THE RUELLE ZETA FUNCTION FOR ANOSOV FLOWS

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Abstract

In this memoire, we will explain the meromorphic continuation of the Ruelle zeta function for Anosov flows using microlocal analysis. This extension is done considering the generator of the flow acting on some anisotropic Sobolev spaces introduced by Faure-Sjöstrand. Then, we show that the Ruelle zeta function of the geodesic flow on a negatively curved surface vanishes at zero with order the Euler characteristic of the surface. This is a result due to S. Dyatlov and M. Zworski.

1 Introduction

Historical background. In 1956, Selberg [Sel56] introduced a zeta function associated to surface of constant negative curvature $\kappa = -1$

$$\zeta_{\mathrm{Selberg}}(s) = \prod_\gamma \prod_{k=0}^\infty \left(1 - \mathrm{e}^{-(s+k)\ell(\gamma)}\right), \quad \Re(s) > 1,$$

where the first product runs over all primitive closed geodesics γ. He showed that ζ_{Selberg} extends to a meromorphic function on the whole complex plane using a trace formula (today called Selberg's trace formula), which links the lengths $\ell(\gamma)$ of the closed geodesics with the eigenvalues of the hyperbolic Laplacian. For a compact hyperbolic surface $\Sigma = \Gamma\backslash\mathbb{H}^2$, the Selberg's trace formula leads to the following dichotomy on the zeroes of ζ_{Selberg} :

- The non-trivial zeroes of ζ_{Selberg}, which are of the form

$$s = 1, \quad s = \frac{1}{2} \pm i\rho_j, \quad j \in \mathbb{N}_{\geq 1},$$

  where $\rho_j^2 + 1/4 = \lambda_j$ and $0 = \lambda_0 < \lambda_1 < \cdots \to \infty$ are the eigenvalues of the hyperbolic Laplacian Δ. The vanishing order at $s = \frac{1}{2} \pm \rho_j$ is μ_j if $\rho_j \neq 0$ and $2\mu_j$ otherwise, where μ_j is the multiplicity of λ_j as an eigenvalue of Δ ; the vanishing order at $s = 1$ is 1.
- The trivial zeroes of ζ_{Selberg}, which are of the form

$$s = -m, \quad m \in \mathbb{N},$$

  with vanishing order $1 - \chi(\Sigma)$ for $m = 0$ and $-(2m+1)\chi(\Sigma)$ for $m > 0$.

The definition of ζ_{Selberg} of course generalizes to the case of surfaces with variable negative curvature ; however we do not know any analogue of the Selberg's trace formula in the variable curvature setting, which restricts our knowledge about ζ_{Selberg} in this context.

In 1976, D. Ruelle associated to any Anosov flow ϕ^t a zeta function [Rue76]. It can be thought as an analogue of the inverse of the Riemann zeta function

$$\zeta_{\mathrm{Riemann}}(s)^{-1} = \prod_{p \in \mathcal{P}} \left(1 - p^{-s}\right), \quad \Re(s) > 1.$$

where $\mathcal{P}$ is the set of prime numbers, replacing the prime numbers by the exponential of the lengths of the primitive periodic orbits of ϕ^t :

$$\zeta_{\mathrm{Ruelle}}(s) = \prod_{\gamma \text{ primitive}} \left(1 - \mathrm{e}^{-s\ell(\gamma)}\right), \quad \Re(s) \gg 0.$$

Ruelle and Selberg zeta functions are linked by the formulae

$$\zeta_{\mathrm{Ruelle}}(s) = \frac{\zeta_{\mathrm{Selberg}}(s)}{\zeta_{\mathrm{Selberg}}(s+1)}, \qquad \zeta_{\mathrm{Selberg}} = \prod_{p=0}^\infty \zeta_{\mathrm{Ruelle}}(s+p). \tag{1.1}$$

As in number theory, the Ruelle zeta function is very useful to describe the distribution of the primitive periodic orbits.

Ruelle showed that ζ_{Ruelle} extends meromorphically to the whole complex plane provided the flow and its stable and unstable bundles distributions (see definitions in §2.1) are analytic ; then, first Rugh [Rug96] in dimension 3, then Fried [Fri95] in all dimensions showed that we can remove the analyticity assumption on the stable and unstable distributions (but not on the flow). In the case of a $\mathcal{C}^\infty$ flow, we know since a while that ζ_{Ruelle} is analytic and nonzero on $\Re(s) > h(\phi)$, where $h(\phi)$ is the topological entropy of the flow, with a simple pole at $h(\phi)$. It was also known that it extends meromorphically to a neighborhood of $\{\Re(s) \geq h(\phi)\}$ if the flow is weak mixing, see [PP90, Chapter 9] for more references and details. Those results are typically obtained by coding the dynamics using Markov partitions, in order to relate the zeta function to the dynamical determinant, or Fredholm determinant, of some appropriate operators. However those methods do not take into account the smoothness of the dynamics, and we know thanks to Kitaev [Kit99] (at least in the case of Anosov diffeomorphisms) that the smoothness of the map is highly related to how far we can extend the Fredholm determinant.

In the early 2000's, Blank, Keller and Liverani [BKL02] introduced some Banach spaces adapted to an hyperbolic diffeomorphism on which the transfer operator is quasi-compact ; this led to a lot of developements in this direction [Bal04, GL06, BT07]. Also some spaces adapted to Anosov flows [Liv04, BL07] have been developed, on which the generator of the flow has a quasi-compact resolvent. More recently, Faure-Roy-Sjöstrand [FRS08] introduced a microlocal approach to construct anisotropic Sobolev spaces adapted to Anosov diffeomorphisms ; then Faure-Sjöstrand [FS11] constructed such spaces for Anosov flows. Those spaces provide the right regularity to study the generator X of the flow which appears to be Fredholm restricted to them ; this gives the

Theorem 1.1 ([BL07, FS11]). *For an Anosov vector field X, the resolvent $(-iX - \lambda)^{-1} : \mathcal{C}^\infty(M) \to \mathcal{D}'(M)$ has a meromorphic extension to the whole complex plane, with poles of finite multiplicity. Those poles are the Ruelle resonances.*

All those modern techniques finally allowed Giuletti-Liverani-Policott, then Dyatlov-Zworski, to show

Theorem 1.2 ([GLP13, DZ13]). *The Ruelle zeta function of a smooth Anosov flow extends to a meromorphic function on the whole complex plane.*

It is moreover showed in [GLP13] that in $\mathcal{C}^k$ regularity, that ζ_{Ruelle} extends meromorphically to the half plane $\Re(s) > h(\phi) - ck$ where the constant c is determined by the Anosov splitting. This paper is an extension of [GL06, BL07] whereas the approach in [DZ13] uses semiclassical analysis and is based on [FS11]. In this memoire, we will concentrate on this microlocal approach. Let us briefly recall the main components of the proof of Dyatlov-Zworski. The first step is to relate the (log derivative of the) zeta function to the flat trace of the (shifted) resolvent of the generator of the flow for $\Re(s) \gg 0$ thanks to the Guillemin trace formula [Gui77]. Then using the Faure-Sjöstrand spaces and microlocal analysis, they show that the meromorphic extension of the (shifted) resolvent lie in an appropriate space on which the flat trace is continuous, which guarantees the meromorphicity of the zeta function.

Later, Dyatlov-Zworski got interested in the behavior of ζ_{Ruelle} at $s = 0$:

Theorem 1.3 ([DZ17b])**.** *For a smooth contact Anosov flow on a 3-manifold M with orientable stable distribution, the Ruelle zeta function vanishes at zero with order $-\chi(M)$ where $\chi(M)$ is the Euler characteristic of M.*

In particular, for the geodesic flow on a closed negatively curved surface Σ (which is an Anosov flow on the unitary tangent bundle) the meromorphic extension of ζ_{Ruelle} vanishes at $s = 0$, with a zero of order $|\chi(\Sigma)|$, where $\chi(\Sigma)$ is the Euler characteristic of Σ. Note that thanks to (1.1), we recover the special case of compact hyperbolic surfaces earlier mentioned : the pole at zero of ζ_{Selberg} is $1 - \chi(\Sigma)$. This result shows in particular that for a negatively curved surface, the length spectrum of the surface (that is, the set of lengths of primitive geodesics) determines the genus of the surface. This result is proved using the standard factorization of the zeta function and calculating the dimension of the spaces of generalized resonant states of the generator of the flow.

