Counting closed geodesics under intersection constraints

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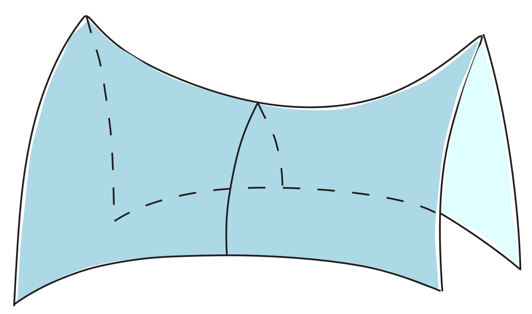
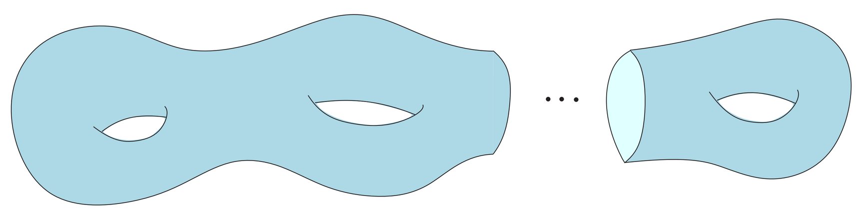
Negatively curved surfaces

Let Σ be a closed (i.e. compact and without boundary) negatively curved Riemannian surface.



Topologically, Σ consists in a surface of genus g (meaning that it has g holes) where $g \geqslant 2$.

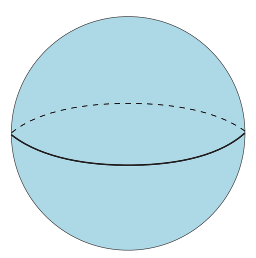


Typical examples are *hyperbolic surfaces* $\Gamma \backslash \mathbb{H}$ where $\mathbb{H}$ is the Poincaré halfplane and Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$; in this case the curvature is constant and equal to -1.

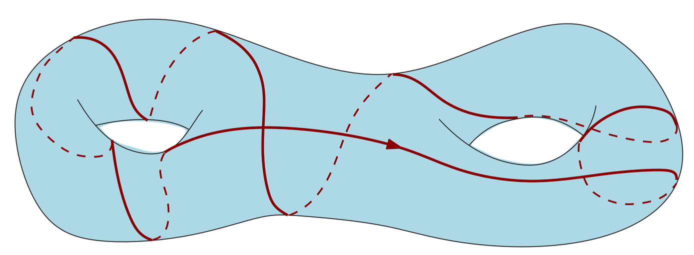
Closed geodesics

On a Riemannian surface, there are remarkable curves called *geodesics* which (locally) minimize lengths.

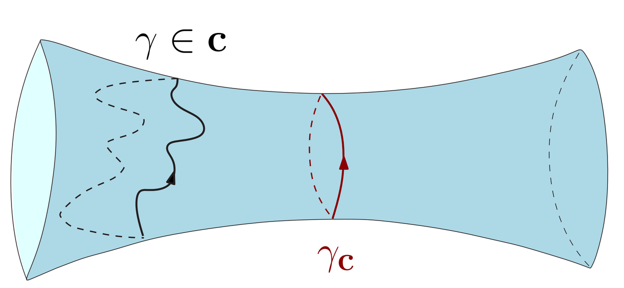
Example. On the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, the geodesics are exactly the great circles; in particular every geodesic is periodic.



The negativeness of the curvature implies that the geodesic flow is *chaotic* (i.e. very sensitive to initial conditions).



However, in every nontrivial class $\mathbf{c}$ of deformation of closed curves $\mathbb{S}^1 \to \Sigma$, there is exactly one *closed geodesic* (i.e. a periodic geodesic trajectory) $\gamma_{\mathbf{c}}$, which minimizes the length in the class $\mathbf{c}$.



Counting primitive geodesics

We denote by $\mathcal{P}$ the set of *primitive* closed geodesics (i.e. closed geodesics which are not a multiple of a shorter one). A famous result of Margulis [Mar69] reads

$$\sharp\{\gamma \in \mathcal{P} : \ell(\gamma) \leqslant L\} \sim \frac{e^{hL}}{hL}$$

as $L \to \infty$, where $\ell(\gamma)$ denotes the *length* of the closed geodesic γ and $h > 0$ is the *topological entropy* of the geodesic flow (h measures the chaos; if Σ is hyperbolic then $h = 1$).

