sigma_m^{-1}(I + \lambda_m(\xi_m + N_m))u, -\sigma_m^{-1}(\xi_m + N_m)u\Big) \ : \ u \in \Pi_0\Big\} \\ &= \{(u, -N_{m-1}u) : u \in \Pi_{m-1}\},\end{aligned}$$

where

$$N_{m-1} = \sigma_m^{-1}(\xi_m + N_m)\Big(I + \lambda_m(\xi_m + N_m)\Big)^{-1} \sigma_m : \Pi_{m-1} \longrightarrow \Pi_{m-1}.$$

By induction, introduce the Lagrangian spaces

$$Q_k = \{(u, -N_k u) : u \in \Pi_k\}, \quad N_k = \mathrm{B}_k(N_{k+1}), \quad k = 0, \ldots, m-1,$$

where

$$\mathrm{B}_k(M) = \sigma_{k+1}^{-1}(\xi_{k+1} + M)\Big(I + \lambda_{k+1}(\xi_{k+1} + M)\Big)^{-1} \sigma_{k+1} : \Pi_k \longrightarrow \Pi_k.$$

It is easy to see that B_k are contractions from $\mathrm{S}_{k+1}^+(\varepsilon)$ to $\mathrm{S}_k^+(\varepsilon)$ since

$$\begin{aligned}&\sigma_{k+1}\Big(\mathrm{B}_k(M_1) - \mathrm{B}_k(M_2)\Big)\sigma_{k+1}^{-1} \\ &\qquad = (I + \lambda_{k+1}(\xi_{k+1} + M_1))^{-1}(M_1 - M_2)(I + \lambda_{k+1}(\xi_{k+1} + M_2))^{-1}.\end{aligned}$$

Therefore, $\mathrm{B} = \mathrm{B}_0 \circ \cdots \circ \mathrm{B}_{m-1}$ will be contraction from $\mathrm{S}_0^+(\varepsilon)$ to $\mathrm{S}_0^+(\varepsilon)$ and there exists a fixed point $N_m \in \mathrm{S}_0^+(\varepsilon)$ of B. Moreover,

$$P_\gamma^{-1}\begin{pmatrix} u \\ -N_m u \end{pmatrix} = \begin{pmatrix} \widetilde{S}u \\ -N_m \widetilde{S}u \end{pmatrix}, \quad u \in \Pi_0,$$

where

$$\widetilde{S} = \sigma_1^{-1}(I + \lambda_1(\xi_1 + \mathrm{B}_1'(N_m))) \circ \sigma_2^{-1}(I + \lambda_2(\xi_2 + \mathrm{B}_2'(N_m))) \circ \cdots \circ \sigma_m^{-1}(I + \lambda_m(\xi_m + N_m))$$

and $\mathrm{B}_k' = \mathrm{B}_k \circ \cdots \circ \mathrm{B}_{m-1},\ k = 1, \dots, m-1$. Clearly,

$$\|\widetilde{S}u\| \geqslant (1 + d_0\varepsilon)^m \|u\|,\ u \in \Pi_0,$$

where $\varepsilon > 0$ depends of the sectional curvatures of D. Thus the stable manifold at $\phi_\sigma(z), -\lambda_{m-1} < \sigma < 0$ can be defined as $E_s(\phi_\sigma(z)) = \mathrm{d}\phi_\sigma(z)(Q_m)$ and we may repeat the above argument for the estimate of $\mathrm{d}\phi_t(\phi_\sigma(z))$ acting on $E_s(\phi_\sigma(z))$ for $t < 0$.

The intersection of the unstable and stable manifolds at $y = \phi_t(z),\ 0 < t < \lambda_p$ is $(0, 0)$. Indeed, we have

$$E_u(y) = \mathrm{d}\phi_t(z)(L_{p-1}), \quad E_s(y) = \mathrm{d}\phi_{t-\lambda_p}(\phi_{\lambda_p}(z))(Q_p),$$

where

$$L_{p-1} = \{(u, M_{p-1}u) :\ u \in \Pi_{p-1} \times \Pi_{p-1}\} \quad \text{and} \quad Q_p = \{(-u, -N_p u) :\ u \in \Pi_p \times \Pi_p\}.$$

We proceed by contradiction, and assume that $E_u(y) \cap E_s(y) \neq (0, 0)$. Then there exists $0 \neq v \in L_{p-1} \cap \mathrm{d}\phi_{-\lambda_p}(\phi_{\lambda_p}(z))(Q_p)$. By the above argument,

$$\mathrm{d}\phi_{-\lambda_p}(\phi_{\lambda_p}(z))(Q_p) = \{(u, -N_{p-1}u) : u \in \Pi_{p-1} \times \Pi_{p-1}\}.$$

This implies the existence of $u \neq 0$ for which $(M_{p-1} + N_{p-1})u = 0$ which is impossible since $M_{p-1} + N_{p-1}$ is a definite positive map. Consequently, $E_u(y)$ and $E_s(y)$ are transversal subspaces of dimension $d - 1$ of Σ_y and we have a direct sum $\Sigma_y = E_u(y) \oplus E_s(y)$.