Further extensions. Very recently, the meromorphic extension of the zeta function for Axiom A flows have been established by Dyatlov-Guillarmou [DG18], answering positively to a conjecture of Smale [Sma67]. This is based on their previous work [DG16] about the zeta function for open systems. The meromorphic extension of the zeta function was already proved in the case of Grassmanian extensions of contact Anosov flows, which is a special case of Axiom A flows, by Faure-Tsuji [FT17].

Guillarmou-Hilgert-Weich [GHW18] exhibited a correspondence between classical resonant states (that is, Ruelle resonant states) and quantum resonant states (that is, eigenfunctions of the Laplacian) for convex co-compact hyperbolic surfaces. This highlights the deep link existing between classical and quantum mechanics on hyperbolic surfaces ; such a link was already known, as witnessed by the Selberg's trace formula (see [PP$^+$01] for the convex co-compact case), but only for resonances.

Also, Hadfield [Had18] proved that the Ruelle zeta function of the geodesic flow on a negatively curved surface Σ with strictly convex boundary (which is Axiom A) vanishes at zero with order $1 - \chi(\Sigma)$. We refer to the introduction of [DG16] for a more exhaustive overview of results about Policott-Ruelle resonances and dynamical zeta functions.

Structure of the memoire. In section 2 we recall the definitions of some dynamical notions and the Guillemin trace fomula. We give the construction of the microlocal Faure-Sjöstrand spaces in section 3 and prove that the resolvent of the generator of the flow extends meromorphically to the whole complex plane. We give in section 4 some semiclassical estimates that allows to compute the wavefront set of the extension of the resolvent. In section 5, we show the meromorphic continuation of the Ruelle zeta function for Anosov flows whose stable and unstable bundles are orientable. We compute the vanishing order of ζ_{Ruelle} at zero in section 6. In appendix A we recall basic definitions about pseudo-differential operators, semiclassical calculus and wavefront sets. In appendix B we recall standard facts about operator theory and flat traces. In appendix C we show some recurrence estimates to get a bound on the growth of the number of periodic orbits.

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Contents

2 Preliminaries on dynamical systems

2.1 Anosov flows and dynamic on the cotangent bundle

Let M be a smooth compact manifold.

Definition 2.1. A smooth vector field X on M will be called to be Anosov if for each x in M, we have a decomposition

$$T_xM = \mathrm{Ran}(X_x) \oplus E_s(x) \oplus E_u(x),$$

preserved by the flow, and satisfying that for any metric $|\cdot|$ on TM there exists $C, \theta > 0$ such that :

$$|\mathrm{d}\phi_x^t(v_s)| \le C\mathrm{e}^{-\theta t}|v_s|, \quad t \ge 0, \quad v_s \in E_s(x)$$
$$|\mathrm{d}\phi_x^t(v_u)| \le C\mathrm{e}^{-\theta|t|}|v_u|, \quad t \le 0, \quad v_u \in E_u(x).$$

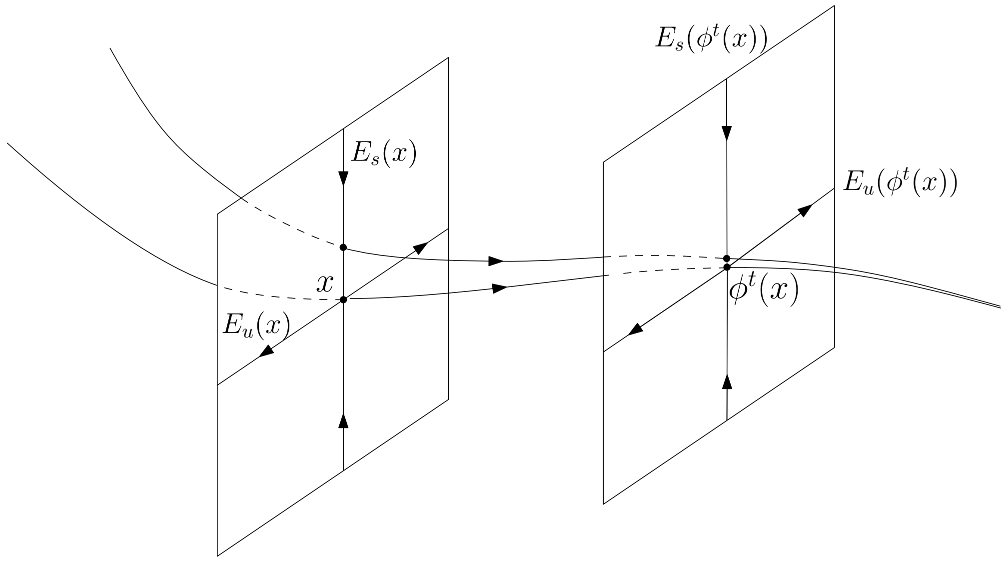

Figure 2.1. An illustration of an Anosov flow.

Remark 2.2. A typical situation is given by the case where $M = S^*\Sigma$ is the cosphere bundle over a negatively curved surface Σ; in that case, this is a well-known fact that the geodesic flow on M is Anosov [Ano67].

Remark 2.3. One can always find a norm such that the previous constant C is equal to 1, with a new constant $\widetilde{\theta}$ arbitrarily close to the first one. Indeed, it suffices to average the metric along the flow : for T large enough the metric $\|v\|_T = \int_0^T |\mathrm{d}\phi^s v|\mathrm{d}s$ satisfy those properties.

If $\varphi \in \mathrm{Diff}(M)$, we will denote by $\widetilde{\varphi}$ its lift to T^*M :

$$\widetilde{\varphi}(x,\xi) = (\varphi(x),\ {}^T(\mathrm{d}\varphi)_x^{-1}\cdot\xi), \quad (x,\xi) \in T^*M,$$

where ^{T} denote the transposition. Let Ω be the canonical symplectic form on T^*M, that is $\Omega = \mathrm{d}p$ where p is the 1-form defined by $p_{(x,\xi)}(v) = \xi(\mathrm{d}\pi_{(x,\xi)}\cdot v)$ for $v \in T_{(x,\xi)}T^*M$. Here $\pi : T^*M \to M$ is the natural projection. Note that for any $\varphi \in \mathrm{Diff}(M)$, one has $\widetilde{\varphi}^*\Omega = \Omega$. Let X be a smooth vector field on M, and denote by $H \in \mathcal{C}^\infty(T^*M)$ the Hamiltonian defined by $H(x,\xi) = \xi(X_x)$. Let $\mathbf{X}$ be the Hamiltonian vector field of H with respect to Ω, that is, $\iota_{\mathbf{X}}\Omega = \mathrm{d}H$, where ι denotes the interior product. Differentiating with respect to t, we get the following lemma which tells us that the lift of the flow of X on T^*M is the Hamiltonian flow of H.

Lemma 2.4. *Let ϕ^t be the flow of X on M and denote its lift $\widetilde{\phi^t}$ on T^*M by Φ^t. Then Φ^t is the flow generated by the Hamiltonian vector field $\mathbf{X}$.*

We will denote by E_0^*, E_s^* and E_u^* the dual decomposition of E_0, E_s and E_u in the following sense : $E_0^*\left(E_u \oplus E_s\right) = 0$, $E_s^*\left(E_s \oplus E_0\right) = 0$ and $E_u^*\left(E_u \oplus E_0\right) = 0$. Note that E_u^* and E_s^* do not correspond to the usual definition of dual spaces (they are exchanged) but one would rather prefer this convention since Φ^t preserves this decomposition and we have, with the norm $|\cdot|$ on T^*M induced by the one on TM :

$$\begin{aligned} &|\Phi^t(x,\xi_s)| \leq C\mathrm{e}^{-\theta t}, \quad t \geq 0, \quad \xi_s \in E_s^*(x), \\ &|\Phi^t(x,\xi_u)| \leq C\mathrm{e}^{-\theta|t|}, \quad t \leq 0, \quad \xi_u \in E_u^*(x). \end{aligned} \tag{2.1}$$

Those estimates motivate the following definition.