Note the analogy with the prime number theorem which reads $\pi(x) \sim x/\log x$ as $x \to \infty$ where $\pi(x)$ is the number of primes which are smaller than x.

Imposing constraints

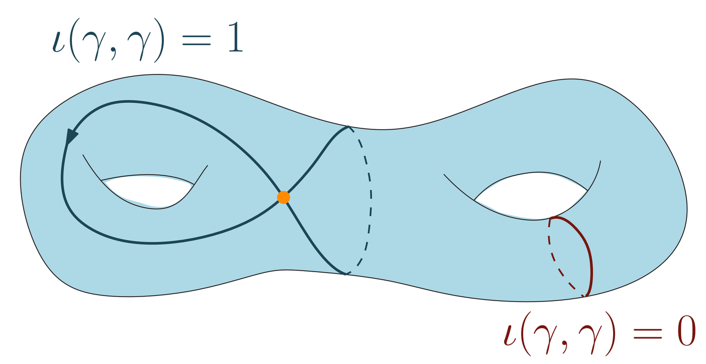
A natural question to investigate is whether we can understand the asymptotic growth of the number of geodesics that satisfy a certain (geometric or topological) constraint.

We will be interested in the following features:

- self–intersection numbers;
- homology classes;
- geometric intersection numbers.

Self–intersection numbers

For $\gamma \in \mathcal{P}$ we denote by $\iota(\gamma, \gamma)$ its *self–intersection number*.



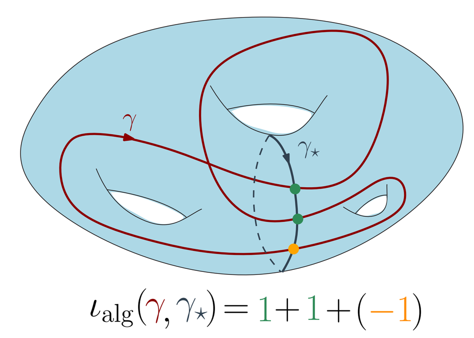
Mirzakhani [Mir08, Mir16] showed that, provided Σ is a hyperbolic, we have for any $k \in \mathbb{Z}_{\geqslant 0}$,

$$\sharp\{\gamma \in \mathcal{P} : \ell(\gamma) \leqslant L,\ \iota(\gamma, \gamma) = k\} \sim c_k L^{6g-6}$$

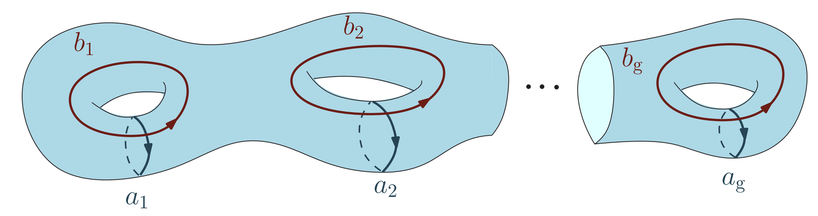
as $L \to \infty$, for some $c_k > 0$ independent of L.

Homology classes

Each closed geodesic γ gives rise to a *homology class* $[\gamma] \in H_1(\Sigma, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$.



This homology class is determined by the *algebraic* intersection numbers of γ with a basis $(a_1, b_1, \ldots, a_g, b_g)$ of the first homology group.



Lalley [Lal88] and Pollicott [Pol91] showed that for any $\xi \in H_1(\Sigma, \mathbb{Z})$, we have

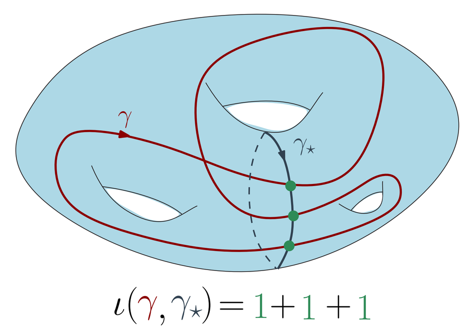
$$\sharp\{\gamma \in \mathcal{P} : \ell(\gamma) \leqslant L,\ [\gamma] = \xi\} \sim c \frac{e^{hL}}{L^g}$$

as $L \to \infty$, where $c > 0$ is independent of ξ.