Now we pass to the estimates of $\mathrm{d}\phi_t(z)|_{E_u(z)}$, where $z \in \mathring{B} \cap K_e$ is not a periodic point. Since $z \in K_e$, the trajectory $\gamma = \{\phi_t(z) : t \in \mathbb{R}\}$ has infinite number successive reflection points $q_k \in \partial D_{i_k},\ k \in \mathbb{Z}$, with an infinite sequence

$$J_0 = (i_j)_{j \in \mathbb{Z}}, \quad i_j \neq i_{j+1}.$$

For every $p \geqslant p_0 \gg 1$ define the configuration

$$\alpha_p = \begin{cases} (i_{-p}, \dots, i_0, \dots, i_p) & \text{if } i_p \neq i_{-p}, \\ (i_{-p}, \dots, i_0, \dots, i_{p+1}) & \text{if } i_p = i_{-p}. \end{cases}$$

Repeating α_p infinite times, one obtains an infinite configuration and following the arguments of the proof of Proposition 10.3.2 in [PS17], there exists a periodic ray γ_p following this configuration. Thus we obtain a sequence of periodic rays $(\gamma_{p_0+k})_{k \geqslant 0}$. Let $\{q_{p,k} \in \partial D_{i_k}\}$ be the reflexion points of γ_p. For the periodic ray γ_p passing through $q_{p,0} \in \partial D_{i_0}$ consider the linear space

$$L_{p,0} = \{(u, M_{p,0}u) :\ u \in \Pi_{p,0}\} \subset \Pi_{p,0} \times \Pi_{p,0}.$$

Our purpose is to show that the symmetric linear maps $M_{p,0} \in \mathrm{S}_{p,0}^+(\varepsilon)$ composed by some unitary maps converge as $p \to \infty$ to a symmetric linear map $\widetilde{M}_0 \in \mathrm{S}_0^+(\varepsilon)$ on Π_0.

This composition is necessary since the maps $M_{p,0},\ \ p \geqslant p_0$, are defined on different spaces. To do this, we will use Lemmas 10.2.1, 10.4.1 and 10.4.2 in [PS17]. Consider the rays $\gamma_{p_0+q},\ q \geqslant 1$, and γ. These rays have reflection points passing successively through the obstacles

$$D_{i_{-p_0-1}}, D_{i_{-p_0}}, \dots, D_{i_0}, \dots, D_{i_{p_0}}, D_{i_{p_0+1}}.$$

According to Lemma 10.2.1 in [PS17], there exist uniform constants $C > 0$ and $\delta \in (0, 1)$ such that for any $|k| \leqslant p_0$ and $j = 1, \dots, q$, one has

$$\|q_{p_0+1,k} - q_{p_0+j,k}\| \leqslant C(\delta^{p_0+k} + \delta^{p_0-k}) \quad \text{and} \quad \|q_{p_0+j,k} - q_k\| \ \leqslant C(\delta^{p_0+k} + \delta^{p_0-k}).$$

We need to introduce some notations from Section 10.4 in [PS17]. Let $x \in \partial D_i$ and $y \in \partial D_j$ with $i \neq j$, and assume that the segment $[x, y]$ is transversal to both ∂D_i and ∂D_j. Let Π be the plane orthogonal to $[x, y]$, passing through x. Let $e = (x - y)/\|x - y\|$, and introduce the projection $\pi : \Pi \longrightarrow T_x(\partial D)$ along the vector e. As above, we define the symmetric linear map $\widetilde{\psi} : \Pi \to \Pi$ by

$$\langle \widetilde{\psi}(u), u\rangle = 2\langle e, n(x)\rangle \langle G_x(\pi(u)), \pi(u)\rangle, \quad u \in \Pi,$$

and notice that

$$\operatorname{spec} \widetilde{\psi} \subset [\mu_1, \mu_2], \quad 0 < \mu_1 < \mu_2.$$

Setting $D_0 = 2C$, we have the estimates

$$\|q_{p_0+j,k} - q_k\| \leqslant D_0 \delta^{p_0+k}, \quad k = -p_0 + 1, \dots, 0, \quad j = 1, \dots, q.$$

Fix $1 \leqslant j \leqslant q$ and introduce the vectors

$$e_k = \frac{q_{k+1} - q_k}{\|q_{k+1} - q_k\|}, \quad e'_k = \frac{q_{p_0+j,k+1} - q_{p_0+j,k}}{\|q_{p_0+j,k+1} - q_{p_0+j,k}\|}.$$