Definition 2.5. Let $\rho : T^*M \setminus 0 \to S^*M$ the natural projection, where S^*M is the unitary cotangent bundle, namely $S^*M = \{(x,\xi) \in T^*M,\ |\xi| = 1\}$. Suppose L is a closed conic set invariant under the flow Φ^t. L will be called a radial source if there exists an open conical neighborhood U of L in $T^*M \setminus 0$ and $C, \theta > 0$ such that

$$\begin{aligned} &\mathrm{dist}\left(\rho\left(\Phi^{-t}(U)\right), \rho(L)\right) \underset{t\to+\infty}{\longrightarrow} 0; \\ &|\Phi^{-t}(x,\xi)| \geq C^{-1}\mathrm{e}^{\theta t}|\xi|, \quad (x,\xi) \in U, \quad t \geq 0. \end{aligned}$$

Reversing the time of the flow we get the definition of a radial sink.

Remark 2.6. It is obvious from (2.1) that E_s^* is a radial source whereas E_u^* is a radial sink.

2.2 Periodic orbits and the Ruelle zeta function

In the following, M is a compact manifold and X is an Anosov flow on M, with ϕ^t its flow.

Lemma 2.7 (Growth of the number of periodic orbits)**.** *Let $N(T)$ be the number of periodic orbits of ϕ^t of length smaller than T. Then there exists $C, L > 0$ such that*

$$N(T) \leq C\mathrm{e}^{LT}. \tag{2.2}$$

An elementary proof of this fact is given in section C. In fact, one has the more precise statement (which we won't need here, [Mar69]) : $N(T)$ grows exponentially fast and there is $h(\phi) > 0$ such that

$$N(T) \sim \frac{\mathrm{e}^{h(\phi)T}}{h(\phi)T}.$$

The number $h(\phi)$ is the topological entropy of the flow ϕ^t.

Definition 2.8. Let $\{\gamma^\#\}$ be the primitive periodic orbits of X and denote $\{\ell(\gamma^\#)\}$ their periods. Then the Ruelle zeta function ζ of X is defined by the formula

$$\zeta(s) = \prod_{\gamma^\#} \left(1 - \mathrm{e}^{-s\ell(\gamma^\#)}\right), \tag{2.3}$$

where $s \in \mathbb{C}$ has a big enough real part.

If $\tau_1 \leq \tau_2 \leq ...$ are the periods of ϕ^t, one has

$$\tau_n \gtrsim \log n \tag{2.4}$$

by (2.2), and this guarantees the convergence of (2.3) for $\Re(s) \gg 0$.

Let us now introduce the notion of linearized Poincaré map of a periodic orbit.

Definition 2.9. Let $\gamma(t) = \phi^t(x)$ be a periodic orbit of period τ. Then the linearized Poincaré map of γ is defined by

$$P_\gamma = \left(\mathrm{d}\phi^{-\tau}\right)_x |_{E_s(x) \oplus E_u(x)}.$$

The Anosov property of the flow implies that periodic orbits are non degenerate in the sense that $I - P_\gamma$ is invertible. Indeed, if $v \in E_s(x) \oplus E_u(x)$ satisfies $v = P_\gamma v$, then $v = P_\gamma^n v$ for all $n \in \mathbb{Z}$. Writing $v = v_u + v_s$ with $v_u \in E_u(x)$ and $v_s \in E_s(x)$, and noting that P_γ preserves E_u and E_s, we get $|v_u| \leq Ce^{-n\tau}|v_u|$ for all n so $v_u = 0$. Similarly $v_s = 0$, so $v = 0$.

If we choose an other base point $x' = \phi^s(x)$, then the new linearized Poincaré map P'_γ will be conjugated by $\mathrm{d}\phi^s_x$ to P_γ. In particular, the determinant $\det(I - P_\gamma)$ is well defined and does not depend of the base point. Moreover, we have $\det(I - P_\gamma|_{E_u}) > 0$ since the eigenvalues of $P_\gamma|_{E_u}$ are of modulus strictly less than 1 and therefore $I - P_\gamma|_{E_u}$ and I lie in the same connected component of $\mathrm{GL}(E_u)$. Similarly, we have $\det(I - P_\gamma^{-1}|_{E_s}) > 0$. If E_s is orientable, we obtain $\det(P_\gamma|_{E_s}) > 0$ since $\mathrm{d}\phi^t$ preserves the orientation of the stable bundle. Letting $q = \dim E_s$, we get $\det(I - P_\gamma|_{E_s}) = \det(P_\gamma^{-1}|_{E_s} - I)\det(P_\gamma|_{E_s}) = (-1)^q \underbrace{\det(-P_\gamma^{-1}|_{E_s} + I)}_{>0}\underbrace{\det(P_\gamma|_{E_s})}_{>0}$ so that $|\det(I - P_\gamma|_{E_s})| = (-1)^q \det(I - P_\gamma|_{E_s})$. Writing $\det(I - P_\gamma) = \det(I - P_\gamma|_{E_s})\det(I - P_\gamma|_{E_u})$ we get

$$|\det(I - P_\gamma)| = (-1)^q \det(I - P_\gamma). \tag{2.5}$$

2.3 The Ruelle transfer operator and the Guillemin trace formula

For $0 \leq k \leq n$, let Ω^k be the vector bundle of k-forms on M and define the operator $\mathbf{P}_k : \mathcal{C}^\infty(M, \Omega^k) \to \mathcal{C}^\infty(M, \Omega^k)$ by $\mathbf{P}_k\alpha = -i\mathcal{L}_X\mathbf{f}$. We will note for simplicity $\Omega = \bigoplus_k \Omega^k$ and $\mathbf{P} = \bigoplus_k \mathbf{P}_k$. We also define Ω_0^k, the vector bundle of k-forms $\mathbf{f}$ with $\iota_X\mathbf{f} = 0$ where ι is the interior product. Let

$$\mathbf{T}_k : \mathcal{C}^\infty(M, \Omega_0^k) \to \mathcal{C}^\infty(\mathbb{R}_{>0} \times M, \pi_M^*\Omega_0^k)$$

the operator defined by $\mathbf{T}_k(\mathbf{f}) = (\mathbf{f}_t)_{t\in\mathbb{R}_{>0}}$ where $\mathbf{f}_t = \mathrm{e}^{-it\mathbf{P}_k}\mathbf{f} := (\phi^{-t})^*\mathbf{f}$ and π_M is the projection over M. The operator $\mathrm{e}^{-it\mathbf{P}_k}$ is called the Ruelle transfer operator. The purpose of the Guillemin trace formula is to link the Ruelle transfer operator to the linearized Poincaré maps of the periodic orbits:

Theorem 2.10 (Guillemin trace formula [Gui77]). *The flat trace of $\mathbf{T}_k$ is well defined, and we have in $\mathcal{D}'(\mathbb{R}_{>0})$:*

$$\mathrm{tr}^\flat \mathbf{T}_k = \sum_\gamma \frac{\ell(\gamma^\#)\mathrm{tr}(\bigwedge^k P_\gamma)\delta_{\ell(\gamma)}}{|\det(I - P_\gamma)|}, \tag{2.6}$$

where the sum goes over all periodic orbits γ, $\ell(\gamma)$ is the period of γ, $\ell(\gamma^\#)$ its primitive period, $\delta_{\ell(\gamma)}$ is the Dirac distribution at $\ell(\gamma)$, and P_γ is the linearized Poincaré map of γ.

See § B.3 for the definition of the flat trace.