Geometric intersection numbers

Let $\gamma_\star$ be a *simple* and *non-separating* closed geodesic.

We denote by $\iota(\gamma, \gamma_\star)$ the *geometric intersection number* between γ and $\gamma_\star$.

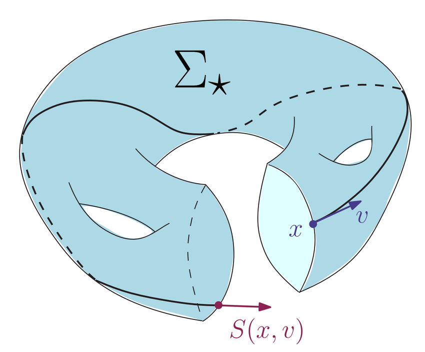


Then one can show [Cha21] that there are $c_\star > 0$ and $h_\star \in]0, h[$ such that, as $L \to \infty$, we have

$$\sharp\{\gamma \in \mathcal{P} : \ell(\gamma) \leqslant L,\ \iota(\gamma, \gamma_\star) = k\} \sim \frac{(c_\star L)^k}{k!} \frac{e^{h_\star L}}{h_\star L}$$

for every $k \geqslant 0$.

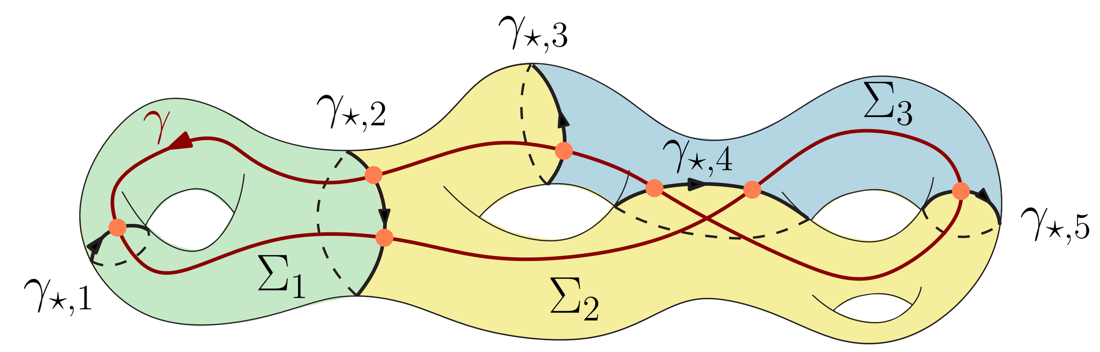
In fact, $h_\star$ is the entropy of the surface with boundary $\Sigma_\star$ obtained by cutting Σ along $\gamma_\star$.



The proof uses tools from microlocal analysis to study the *transfer operator* $f \mapsto f \circ S$ associated to the *dynamical scattering map* S.

Multiple curves

We fix a family of pairwise disjoint simple closed geodesics $\gamma_{\star,1}, \ldots, \gamma_{\star,r}$. Those curves separate Σ into sub-surfaces $\Sigma_1, \ldots, \Sigma_q$.



Then [Cha21] for every $\mathbf{n} = (n_1, \ldots n_r)$, there is $c_{\mathbf{n}} > 0$, $d_{\mathbf{n}} \in \mathbb{Z}_{\geqslant 0}$ and $h_{\mathbf{n}} \in]0, h[$ such that, as $L \to \infty$,

$$\sharp\{\gamma \in \mathcal{P} : \ell(\gamma) \leqslant L,\ \mathbf{i}(\gamma, \vec{\gamma}_\star) = \mathbf{n}\} \sim c_{\mathbf{n}} L^{d_{\mathbf{n}}} e^{h_{\mathbf{n}} L}$$

where $\mathbf{i}(\gamma, \vec{\gamma}_\star) = (\iota(\gamma, \gamma_{\star,1}), \ldots, \iota(\gamma, \gamma_{\star,r}))$ — here we need $\mathbf{n}$ to be *admissible*, that is $\mathbf{n} = \mathbf{i}(\gamma, \vec{\gamma}_\star)$ for some γ.

The number $h_{\mathbf{n}}$ is the maximum of the entropies of the surfaces Σ_j that are encountered by any γ satisfying $\mathbf{i}(\gamma, \vec{\gamma}_\star) = \mathbf{n}$, while $d_{\mathbf{n}}$ is the number of times such a γ travels through a surface of maximal entropy.

References

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