Consider the maps $\widetilde{\xi}_k : \Pi_k \longrightarrow \Pi_k$ and $\widetilde{\psi}'_k : \Pi'_k \longrightarrow \Pi'_k$ related to the segments $[q_{k-1}, q_k]$ and $[q_{p_0+j,k-1}, q_{p_0+j,k}]$, respectively. Let $M_{-p_0+1} : \Pi_{-p_0+1} \longrightarrow \Pi_{-p_0+1}$ and $M'_{-p_0+j} : \Pi'_{-p_0+j} \longrightarrow \Pi'_{-p_0+j}$ be symmetric non-negative definite linear operators. By induction, define

$$M_k = \sigma_k M_{k-1}(I + \lambda_k M_{k-1})^{-1}\sigma_k + \widetilde{\xi}_k, \quad k = -p_0 + 2, \dots, 0,$$

where $\lambda_k = \|q_{k-1} - q_k\|$ and σ_k is the symmetry with respect to $T_{q_k}\partial D$. Similarly, we define $M'_k,\ k = -p_0 + 2, \dots, 0$, by replacing $\widetilde{\xi}_k, \sigma_k$ and λ_k by $\widetilde{\xi}'_k,\ \sigma'_k$, and

$$\lambda_{p_0+j,k} = \|q_{p_0+j,k-1} - q_{p_0+j,k}\|$$

respectively. Next, introduce the constants

$$b = (1 + 2\mu_1 \kappa d_0)^{-1} < 1, \quad a_1 = \max\{\delta, b\} < 1,$$

where $d_0 > 0$ and $\kappa > 0$ were defined above. If we choose M_{-p_0+1} such that $\|M_{-p_0+1}\| \leqslant B_0$, then by induction, one deduces $\|M_k\| \leqslant B_0$. Here $B_0 > 0$ is the constant in (8.6.3). We have uniform estimates

$$\|M_k\| \leqslant B_0, \quad \|M'_k\| \leqslant B_0, \quad k = -p_0 + 1, \dots, 0. \tag{8.6.4}$$

Applying Lemma 10.4.1 in [PS17], there exists a linear isometry $H_k : \mathbb{R}^d \to \mathbb{R}^d$ such that $H_k(\Pi'_k) = \Pi_k$, and H_k satisfies the estimates

$$\|H_k - I\| \leqslant C_1 D_0(1+\delta)\delta^k, \quad \|\widetilde{\xi}_k - H_k \widetilde{\xi}'_k H_k^{-1}\| \leqslant C_2 D_0 (1+\delta)\delta^k, \tag{8.6.5}$$

for any $k = -p_0+1, \dots, 0$. Now we are in position to apply Lemma 10.4.2 in [PS17] saying that with some constant $E > 0$, depending only on D, κ, δ and b, it holds, for $k = -p_0+1, \dots, 0$,

$$\|M_k - H_k M'_k H_k^{-1}\| \leqslant D_0 E a_1^{p_0+k} + b^{2(k+p_0-1)} \|M_{-p_0+1} - H_{-p_0+1} M'_{-p_0+1} H^{-1}_{-p_0+1}\|. \tag{8.6.6}$$

The norm of the second term on the right hand side is bounded by $2B_0 b^{2(k+p_0-1)}$ and for $k=0$ one gets

$$\|M_0 - H_0 M'_0 H_0^{-1}\| \leqslant D_0 E a_1^{p_0} + 2B_0 b^{2(p_0-1)}.$$

Applying the above estimate for the rays γ_{p_0+q}, the maps M'_0, H_0 will depend on the ray γ_{p_0+q} and for this reason we denote them by $M'_{q,0}, H_{q,0}$. Now we use these estimates for the maps $M'_{q,0}$, $M'_{q',0}$ related to the rays γ_{p_0+q} and $\gamma_{p_0+q'}$ and by the triangle inequality one deduces

$$\left\|H_{q,0} M'_{q,0} H^{-1}_{q,0} - H_{q',0} M'_{q',0} H^{-1}_{q',0}\right\| \leqslant 2D_0 E a_1^{p_0} + 4B_0 b^{2(p_0-1)}. \tag{8.6.7}$$

Here $H_{q,0}(\Pi'_{q,0}) = \Pi_0$ and $H_{q',0}(\Pi'_{q',0}) = \Pi_0$ are some isometries satisfying the estimates (8.6.5). Clearly, one obtain a Cauchy sequence $(H_{q,0} M'_{q,0} H^{-1}_{q,0})_{q \geqslant 1}$ which converges to a symmetric non-negative linear map $\widetilde{M}_0$ in Π_0. Moreover, if for every q we have $M'_{q,0} \geqslant \varepsilon I$, hence $\widetilde{M}_0 \geqslant \varepsilon I$.

After this preparation we define the unstable manifold at $\phi_t(z_0)$ for some $0 < \tau < \|q_1 - q_0\|$ as the subspace

$$E_u(\phi_\tau(z)) = \mathrm{d}\phi_\tau(z)\{(u, \widetilde{M}_0 u) \in \Pi_0 \times \Pi_0 : \ u \in \Pi_0\} \subset \Sigma_{\phi_\tau(z)}.$$

It is important to note that the procedure leading to the estimate (8.6.6) can be repeated starting with $\widetilde{M}_0$ instead of M_{-p_0+1}. Then if $\widetilde{M}_k$ are the maps obtained from $\widetilde{M}_0$ after successive reflexions, we obtain an estimate

$$\|\widetilde{M}_k - H_k M'_k H_k^{-1}\| \leqslant D_0 E a_1^{p_0+k} + b^{2(k+p_0-1)} \|\widetilde{M}_0 - H_0 \widetilde{M}'_0 H_0^{-1}\|$$

for $k = 1, \dots, p_0/2$.