Proof of Theorem 2.10. Denote by $\mathbf{K}$ is the Schwartz kernel of $\mathbf{T}$ with respect to some density vol. Now note that $\mathbf{K}$ acting on 0-forms is a density on the surface $S = \{(t,y,x) \in \mathbb{R}_{>0} \times M \times M,\ y = \phi^t(x)\}$. Therefore [Hör90, Example 8.2.5], $\mathrm{WF}(\mathbf{K}) \subset N^*S$, that is

$$\mathrm{WF}(\mathbf{K}) \subset \left\{ \left(t, -\eta \cdot X_x, \phi^t(x), \eta, x, -\,{}^T\mathrm{d}\phi^t_x \cdot \eta\right) \mid t \in \mathbb{R}_{>0},\ x \in M,\ \eta \in T_{\phi^t(x)}M \setminus 0 \right\}.$$

Now to show the condition (B.2) (in order to define the flat trace of $\mathbf{T}$), we have to verify that $\mathrm{WF}(\mathbf{K}) \cap \{(t,0,x,\xi,x,-\xi) | t > 0,\ (x,\xi) \in T^*M \setminus 0\} = \emptyset$. But this is straightforward since $\eta \cdot X_x = 0$, $\phi^t(x) = x$ and $(I - {}^T\mathrm{d}\phi^t_x) \cdot \eta$ implies $\eta = 0$. Indeed the Poincaré return map of a closed orbit is invertible restricted to $E_s \oplus E_u$ (see the discussion below Definition 2.9).

Let us now interest ourselves to the case $k = 0$. Let $T = \mathbf{T}_0$, $K = K_T$ and $j : \mathbb{R}_{>0} \times M \to \mathbb{R}_{>0} \times M \times M, (t,x) \mapsto (t,x,x)$.

Lemma 2.11. *Let $t_0 > 0$ and $x_0 \in M$ such that $\phi^{t_0}(x_0) = x_0$, and γ the associated orbit. Then there exists a small neighborhood U of x_0 and $\delta > 0$ such that all $\chi \in \mathcal{C}^\infty_c((t_0 - \delta, t_0 + \delta) \times U)$ we have*

$$\langle j^*T, \chi \rangle = \frac{1}{|\det(I - P_\gamma)|} \int_{t_0-\delta}^{t_0+\delta} \chi(t_0, \phi^s(x_0)) \mathrm{d}s.$$

Proof. Take some local coordinates $w : U_1 \to B(0,\varepsilon) \subset \mathbb{R}^n$ such that $w_* X = \partial_{w_1}$ and $w(x_0) = 0$. Let $\widetilde{\phi}^t = w \circ \phi^t \circ w^{-1}$ (this is well defined for t near t_0) and take some $W \subset B(0,\varepsilon)$ and $\delta > 0$ satisfying $\widetilde{\phi}^{-t}(W) \subset B(0,\varepsilon)$ for every $|t - t_0| < \delta$. Let $\rho \in \mathcal{C}^\infty_c(\mathbb{R}^n, [0,1])$ such that $\int_{\mathbb{R}^n} \rho = 1$. For $d \in \mathbb{N}$, $w = (w_1, \cdots, w_d)$ and $\overline{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_d)$, set $\rho_{\overline{\varepsilon}}(w) = \Pi_{j=1}^d \varepsilon_j^{-1} \rho(w_j/\varepsilon_j)$. Now define for $\overline{\varepsilon}, \overline{\varepsilon}' \in (0,\infty)^n$ and $\varepsilon_0 > 0$, $\theta_{\varepsilon_0, \overline{\varepsilon}, \overline{\varepsilon}'} = \rho_\varepsilon \otimes \rho_{\overline{\varepsilon}} \otimes \rho_{\overline{\varepsilon}'}$. Since the flat trace does not depend of the density, we can assume that w_* vol is the Lebesgue measure on $\mathbb{R}^n$. We have for $\psi \in \mathcal{C}^\infty_c(W)$ and $\widetilde{\chi} \in \mathcal{C}^\infty_c((t_0 - \delta, t_0 + \delta)$:

$$\begin{aligned}\langle j^* \left((w_* K) * \theta_{\varepsilon_0, \overline{\varepsilon}, \overline{\varepsilon}'} \right), \widetilde{\chi} \otimes \psi \rangle &= \int \theta_{\varepsilon_0, \overline{\varepsilon}, \overline{\varepsilon}'}(t - t', x - \widetilde{\phi}^{-t}(x'), x - x') \widetilde{\chi}(t) \psi(x) \mathrm{d}x' \mathrm{d}t' \mathrm{d}x \mathrm{d}t \\ &= \int \rho_{\overline{\varepsilon}'}(x - x') \psi(x) \widetilde{\chi}(t) \int \rho_{\varepsilon_0}(t - t') \rho_{\overline{\varepsilon}}(x - \widetilde{\phi}^{-t'}(x')) \mathrm{d}t' \mathrm{d}x \mathrm{d}x' \mathrm{d}t.\end{aligned}$$

Letting $\varepsilon_0, \overline{\varepsilon}' \to 0$ this reads

$$\langle j^* \left((w_* K) * (\delta_0^1 \otimes \rho_{\overline{\varepsilon}} \otimes \delta_0^n) \right), \widetilde{\chi} \otimes \psi \rangle = \int \rho_{\overline{\varepsilon}}(x - \widetilde{\phi}^{-t}(x)) \widetilde{\chi}(t) \psi(x) \mathrm{d}x \mathrm{d}t, \quad (2.7)$$

where δ_0^d is the Dirac at 0 on $\mathbb{R}^d$ for $d \in \mathbb{N}$. Write $\widetilde{\phi}^{-t}(x) = (-t + t_0 + x_1 + a(z), b(z))$, where $x = (x_1, z)$, for some smooth functions a and b. If $\overline{\varepsilon} = (\varepsilon_1, \overline{h})$, we obtain

$$\int \rho_{\epsilon_1}(t - t_0 + a(z)) \rho_{\overline{h}}(z - b(z)) \widetilde{\chi}(t) \psi(x) \mathrm{d}t \mathrm{d}x_1 \mathrm{d}z.$$

Letting $\varepsilon_1 \to 0$, one obtains $\int \rho_{\overline{h}}(z - b(z))\widetilde{\chi}(t_0 - a(z))\mathrm{d}x_1\mathrm{d}z$. The change of variable $z' = z - b(z) = \nu(z)$ gives

$$\int \frac{1}{|\det(I - (\mathrm{d}b)_{\nu^{-1}(z')})|}\rho_{\overline{h}}(z')\widetilde{\chi}(t_0 - a \circ \nu^{-1}(z'))\psi(x_1, \nu^{-1}(z'))\mathrm{d}x_1\mathrm{d}z'.$$

Finally, letting $\overline{h} \to 0$, we get since $\nu(0) = 0$ and $a(0) = 0$ (indeed $\phi^{t_0}(x_0) = x_0$),

$$\widetilde{\chi}(t_0)\frac{1}{|\det(I - (\mathrm{d}b)_0)|}\int \psi(x_1, 0)\mathrm{d}x_1.$$

We have $w_*X = \partial_{w_1}$ and $(\mathrm{d}b)_0 = (\mathrm{d}\widetilde{\phi^{-t}})_0|_{w_1=0}$ so $(\mathrm{d}b)_0$ is conjugated to P_γ. This concludes the proof of the lemma, since $j^*\left((w_*K) * (\delta_0^1 \otimes \rho_{\overline{\varepsilon}} \otimes \delta_0^d)\right) \underset{\overline{\varepsilon}\to 0}{\longrightarrow} j^*w_*K$ in $\mathcal{D}'((t_0-\delta, t_0+\delta)\times W)$. Indeed, convolution preserves the spaces $\mathcal{D}'_\Gamma$; moreover we can take the limit $\overline{\varepsilon} \to 0$ in an arbitrary order (of the ε_i) because of the formula $\langle u*(\otimes_j\rho_j), \otimes_j\psi_j\rangle = \langle u, \otimes_j(\rho_j*\psi_j)\rangle$ for $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\rho_j, \psi_j \in \mathcal{C}_c^\infty(\mathbb{R})$, where $(\otimes_j\psi_j)(x_1, \cdots, x_d) = \Pi_j\psi_j(x_j)$. □

Remark 2.12. Since K is a delta function on S, if $\psi \in \mathcal{C}_c^\infty(U)$ with U an open set satisfying $\phi^{-t}(U) \cap U = \emptyset$ for $t \in (t_0 - \delta, t_0 + \delta)$, then $\langle j^*T, \chi \otimes \psi\rangle = 0$ for every $\chi \in \mathcal{C}_c^\infty(t_0 - \delta, t_0 + \delta)$.

Now let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}_{>0})$ and $\psi \in \mathcal{C}^\infty(M)$. There exists a finite set $\gamma_1, \cdots, \gamma_N$ of periodic orbits of length less than sup supp χ. Thanks to the preceding remark, we can assume that ψ is supported in $\bigcup_{j=1}^n U_j$ with $U_j \cap U_i = \emptyset$ and $\gamma_j \subset U_j$ for all $1 \leq i, j \leq N$, and that χ is supported near the values $\{\ell(\gamma_i)\}$. Now decompose U_i in neighborhoods that satisfy the assumptions of Lemma 2.11 (up to shrinking them a little) and take a partition of unity along this decomposition to get

$$\langle j^*T, (\chi \otimes \psi)|_{(\ell(\gamma_i)-\delta_i, \ell(\gamma_i)+\delta_i)\times U_i}\rangle = \frac{\chi(\ell(\gamma_i))}{|\det(I - P_{\gamma_i})|}\int_0^{\ell(\gamma_i^\#)}\psi(\gamma_i(t))\mathrm{d}t.$$

This finally shows that for every χ, ψ, we have

$$\langle j^*T, \chi \otimes \psi\rangle = \sum_\gamma \frac{\chi(\ell(\gamma))}{|\det(I - P_\gamma)|}\int_0^{\ell(\gamma^\#)}\psi(\gamma(t))\mathrm{d}t,$$

which is exactly (2.6) for $k = 0$.

To deal with the general case, we shall take the notations of appendix B.3 and calculate the Schwartz kernel of $S_{\mathbf{T}_k}$. Take $x_0 \in M$, $t_0 > 0$ such that $\phi^{t_0}(x_0) = x_0$, and $(\mathbf{e}_j)$ a local basis of Ω_0^k on a small neighborhood U of x_0. Let $\mathbf{f} = \sum_j f_j\mathbf{e}_j \in \mathcal{C}^\infty(U, \Omega_0^k)$, $r = \dim \Omega_0^k$ and define (locally) the functions $b_{jl} : \mathcal{C}^\infty((t_0 - \delta, t_0 + \delta) \times U)$ by

$$\left(\left(\phi^{-t}\right)^*\mathbf{f}\right)(x) = \sum_{j,l}\left((\phi^{-t})^*f_j\right)(x)b_{jl}(t, x)\mathbf{e}_l(x). \tag{2.8}$$

Then for $\psi \in \mathcal{C}^\infty(U)$, we have $S_{\mathbf{T}_k}\psi(t,x) = \sum_j b_{jj}(t,x)\left((\phi^{-t})^*\psi\right)(x)$, which implies that

$$K_k = \left(\sum_j b_{jj}\right) K \tag{2.9}$$

locally in the sense of distributions, where K_k is the Schwartz kernel of $S_{\mathbf{T}_k}$. Applying the previous lemma, one obtains

$$\langle j^* S_{\mathbf{T}_k}, \chi \otimes \psi\rangle = \frac{\chi(t_0)}{|\det(I - P_\gamma)|}\int_{t_0-\delta, t_0+\delta} \psi(\phi^s(x_0))\left(\sum_j b_{jj}(t_0, \phi^s(x_0))\right) \mathrm{d}s,$$

for $\chi \in \mathcal{C}^\infty((t_0-\delta, t_0+\delta))$. On the other hand, since $\iota_X \mathbf{e}_j = 0$, (2.8) shows that

$$\sum_j b_{jj}(t_0, \phi^s(x_0)) = \operatorname{tr} \bigwedge^k \left(\left({}^T\mathrm{d}\phi^{-t_0}_{\phi^s(x_0)}\right)|_{E_s^* \oplus E_u^*(\phi^s(x_0))}\right).$$

But now for all s, $\mathrm{d}\phi^{-t_0}_{\phi^s(x_0)}|_{E_s^*\oplus E_u^*(\phi^s(x_0))}$ is conjugated to P_γ, which concludes the proof of Theorem 2.10. $\square$

For $\chi \in \mathcal{C}_0^\infty(\mathbb{R}_{>0})$ let $\mathbf{M}_\chi : \mathcal{C}^\infty(M, \Omega_0) \to \mathcal{D}'(M, \Omega_0)$ the operator defined by

$$\mathbf{M}_\chi \mathbf{f} = \int_0^\infty \chi(t)\left(\phi^{-t}\right)^* \mathbf{f}\mathrm{d}t.$$

We shall prove the following lemma which will be useful later.

Lemma 2.13. $\mathbf{M}_\chi$ *has a well defined flat trace and we have*

$$\langle \operatorname{tr}^\flat \mathbf{T}, \chi\rangle = \operatorname{tr}^\flat \mathbf{M}_\chi. \tag{2.10}$$

Proof. Fix a density vol on M and let $\mathbf{K}_\chi \in \mathcal{D}'(M \times M, \pi_1^*\Omega_0 \otimes \pi_2^*\Omega_0^*)$ be the Schartz kernel of $\mathbf{M}_\chi$ with respect to vol, where $\pi_j : M \times M \to M$ denotes the projection on the j-th factor, $j = 1, 2$. Define also the application $L : \mathcal{C}_c^\infty(\mathbb{R}_{>0}) \to \mathcal{D}'(M \times M, \pi_1^*\Omega_0 \otimes \pi_2^*\Omega_0^*)$ defined by $L(\chi) = \mathbf{K}_\chi$. Then [Hör90, Theorem 8.2.12] shows that $\mathrm{WF}(L(\chi)) \subset \{(x,\xi,y,\eta) \in T^*(M\times M) | \exists t \in \operatorname{supp}\chi, (t, 0, x, \xi, y, \eta) \in \mathrm{WF}(K_L)\}$, where K_L is the Schwartz kernel of L with respect to vol. Using vol, we identify $\mathcal{C}^\infty(M, \Omega_0)$ with $\mathcal{C}^\infty(M, \Omega_0 \otimes D)$ where D is the line bundle of densities. Take $\mathbf{f} \in \mathcal{C}^\infty(M, \Omega_0)$ and $\mathbf{g} \in \mathcal{C}^\infty(M, \Omega_0^*)$. We have by definition of the Schwartz kernels previously involved :

$$\begin{aligned}
\langle K_L, \chi \otimes \mathbf{f} \otimes \mathbf{g}\rangle &= \langle L(\chi), \mathbf{f} \otimes \mathbf{g}\rangle \\
&= \langle \mathbf{K}_\chi, \mathbf{f} \otimes \mathbf{g}\rangle \\
&= \langle \mathbf{M}_\chi \mathbf{f}, \mathbf{g}\rangle \\
&= \langle \mathbf{T}\mathbf{f}, \chi \otimes \mathbf{g}\rangle \\
&= \langle \mathbf{K}, \chi \otimes \mathbf{g} \otimes \mathbf{f}\rangle,
\end{aligned}$$

which implies (up to reordering) that $K_L = \mathbf{K}$. Now recall from the proof of Theorem 2.10 that $\mathrm{WF}(\mathbf{K}) \cap \{(t, 0, x, \xi, x, -\xi) | t > 0,\ (x, \xi) \in T^*M \setminus 0\} = \emptyset$, which implies that $\mathbf{M}_\chi$ has a well defined flat trace. Finally, (2.10) is implied by (2.7), letting $\overline{\varepsilon} \to 0$. $\square$

3 Policott-Ruelle resonances

In what follows, M is a compact manifold and X is an Anosov vector field on M. We start in §3.1 by constructing an escape funtion that decreases along the flow (Proposition 3.5). We define anisotropic Sobolev spaces in §3.2. We finish by showing in §3.3 that the resolvant $(\mathbf{P}-\lambda)^{-1}$ extends meromorphically to the whole complex plane (Theorem 3.15). We follow here [FS11].