We can repeat the above argument for $\rho = \phi_\tau(z)$, $v \in E_u(\rho)$, and $t = -\tau + \sum_{j=1}^p \lambda_j + \sigma$, where $0 < \tau < \lambda_1$ and $0 < \sigma < \lambda_{p+1}$, to estimate $\|\mathrm{d}\phi_t(\rho) \cdot v\|$. We apply (8.6) with the expansion map $\widetilde{S}_p$ defined as the composition of the maps $(I + \lambda_k \mathrm{A}'_{k-1}(\widetilde{M}_0))$ and we get (8.6.4). Finally, the construction of the stable space $E_s(\phi_\sigma(z))$, $-\|q_{-1} - q_0\| < \sigma < 0$ can be obtained by a similar argument and we omit the details.

Annexe A

Un théorème taubérien

A.1 A basic result

We present a first basic result which will be useful to obtain a weak version of Delange's theorem. A simple proof of this result based on Newman's approach [New80] for proving the prime number theorem can be found in [Vat15].

Theorem A.1.1. *Let* $g : [0, \infty[\to \mathbb{R}$ *be a bounded and measurable function. For* $\mathrm{Re}(s) > 0$ *we set*

$$G(s) = \int_0^\infty g(t)\mathrm{e}^{-st}\mathrm{d}t.$$

Assume that G *extends to a continuous fonction on* $\{s \in \mathbb{C} \ : \ \mathrm{Re}(s) \geqslant 0\}$*, which we still denote by* G*. Then*

$$\lim_{T\to\infty} \int_0^T g(t)\mathrm{d}t = G(0).$$

A.2 A weak version of a theorem by Delange

In this section we state a Tauberian theorem which is a weak version of a theorem of Delange [Del54, Théorème III]. The latter theorem is itself a generalization of the classical Ikehara's theorem [Ike31] to the case where the singularity of the Laplace transform of the studied function is not necessarily a simple pole. The result goes as follows.

Theorem A.2.1. *Let* $g : [0, \infty[\to \mathbb{R}_{\geqslant 0}$ *be a nondecreasing function such that for some* $C > 0$ *and* $n \geqslant 1$ *it holds*

$$|g(t)| \leqslant C(1+t)^{n-1}\exp(t), \quad t \geqslant 0. \tag{A.2.1}$$

Assume that there are $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ *with* $\alpha_n > 0$ *such that the function* G *defined by*

$$G(s) = \int_0^\infty g(t)\mathrm{e}^{-st}\mathrm{d}t - \sum_{j=1}^n \frac{\alpha_j}{(s-1)^j}, \quad \mathrm{Re}(s) > 1,$$

extends to a continuous function on $\{\mathrm{Re}(s) \geqslant 1\}$*. Then it holds*

$$g(t) \sim \frac{\alpha_n t^{n-1}}{(n-1)!}\exp(t), \quad t \to \infty. \tag{A.2.2}$$

For convenience of the reader, we provide a short proof based on Theorem A.1.1, adapting the strategy of [Vat15] (the latter work only deals with simple poles).

Proof. We consider the function

$$h(t) = g(t)\mathrm{e}^{-t} - \sum_{j=1}^{n} \frac{\alpha_j}{(j-1)!} t^{j-1}, \quad t \geqslant 0, \tag{A.2.3}$$

and

$$H(s) = \int_0^\infty h(t)\mathrm{e}^{-st}\mathrm{d}t, \quad \mathrm{Re}(s) > 0.$$

Then using the identity $\int_0^\infty t^{j-1}\mathrm{e}^{-st}\mathrm{d}t = \Gamma(j)s^{-j} = (j-1)!s^{-j}$ we get

$$H(s) = \int_0^\infty g(t)\mathrm{e}^{-(s+1)t}\mathrm{d}t - \sum_{j=1}^{n} \frac{\alpha_j}{(j-1)!} \int_0^\infty t^{j-1}\mathrm{e}^{-st}\mathrm{d}t = G(s+1),$$

and thus H extends to a continuous function on $\{\mathrm{Re}(s) \geqslant 0\}$. Of course the same holds for the function $s \mapsto F(s) = \int_0^\infty \chi(t)h(t)\mathrm{e}^{-st}\mathrm{d}t$, where χ is a smooth function $\mathbb{R}_{\geqslant 0} \to [0,1]$ chosen so that $\chi \equiv 0$ on $[0,1]$ and $\chi \equiv 1$ on $[2,\infty[$. Next, define

$$A(s) = \int_0^\infty \chi(t)h(t)t^{-(n-1)}\mathrm{e}^{-st}\mathrm{d}t, \quad \mathrm{Re}(s) > 0.$$