3.1 The escape function

The purpose of this subsection is to construct an "escape function" g on the cotangent space. This escape function decreases in the direction of the flow, which will guarantee nice properties of the operator $\mathrm{e}^{Op(g)}$. Recall from appendix A.5 the definition of the fiber-radially compactified cotangent space $\overline{T}^*M$. Fix a norm $|\cdot|$ on T^*M and denote the projection $T^*M\setminus 0 \to \partial\overline{T}^*M = S^*M = \{|\xi| = 1\}$ by ρ as in section 2. The Hamiltonian vector field $\mathbf{X}$ defined in subsection 2.1 satisfies $\mathrm{d}\rho_{(x,\lambda\xi)}(\mathbf{X}(x,\lambda\xi)) = \mathrm{d}\rho_{(x,\xi)}(\mathbf{X}(x,\xi))$ for $(x,\xi)\in T^*M\setminus 0$ and $\lambda\neq 0$ so it descends to a vector field $\widetilde{\mathbf{X}}$ on S^*M. For $x\in M$ and $\xi\in T^*M\setminus 0$, we'll denote $[\xi]=\rho(\xi)$. The following lemma will be useful to define an order function m:

Lemma 3.1. *Suppose that v is a vector field on a compact manifold N, and K_u and K_s are e^{tv}-invariant compact disjoint subsets such that*

$$\begin{aligned} &\mathrm{dist}(\mathrm{e}^{tv}(x), K_s) \underset{t\to+\infty}{\longrightarrow} 0, \quad x\notin K_u \\ &\mathrm{dist}(\mathrm{e}^{tv}(x), K_u) \underset{t\to-\infty}{\longrightarrow} 0, \quad x\notin K_s. \end{aligned} \tag{3.1}$$

We moreover ask that the convergence is locally uniform in $N\setminus(K_s\cup K_u)$. Let V_s and V_u be open neighborhoods of K_s and K_u, and fix $\varepsilon>0$.
Then there exists $W_s\subset V_s$ and $W_u\subset V_s$, neighborhoods of K_s and K_u, a function $m\in\mathcal{C}^\infty(N,[0,1])$ taking the values 1 and 0 on neighborhoods of K_s and K_u, and $\eta>0$ satisfying that $v(m)\geq 0$ on N, $v(m)>\eta>0$ on $N\setminus(W_u\cup W_s)$, $v(m)\geq 1-\varepsilon$ on W_s and $v(m)\leq\varepsilon$ on W_u.

Proof. Schrinking a little V_u and V_s we can assume that $\overline{V_s}\cap\overline{V_u}=\emptyset$. For $\delta>0$ and $K\subset N$, we set $K^\delta := \{x\in N|\mathrm{dist}(x,K)<\delta\}$ the δ-neighborhood of K. We claim that there exist $\delta>0$ and $R>0$ such that $K_s^\delta\subset V_s$, $K_u^\delta\subset V_u$ and for all $t\geq R$, one has $\mathrm{e}^{tv}(V_s)\subset K_s^\delta$ and $\mathrm{e}^{-tv}(V_u)\subset K_u^\delta$. Indeed, this follows immediately from the fact that the convergences in (3.1) are locally uniform in $M\setminus(K_s\cup K_u)$. Thus, considering $\widetilde{V_s} := \bigcup_{t\geq R}\mathrm{e}^{tv}(V_s)$ and doing the same for V_u, one can assume that

$$\mathrm{e}^{tv}(V_s)\subset V_s, \quad \mathrm{e}^{-tv}(V_u)\subset V_u, \quad t\geq 0.$$

For $T > 0$ we set $W_s^T = \mathrm{e}^{Tv}(N \setminus V_u)$ and $W_u^T = \mathrm{e}^{-Tv}(N \setminus V_s)$. According to what precedes, if T is large enough, one has $W_u^T \subset V_u$ and $W_s^T \subset V_s$. Choose some $m_0 \in \mathcal{C}^\infty(N, [0,1])$ satisfying $m_0 \equiv 1$ on V_s and $m_0 \equiv 0$ on V_u. Then define $m_T \in \mathcal{C}^\infty(N)$ by

$$m_T(x) = \frac{1}{2T} \int_{-T}^{T} m_0(\mathrm{e}^{tv}(x))\mathrm{d}t, \quad x \in N.$$

We have

$$v(m_T) = \frac{1}{2T}(m_0 \circ \mathrm{e}^{Tv} - m_0 \circ \mathrm{e}^{-Tv}). \tag{3.2}$$

For $x \in N$, let $\mathcal{I}(x) = \{t \in \mathbb{R} \mid \mathrm{e}^{tv}(x) \in N \setminus (V_s \cup V_u)\}$ and $\tau(x)$ be the travel time between V_u and V_s, that is $\tau(x) = \sup \mathcal{I}(x) - \inf \mathcal{I}(x)$. One has $\tau = \sup_{x \in N} \tau(x) < \infty$, again because the convergence in the hypothesis is locally uniform. From now on, we assume $T > \tau$, which implies with (3.2) that $v(m)(x) = \frac{1}{2T} > 0$ for $x \notin W_s^T \cup W_u^T$.

For $x \in W_u^T$, one has $\mathrm{e}^{Tv}(x) \notin V_s$ thus for $t \leq T - \tau$, we have $e^{tv}(x) \in V_u$ and $m_0(e^{tv}(x)) = 0$. Therefore,

$$m(x) = \frac{1}{2T}\left(\int_{-T}^{T-\tau} \underbrace{m_0(e^{tv}(x))\mathrm{d}t}_{=0} + \int_{T-\tau}^{T} m_0(e^{tv}(x))\right) \leq \frac{\tau}{2T}.$$

Moreover, (3.2) gives $v(m_T)(x) \geq 0$ since $m_0(e^{-Tv}(x)) = 0$.

We show identically that for $x \in W_u^T$, we have $m(x) \geq 1 - \frac{\tau}{2T}$ and $v(m_T)(x) \geq 0$. Thus, for T chosen large enough so that $\frac{\tau}{2T} < \varepsilon$, we get the desired objects with $m = m_T$, $W_u = W_u^T$, $W_s = W_s^T$ and $\eta = \frac{1}{2T}$. □

Now we are in position to make explicit the construction of our order function m. In what follows, if C is a conical subset of T^*M, its projection on S^*M will be denoted by $\widetilde{C}$.

Lemma 3.2 (The order function)**.** *Fix $u < 0 \leq n_0 < s$. We can find arbitrarily small conical neighborhoods Γ_u, Γ_0 and Γ_s of E_u^*, E_0^* and E_s^* in T^*M and a smooth order function $m \in \mathcal{C}^\infty(T^*M, [u, s])$ such that for $|\xi| \geq 1$, $m(x,\xi)$ depends only of $[\xi] \in S^*M$ and :*

(i) *m is equal to u (resp. s and n_0) near $\widetilde{E_u^*}$ (resp. $\widetilde{E_s^*}$ and $\widetilde{E_0^*}$),*

(ii) *$\mathbf{X}(m) \leq 0$ globally,*

(iii) *$\mathbf{X}(m)(x,\xi) < -\eta c_m$ for $(x,\xi) \in S^*M \setminus (\Gamma_0 \cup \Gamma_s \cup \Gamma_u)$, where $\eta > 0$ does not depend on u, n_0, s, and $c_m = \min(s - n_0, n_0 - u)$.*

(iv) *$m > s/2$ (resp. $m < u/2$) on $\widetilde{\Gamma_s}$ (resp. $\widetilde{\Gamma_u}$).*

Proof. Let $N = S^*M$ and $v = \widetilde{\mathbf{X}}$. Fix $\varepsilon > 0$ and construct $\widetilde{m_1}$ and $\widetilde{m_2}$ functions on N by applying lemma 6.2 to the two following situations :

1. $K_u^1 = \widetilde{E_s^*}$, $K_s^1 = \widetilde{E_u^* \oplus E_0^*}$.
2. $K_u^2 = \widetilde{E_s^* \oplus E_0^*}$, $K_s^2 = \widetilde{E_u^*}$.

Lemma 2.4 and (2.1) guarantees the locally uniform convergence required to apply lemma 6.2. Indeed, for example in situation 1, let K be a compact subset of $S^*M \setminus (K_u^1 \cup K_s^1)$. There exists $\alpha > 0$ such that

$$K \subset \{[\xi] \in S^*M \; |\alpha^{-1}|\xi_s| \leq |\xi_0 + \xi_u| \leq \alpha|\xi_s|\}.$$

Now (2.1) gives

$$|\Phi^{-t}(\xi_s)| \geq C^{-1}\mathrm{e}^{\theta t}|\xi_s| \geq C^{-1}\mathrm{e}^{\theta t}\alpha^{-1}|\xi_0 + \xi_u|, \quad [\xi] \in K, \quad t > 0.$$

Again using (2.1) and the fact that E_0^* and E_u^* are in direct sum we get $c > 0$ such that

$$|\xi_0 + \xi_u| \geq c|\Phi^{-t}(\xi_0) + \Phi^{-t}(\xi_u)|, \quad t > 0.$$

We therefore obtain

$$|\Phi^{-t}(\xi_s)| \geq C^{-1}\mathrm{e}^{\theta t}\alpha^{-1}c|\Phi^{-t}(\xi_0) + \Phi^{-t}(\xi_u)|, \quad t > 0,$$

which gives the locally uniform convergence towards K_u^1 in backward times ; this is exactly the same in the future.

For $j = 1, 2$, we thus obtain arbitrarily small neighborhoods W_u^j and W_s^j of K_u^j and K_s^j, a function $\widetilde{m_j}$ and a constant η_j such that $\widetilde{m_j} < \varepsilon$ on W_u^j, $\widetilde{m_j} > 1 - \varepsilon$ on W_s^j, $\widetilde{\mathbf{X}}(\widetilde{m_j}) > \eta_j$ on $S^*M \setminus (W_u^j \cup W_s^j)$ and $\widetilde{\mathbf{X}}(\widetilde{m_j}) \geq 0$ globally.
We then define

$$\widetilde{m} = s + (n_0 - s)\widetilde{m_1} + (u - n_0)\widetilde{m_2},$$

and we choose $m \in \mathcal{C}^\infty(T^*M)$ such that $m(x, \xi) = \widetilde{m}(x, [\xi])$ for $|\xi| \geq 1$ and $m(x, \xi) = 0$ for $|\xi| \leq 1/2$ (for example by letting $m(x, \xi) = \widetilde{m}(x, [\xi])\chi(|\xi|)$ where $\chi \in \mathcal{C}^\infty(\mathbb{R}_{>0})$ satisfies $\chi \equiv 1$ on $[1, +\infty[$ and $\chi \equiv 0$ on $(0, 1/2)$). Since $\widetilde{m}$ takes the values s, n_0 and u on $\widetilde{E_s^*}$, $\widetilde{E_0^*}$ and $\widetilde{E_u^*}$, m satisfies the point (i). The point (ii) comes from the fact that $\widetilde{\mathbf{X}}(\widetilde{m}) = (n_0 - s)\widetilde{\mathbf{X}}(\widetilde{m_1}) + (u - n_0)\widetilde{\mathbf{X}}(\widetilde{m_2}) \leq 0$.
Now let

$$\widetilde{\Gamma_s} = W_u^1 \cap W_u^2, \quad \widetilde{\Gamma_0} = W_s^1 \cap W_u^2, \quad \text{and} \quad \widetilde{\Gamma_u} = W_s^1 \cap W_s^2.$$

Let $\eta = \min(\eta_1, \eta_2)$. On $S^*M \setminus (\widetilde{\Gamma_0} \cup \widetilde{\Gamma_s} \cup \widetilde{\Gamma_u})$, we have $\widetilde{\mathbf{X}}(\widetilde{m_1}) > \eta_1$ or $\widetilde{\mathbf{X}}(\widetilde{m_2}) > \eta_2$. Using $\widetilde{\mathbf{X}}(\widetilde{m}) = (n_0 - s)\widetilde{\mathbf{X}}(\widetilde{m_1}) + (u - n_0)\widetilde{\mathbf{X}}(\widetilde{m_2})$, one gets $\widetilde{\mathbf{X}}(\widetilde{m}) < -\eta c_m$ where c_m is defined above, so m satisfies (iii) : indeed, one has $\mathbf{X}(m)(x, \xi) = \widetilde{\mathbf{X}}(\widetilde{m})(x, [\xi])$ for $|\xi| \geq 1$. Now, since $\widetilde{m_j} < \varepsilon$ on W_u^j and $\widetilde{m_j} > 1 - \varepsilon$ on W_s^j for $j = 1, 2$, we obtain (iv) by letting ε small enough. $\square$

We now introduce a norm f decreasing (resp. increasing) strictly along the flow near the source E_s^* (resp. the sink E_u^*).

Lemma 3.3. *For small enough conical neighborhoods Γ_s, Γ_u and Γ_0 as in the previous lemma, there exists $C > 0$ and $f \in \mathcal{C}^\infty(T^*M \setminus 0, \mathbb{R}_{>0})$, homogeneous of degree 1, positive everywhere, such that*

$$\mathbf{X}(f) < -Cf \text{ on } \Gamma_s, \quad \mathbf{X}(f) > Cf \text{ on } \Gamma_u, \quad \text{and} \quad \mathbf{X}(f) = 0 \text{ on } \Gamma_0.$$

Proof. By estimates (2.1), there exists small conical neighborhoods $\Gamma_s \subset V_s$ and $\Gamma_u \subset V_u$ of E_s^* and E_u^*, and $T > 0$ such that :

$$|\Phi^t(x,\xi)| \leq \frac{1}{2}|\xi|, \quad (x,\xi) \in V_s, \quad t \geq T$$
$$|\Phi^{-t}(x,\xi)| \leq \frac{1}{2}|\xi|, \quad (x,\xi) \in V_u, \quad t \leq T,$$

where Φ^t is the flow of $\mathbf{X}$ on T^*M. Let $f_1 \in \mathcal{C}^\infty(T^*M \setminus 0, \mathbb{R}_{>0})$ defined as follows :

$$f_1(x,\xi) = \int_0^T |\Phi^s(x,\xi)| \mathrm{d}s, \quad (x,\xi) \in T^*M.$$

Then f_1 is an homogeneous function of degree 1 and there is $c > 0$ such that $c^{-1}|\xi| \leq f_1(x,\xi) \leq c|\xi|$ for all (x,ξ). We have that $\mathbf{X}(f_1)(x,\xi) = |\Phi^T(x,\xi)| - |\xi|$ which implies $\mathbf{X}(f_1)(x,\xi) \leq -\frac{1}{2}|\xi| \leq -\frac{c^{-1}}{2} f_1(x,\xi)$ for $(x,\xi) \in V_s$ and similarly $\mathbf{X}(f_1)(x,\xi) \geq \frac{c^{-1}}{2} f_1(x,\xi)$ for $(x,\xi) \in V_u$. Now choose f an homogenous function of degree 1 such that

$$f \equiv f_1 \text{ on } \Gamma_s \cup \Gamma_u, \quad f \equiv H \text{ on } \Gamma_0,$$

where H is the Hamiltonian that $\mathbf{X}$ is derived from. Then f satisfies the requirements of the lemma with $C = \frac{c^{-1}}{2}$, since $\mathbf{X}(H) = 0$. □

Definition 3.4 (The escape function)**.** Let m be an order function as in lemma 3.2 and f as in the previous lemma. Let

$$g_m = m \log\langle f \rangle.$$

The function g_m is called the escape function.

Proposition 3.5 (Uniform decreasing of the escape function outside Γ_0)**.** *The escape function, subordinate to a choice of vicinities $\Gamma_s, \Gamma_u, \Gamma_0$ and a choice of s, u, n_0, satisfies that $\mathbf{X}(g_m) \leq 0$ everywhere. Moreover, there is $R > 0$ such that*

$$\mathbf{X}(g_m)(x,\xi) \leq -C_m, \quad (x,\xi) \notin \Gamma_0, \quad |\xi| \geq R,$$

with $C_m = c\min(|u|, s)$ for some $c > 0$ independent of s, u, n_0.

Proof. We have $\mathbf{X}(g_m) = \mathbf{X}(m)\log\langle f\rangle + m\mathbf{X}\left(\log\langle f\rangle\right)$.