Then it holds

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{n-1} A(s) = (-1)^{n-1}F(s),$$

and in particular the function A also extends to a continuous function on the half plane $\{\mathrm{Re}(s) \geqslant 0\}$. By (A.2.1), the function $t \mapsto \chi(t)h(t)t^{-(n-1)}$ is bounded and we may apply Theorem A.1.1 to obtain that the integral

$$\int_0^\infty \chi(t)h(t)t^{-(n-1)}\mathrm{d}t \tag{A.2.4}$$

converges. Next we proceed by contradiction and we assume that (A.2.2) does not hold. Then we have

$$\liminf_t g(t)\frac{(n-1)!\mathrm{e}^{-t}}{\alpha_n t^{n-1}} < 1 \quad \text{or} \quad \limsup_t g(t)\frac{(n-1)!\mathrm{e}^{-t}}{\alpha_n t^{n-1}} > 1,$$

say $\limsup_t g(t)(n-1)!\mathrm{e}^{-t}\alpha_n^{-1}t^{-n+1} > 1$ (the other case is treated similarly). This means that there is $\lambda > 1$ and infinitely many t's for which

$$g(t) \geqslant \lambda \frac{\alpha_n t^{n-1}}{(n-1)!}\exp(t).$$

Let $\delta > 0$. Since g is nondecreasing, (A.2.3) implies that for any large t as above,

$$\int_t^{t+\delta} \chi(u)h(u)u^{-(n-1)}\mathrm{d}u \geqslant \int_t^{t+\delta} \frac{\alpha_n}{(n-1)!}\left(\lambda \mathrm{e}^{t-u}\left(\frac{t}{u}\right)^{n-1} - 1\right)\mathrm{d}u \\ - \sum_{j=1}^{n-1} \int_t^{t+\delta} \frac{\alpha_j u^{j-n}}{(j-1)!}\mathrm{d}u.$$

Note that every term in the above sum goes to zero when t goes to ∞. Moreover it holds

$$\int_t^{t+\delta}\left(\lambda \mathrm{e}^{t-u}\left(\frac{t}{u}\right)^{n-1}-1\right)\mathrm{d}u \geqslant \delta\left(\lambda\mathrm{e}^{-\delta}\left(\frac{t}{t+\delta}\right)^{n-1}-1\right) \sim_{\delta\to 0} \delta(\lambda-1).$$

In particular if $\delta > 0$ is chosen small enough we have, for any large t as above,

$$\left|\int_t^{t+\delta}\chi(u)h(u)u^{-(n-1)}\mathrm{d}u\right| \geqslant \frac{\alpha_n\delta(\lambda-1)}{2(n-1)!},$$

which contradicts the fact that the integral (A.2.4) converges. □

Annexe B

Noyaux de Schwartz, courants et trace bémol

In this appendix, we review the conventions we will use about currents, Schwartz kernel, and traces. Throughout this chapter, we will consider a smooth oriented manifold M of dimension n and a smooth vector bundle $E \to M$ of dimension d.

B.1 Schwartz kernels as currents

For $k = 0, \ldots, n$, we denote by $\Omega^k(M, E)$ (resp. $\Omega_c^k(M, E)$) the space of differential k-forms (resp. compactly supported differential k-forms) valued in E, that is, the space of smooth section of the bundle $\wedge^k T^*M \otimes E$, and we set

$$\Omega^\bullet(M, E) = \oplus_k \Omega^k(M, E).$$

We let $\mathcal{D}'^k(M, E)$ denote the space of E-valued k-currents, that is, the dual space of $\Omega_c^{n-k}(M, E^\vee)$, and

$$\mathcal{D}'^\bullet(M, E) = \bigoplus_{k=0}^{n} \mathcal{D}'^k(M, E).$$

Note that we have a natural inclusion $\Omega^k(M, E) \hookrightarrow \mathcal{D}'^k(M, E)$ via the non degenerate bilinear pairing

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta, \quad \alpha \in \Omega^k(M, E), \quad \beta \in \Omega_c^{n-k}(M, E^\vee).$$

Here, $\wedge$ denotes usual wedge product $\Omega^k(M, E) \times \Omega^{n-k}(M, E^\vee) \to \Omega^n(M)$.

A continuous linear operator $G : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ is called homogeneous if for some $p \in \mathbb{Z}$, we have $G\Big(\Omega^k(M, E)\Big) \subset \mathcal{D}'^{k+p}(M, E)$ for every $k = 0, \ldots, n$; the number p is called the degree of G and is denoted by $\deg G$. In that case, the Schwartz kernel theorem gives us a twisted current $\mathcal{G} \in \mathcal{D}'^{n+p}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E)$ satisfying

$$\langle Gu, v \rangle_M = \langle \mathcal{G}, \pi_1^* u \wedge \pi_2^* v \rangle_{M \times M}, \quad u \in \Omega^k(M, E), \quad v \in \Omega^{n-k-p}(M, E^\vee),$$

where π_1 and π_2 are the projections of $M \times M$ onto its first and second factors respectively.

B.2 Integration currents

Let N be an oriented submanifold of M of dimension d, possibly with boundary. The associated integration current $[N] \in \mathcal{D}'^{n-d}(M)$ is given by

$$\big\langle [N], \omega \big\rangle = \int_N i_N^* \omega, \quad \omega \in \Omega^d(M),$$

where $i_N : N \to M$ is the inclusion. Then Stokes theorem yields

$$\mathrm{d}[N] = (-1)^{n-d+1}[\partial N]. \tag{B.2.1}$$

For $f \in \mathrm{Diff}(M)$, we will set $\mathrm{Gr}(f) = \{(f(x), x),\ x \in M\}$ the graph of f. Note that $\mathrm{Gr}(f)$ is a n-dimensional submanifold of $M \times M$ which is canonically oriented since M is. Therefore, we can consider the integration current over $\mathrm{Gr}(f)$. By definition, we have for any $\alpha, \beta \in \Omega^\bullet(M)$

$$\big\langle [\mathrm{Gr}(f)], \pi_1^* \alpha \wedge \pi_2^* \beta \big\rangle = \int_M f^* \alpha \wedge \beta.$$

In particular, $[\mathrm{Gr}(f)]$ is the Schwartz kernel of $f^* : \Omega^\bullet(M) \to \Omega^\bullet(M)$.

B.3 Flat traces

B.3.1 Super flat trace

Let $G : \Omega^\bullet(M,E) \to \mathcal{D}'^\bullet(M,E)$ be an operator of degree 0. We denote its Schwartz kernel by $\mathcal{G}$ and we define

$$\mathrm{WF}'(\mathcal{G}) = \big\{(x, y, \xi, \eta),\ (x, y, \xi, -\eta) \in \mathrm{WF}(\mathcal{G})\big\} \subset T^*(M \times M),$$

where WF denotes the classical Hörmander wavefront set, cf [Hör90, §8]. We will also use the notation $\mathrm{WF}(G) = \mathrm{WF}(\mathcal{G})$ and $\mathrm{WF}'(G) = \mathrm{WF}'(\mathcal{G})$. Assume that

$$\mathrm{WF}'(\mathcal{G}) \cap \Delta(T^*M) = \emptyset, \quad \Delta(T^*M) = \{(x, x, \xi, \xi),\ (x, \xi) \in T^*M\}. \tag{B.3.1}$$

Let $\iota : M \to M \times M, x \mapsto (x, x)$ be the diagonal inclusion. Then by [Hör90, Theorem 8.2.4] the pull back $\iota^* \mathcal{G} \in \mathcal{D}'^n(M, E^\vee \otimes E)$ is well defined and we define the *super flat trace* of G by

$$\mathrm{tr}_\mathrm{s}^\flat G = \langle \mathrm{tr}\, \iota^* \mathcal{G}, 1 \rangle,$$

where tr denotes the trace on $E^\vee \otimes E$. We will also use the notation

$$\mathrm{tr}_\mathrm{gr}^\flat G = \mathrm{tr}_\mathrm{s}^\flat NG,$$

where $N : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$ is the number operator, that is, $N\omega = k\omega$ for every $\omega \in \Omega^k(M,E)$.

If $\Gamma \subset T^*M$ is a closed conical subset, we let

$$\mathcal{D}'^\bullet_\Gamma(M,E) = \Big\{ u \in \mathcal{D}'^\bullet(M,E), \mathrm{WF}(u) \subset \Gamma \Big\} \tag{B.3.2}$$

be the space of E-valued current whose wavefront set is contained in Γ, endowed with its usual topology, cf. [Hör90, §8]. If Γ is a closed conical subset of $T^*(M \times M)$ not intersecting the conormal to the diagonal

$$N^*\Delta(T^*M) = \{(x, x, \xi, -\xi),\ (x, \xi) \in T^*M\},$$

then the flat trace is continuous as a map $\mathcal{D}'^{\bullet}_{\Gamma}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E) \to \mathbb{C}$.

B.3.2 Analytic flat trace

If $A : \mathcal{C}^\infty(M, F) \to \mathcal{D}'(M, F)$ be an operator acting on sections of a vector bundle F. If A satisfies (B.3.1), we can also define a flat trace $\operatorname{tr}^\flat A$ as in [DZ16, §2.4]. More precisely, let $B : C^\infty(M) \to \mathcal{D}'(M)$ be a continuous operator satisfying (B.3.1). Let ω be a smooth volume form on M, and let $K_{B,\omega} \in \mathcal{D}'(M \times M)$ be the Schwartz kernel of B with respect to ω, which is defined by

$$\langle K_{B,\omega}, \pi_1^*(u\omega) \wedge \pi_2^*(v\omega)\rangle = \langle Bu, v\omega\rangle, \quad u, v \in C^\infty(M).$$

Then we define the *flat trace* of B by

$$\operatorname{tr}^\flat(B) = \langle \iota^* K_{B,\omega}, \omega\rangle$$

provided that it is well defined. One easily checks that this definition does not depend on the choice of ω.

Next, assume that the kernel K_A of $A : C^\infty(M, F) \to \mathcal{D}'(M, F)$ is compactly supported in $U \times U$, where $U \subset M$ is a chart domain. Take a local basis (f_i) of F; then we have

$$A(uf_i) = \sum_j A_{ij}(u) f_j, \quad u \in C_c^\infty(U),$$

where A_{ij} are operators $C_c^\infty(U) \to \mathcal{D}'_c(U)$. Then we define

$$\operatorname{tr}^\flat A = \sum_i \operatorname{tr}^\flat A_{ii}.$$

To handle the general case, let $(U_\alpha)_\alpha$ be an open cover of M with chart domains, and consider a partition of unity $(\chi_\alpha)_\alpha$ subordinate to $(U_\alpha)_\alpha$. For any α we consider $\widetilde{\chi}_\alpha \in C_c^\infty(U_\alpha)$ such that $\widetilde{\chi}_\alpha = 1$ on $\operatorname{supp}\chi_\alpha$. Then we have

$$A = \sum_\alpha \widetilde{\chi}_\alpha A \chi_\alpha + A'$$

where $\operatorname{supp} K_{A'}$ does not intersect the diagonal, and we define

$$\operatorname{tr}^\flat A = \sum_\alpha \operatorname{tr}^\flat \widetilde{\chi}_\alpha A \chi_\alpha$$

provided that $\operatorname{supp} K_A$ is compact. Again, one easily sees that this definition does not depend on the choice of the partition of unity nor on the choice of the $\widetilde{\chi}_\alpha$'s.

B.3.3 Comparison of the traces

Let $G : \Omega^\bullet(M,E) \to \mathcal{D}'^\bullet(M,E)$ be an operator of degree 0. It gives rise to an operator $G_k : \mathcal{C}^\infty(M,F_k) \to \mathcal{D}'(M,F_k)$ for each $k = 0,\dots,n$, where we set $F_k = \wedge^k T^*M \otimes E$. Then the link between the two notions of flat trace we saw above is given by

$$\operatorname{tr}_s^\flat G = \sum_{k=0}^n (-1)^k \operatorname{tr}^\flat G_k. \tag{B.3.3}$$

Indeed, to prove (B.3.3), it suffices to consider the case where the kernel K_G is smooth and supported in $U \times U$ for some open chart domain U. For simplicity, we assume that E is the trivial bundle $\mathbb{C}$, the general case being handled similarly. Take local coordinates (x^i) and (y^j) on $U \times U$. If $I = (i_1,\dots,i_k)$ with $i_1 < \cdots < i_k$ we write $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \mathrm{d}x^{i_k}$. Then we have

$$K_G = \sum_{I,J} g_{IJ}(x,y)\mathrm{d}x^I \wedge \mathrm{d}y^J$$

for some smooth functions $g_{IJ} \in C_c^\infty(U \times U)$, where the sum runs over all multi-indexes I, J. In particular it holds

$$\operatorname{tr}_s^\flat G = \sum_I \int_U g_{I,\complement I}(x,x)\mathrm{d}x^I \wedge \mathrm{d}x^{\complement I},$$

where $\complement I$ is the unique multi-index $(j_1,\dots j_{n-k})$ with $j_1 < \cdots < j_{n-k}$ and such that $\{i_1,\dots,i_k,j_1,\dots j_{n-k}\} = \{1,\dots,n\}$. On the other hand, we have by definition of K_G, for any (I,J) with $|I| = n-k$, $J = |k|$, and $u_I, v_J \in C_c^\infty(U)$,

$$\begin{aligned}
&\int_U G(u_J \mathrm{d}y^J) \wedge v_I(x)\mathrm{d}x^I \\
&\quad = \int_{U\times U} K_G \wedge u_J(y)\mathrm{d}y^J \wedge v_I(x)\mathrm{d}x^I \\
&\quad = \sum_{K,L} \int_{U\times U} g_{KL}(x,y)\mathrm{d}x^K \wedge \mathrm{d}y^L \wedge u_J(y)\mathrm{d}y^J \wedge v_I(x)\mathrm{d}x^I \\
&\quad = (-1)^k \int_U \left(\int_U g_{\complement I,\complement J}(x,y)u_J(y)\mathrm{d}y \right) \mathrm{d}x^{\complement I} \wedge v_I(x)\mathrm{d}x^I,
\end{aligned}$$

with $\mathrm{d}y = \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$. In particular, we obtain that $G(u_J\mathrm{d}y^J)(x)$ coincides with $(-1)^k \sum_I \left(\int_U g_{\complement I,\complement J}(x,y)u_J(y)\mathrm{d}y\right) \mathrm{d}x^{\complement I}$. Thus with the definition of §B.3.2 we get

$$\operatorname{tr}^\flat G_k = (-1)^k \sum_{|J|=k} \int_U g_{J,\complement J}(x,x)\mathrm{d}x^J \wedge \mathrm{d}x^{\complement J},$$

which concludes the proof.

B.3.4 Cyclicity of the flat trace

Let $G, H : \Omega^\bullet(M,E) \to \mathcal{D}'^\bullet(M,E)$ be two homogeneous operators. We denote by $\mathcal{G}, \mathcal{H}$ their respective kernels. If $\Gamma \subset T^*(M \times M)$ is a conical subset, we define

$$\Gamma^{(1)} = \{(y,\eta) \ : \ \exists x \in M, (x,y,0,\eta) \in \Gamma\},$$

and

$$\Gamma^{(2)} = \{(y, \eta) \ : \ \exists x \in M, (x, y, -\eta, 0) \in \Gamma\}.$$

Then under the assumption

$$\mathrm{WF}(\mathcal{G})^{(2)} \cap \mathrm{WF}(\mathcal{H})^{(1)} = \emptyset,$$

the operator $F = G \circ H$ is well defined by [Hör90, Theorem 8.2.14] and its Schwartz kernel $\mathcal{F}$ satisfies the wave front set estimate

$$\mathrm{WF}(\mathcal{F}) \subset \big\{(x, y, \xi, \eta) \ : \exists (z, \zeta), (x, z, \xi, \zeta) \in \mathrm{WF}'(\mathcal{G}) \\ \text{and } (z, y, \zeta, \eta) \in \mathrm{WF}(\mathcal{H})\big\}.$$

If both compositions $G \circ H$ and $H \circ G$ are defined, we will denote by

$$[G, H] = G \circ H - (-1)^{\deg G \deg H} H \circ G$$

the graded commutator of G and H. We have the following

Proposition B.3.1. *Let G, H be two homogeneous operators with $\deg G + \deg H = 0$ and such that both compositions $G \circ H$ and $H \circ G$ are defined and satisfy the bound (B.3.1). Then we have*

$$\mathrm{tr}_{\mathrm{s}}^{\flat}\, [G, H] = 0.$$

The above result follows from the cyclicity of the L^2-trace, the approximation result [DZ16, Lemma 2.8], the relation

$$\mathrm{tr}_{\mathrm{s}}^{\flat}\, [G, H] = \mathrm{tr}^{\flat} \left[(-1)^N F, G\right],$$

where N is the number operator and $\mathrm{tr}^{\flat}$ is the flat trace with the convention from [DZ16], see §B.3.1, and the fact that the map $(G, H) \mapsto G \circ H$ is continuous

$$\mathcal{D}'^{\bullet}_{\Gamma}(M \times M, \pi_1^* E^{\vee} \otimes \pi_2^* E) \times \mathcal{D}'^{\bullet}_{\widetilde{\Gamma}}(M \times M, \pi_1^* E^{\vee} \otimes \pi_2^* E) \\ \longrightarrow \mathcal{D}'^{\bullet}_{\Upsilon}(M \times M, \pi_1^* E^{\vee} \otimes \pi_2^* E)$$

for any closed conical subsets $\Gamma, \widetilde{\Gamma} \subset T^*(M \times M)$ such that $\Gamma^{(2)} \cap \widetilde{\Gamma}^{(1)} = \emptyset$, and where Υ is a closed conical subset given in [Hör90, 8.2.14].

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Mathématiques Orsay

Titre : Sur quelques applications géométriques de la théorie spectrale des flots hyperboliques

Mots clés : Dynamique hyperbolique, Analyse microlocale, Fonctions zêta, Topologie, Géométrie.

Résumé : Dans cette thèse, nous étudions la distribution des orbites périodiques de certaines dynamiques hyperboliques, et le lien qu'elles entretiennent avec la topologie sous-jacente. Pour cela, nous introduisons certaines fonctions zêta dynamiques que nous étudions via des techniques analytiques et micro-locales développées au cours des dernières décennies - la théorie des résonances de Ruelle. Nous appliquons ces méthodes à divers problèmes géométriques, comme le comptage de géodésiques fermées sous contraintes d'intersection, l'existence d'un lien entre les orbites périodiques des flots d'Anosov de contact et la torsion de Turaev, ou encore la distribution des résonances quantiques pour des systèmes de billards ouverts.

Title : On some geometrical applications of the spectral theory of hyperbolic flows

Keywords : Hyperbolic dynamics, Microlocal analysis, Zeta functions, Topology, Geometry

Abstract : In dynamical systems, one of the main objects or quantities that have been studied are the periodic orbits and their periods. In this thesis we make use of certain dynamical zeta functions to study their distribution and their link with the underlying topology; these zeta functions are studied via recent analytic and micro-local techniques developed in the past decades. We give applications to various geometrical problems, such as counting closed geodesics under intersection constraints, the relation between periodic orbits of contact Anosov flows and the Turaev torsion, or the distribution of quantum resonances for open billiard systems.