On $\mathbf{\Gamma_s \cup \Gamma_u}$. Since $\mathbf{X}\left(\log\langle f\rangle\right) = \frac{f}{\langle f\rangle^2}\mathbf{X}(f)$, one gets that $\mathbf{X}(\log\langle f\rangle) \leq -C\frac{f^2}{\langle f\rangle^2}$ on Γ_s by properties of f. Because $m > s/2$ on Γ_s, we obtain $m(x,\xi)\mathbf{X}(\log\langle f\rangle)(x,\xi) \leq -\widetilde{C}s$ for $|\xi|$ large enough and $(x,\xi) \in \Gamma_s$. Similarly $m(x,\xi)\mathbf{X}(\log\langle f\rangle)(x,\xi) \leq -\widetilde{C}|u|$ for $|\xi|$ large enough and $(x,\xi) \in \Gamma_u$. Since $\mathbf{X}(m) \leq 0$ and $\log\langle f\rangle \geq 0$, we have

$$\mathbf{X}(g_m)(x,\xi) \leq -\widetilde{C}\min(|u|,s)$$

for $|\xi|$ large enough and $(x,\xi) \in \Gamma_s \cup \Gamma_u$.

Outside $\mathbf{\Gamma_s \cup \Gamma_u \cup \Gamma_0}$. One has $\mathbf{X}(m) \leq -\eta c_m$. Moreover, $m\mathbf{X}(\log\langle f\rangle)$ is globally bounded. Therefore, for $|\xi|$ large enough, one has $\mathbf{X}(g_m)(x,\xi) \leq -\eta c_m \log\langle f\rangle(x,\xi) + C \leq -\widetilde{C}\min(|u|,s)$.

On $\mathbf{\Gamma_0}$. We have $\mathbf{X}\langle f\rangle = 0$, since $f(x,\xi) = H(x,\xi)$ for $(x,\xi) \in \Gamma_0$. Therefore, $\mathbf{X}(g_m) = \mathbf{X}(m)\log\langle f\rangle \leq 0$. The constant $\tilde{C}$ involved is independent of s, u, n_0, which completes the proof with $c = \tilde{C}$. $\square$

Remark 3.6. We can rescale g_m and consider $g_m^\delta(x,\xi) = g_m(x,\xi/\delta)$; we have $\mathbf{X}(g_m^\delta)(x,\xi) = \mathbf{X}(g_m)(x,\xi/\delta)$ so g_m^δ satisfies the properties announced in the previous proposition, with δR instead of R (the constant C_m remains identical).

3.2 Anisotropic Sobolev spaces

Since the order function defined in the previous subsection is homogeneous of degree 0, one has $m \in S^0(M)$. The class $S^m_\rho(M)$ is thus well defined (recall from appendix A the definition of the classes S^m_ρ for $m \in S^0(M)$). Let $a_m = \mathrm{e}^{g_m} = \langle f\rangle^m$. We have the following lemma :

Lemma 3.7. *The symbol a_m belongs to the class $S^m_\rho(M)$ for every $\rho < 1$ (we will write $a_m \in S^m_{1-}$ for short), and the symbol g_m lies in $S^\varepsilon(M)$ for all $\varepsilon > 0$.*

Proof. We follow here [FRS08, Lemma 6]. We work in a coordinate chart U and identify $T^*M|_U$ with $U \times \mathbb{R}^n$. We write $f(x,\xi) = c(x,\xi)|\xi|$ with some $c \in \mathcal{C}^\infty(U \times \mathbb{R}^n \setminus U \times 0)$ homogeneous of degree 0. We denote a_m by p for simplicity. We proceed by induction on $|\alpha+\beta|$ to prove the estimates (A.2) ; more precisely we will show that

$$(\partial_\xi^\alpha \partial_x^\beta p)(x,\xi) = q(x,\xi)\langle f\rangle^{m(x,\xi)}, \tag{3.3}$$

with $q \in S_\rho^{-\rho|\alpha|+(1-\rho)|\beta|}$ for all $\rho < 1$. We will use the following

Fact : $\log\langle f\rangle \in S^\varepsilon$ for every $\varepsilon > 0$.

Proof. Indeed for $\tau \in \mathbb{R}$ let $g(x,\tau,\xi) = \sqrt{\tau^2 + f^2(x,\xi)}$. Then g is homogeneous of degree 1, so $\partial_\xi^\alpha \partial_x^\beta g$ is homogeneous of degree $1-|\alpha|$. Taking $\tau = 1$, we obtain that $\partial_\xi^\alpha \partial_x^\beta \langle f \rangle$ lies in $S^{1-|\alpha|}$. Therefore, whenever $(\alpha,\beta) \neq (0,0)$, $\partial_\xi^\alpha \partial_x^\beta \log\langle f \rangle$ lies in $S^{-|\alpha|}$ since the log disappear as soon as we differentiate. Since $\log\langle f \rangle$ is dominated by $\langle \xi \rangle^\varepsilon$ for all $\varepsilon > 0$, we get $\log\langle f \rangle \in S^\varepsilon$ for all $\varepsilon > 0$, which proves the fact. □

Let us now prove (3.3). It is obvious for $|\alpha + \beta| = 0$. Suppose $|\alpha + \beta| = 1$. If $|\alpha| = 1$ and $|\beta| = 0$, then for some $1 \leq i \leq n$, we have

$$\partial_\xi^\alpha \partial_x^\beta p = \partial_{\xi_i} p = \left((\partial_{\xi_i} m) \log\langle f \rangle + m \partial_{\xi_i}(\log\langle f \rangle)\right) \langle f \rangle^m.$$

Let $\rho \in (0,1)$. Since m is homogeneous of degree 0, one has $\partial_{\xi_i} m \in S_\rho^{-1}$. Using the fact, we get that $\partial_{\xi_i} m \log\langle f \rangle$ lies in $S_\rho^{-1} S^{1-\rho} \subset S_\rho^{-1+1-\rho} = S_\rho^{-\rho} = S^{-\rho|\alpha|+(1-\rho)|\beta|}$. Moreover by the fact, $\partial_{\xi_i} \log\langle f \rangle \in S^{-1}$. Since m is in S^0, this implies $m\partial_{\xi_i}(\log\langle f \rangle) \in S^{-1} \subset S_\rho^{-\rho}$, which proves (3.3).
If $|\alpha| = 0$ and $|\beta| = 1$, we have for some $1 \leq i \leq n$:

$$\partial_\xi^\alpha \partial_x^\beta p = \partial_{x_i} p = \left((\partial_{x_i} m) \log\langle f \rangle + m \partial_{x_i}(\log\langle f \rangle)\right) \langle f \rangle^m.$$

Since m is homogeneous of degree 0 we have $\partial_{x_i} m \in S^0$. Again, $\log\langle f \rangle \in S^{1-\rho}$ and $\partial_{x_i} m) \log\langle f \rangle \in S^{0+1-\rho} \subset S_\rho^{1-\rho}$. As before, we have $m\partial_{x_i}(\log\langle f \rangle) \in S_\rho^{1-\rho}$.
Let us now treat the induction step. Suppose (3.3) is true for every (α,β) with $|\alpha + \beta| \leq N$. Suppose now $|\alpha + \beta| = N + 1$.
If $(\alpha,\beta) = (\tilde{\alpha},\beta) + (a,0)$ with $|a| = 1$, let us write $(\partial_\xi^\alpha \partial_x^\beta p)(x,\xi) = (\partial_\xi^a \partial_\xi^{\tilde{\alpha}} \partial_x^\beta p)(x,\xi) = \partial_\xi^a \left(q(x,\xi)\langle f \rangle^{m(x,\xi)}\right)$ for some $q \in S_\rho^{-\rho|\tilde{\alpha}|+(1-\rho)|\beta|}$. Therefore,

$$\partial_\xi^\alpha \partial_x^\beta p = ((\partial_\xi^a q) \log\langle f \rangle + q \partial_\xi^a \log\langle f \rangle)\langle f \rangle^m.$$

Using the fact and the assumption $q \in S_\rho^{-\rho|\tilde{\alpha}|+(1-\rho)|\beta|}$, we get as in the case $|\alpha + \beta| = 1$ that $(\partial_\xi^a q) \log\langle f \rangle + q\partial_\xi^a \log\langle f \rangle$ lies in $S_\rho^{-\rho|\alpha|+(1-\rho)|\beta|}$.
The case $(\alpha,\beta) = (\alpha,\tilde{\beta}) + (0,b)$ with $|b| = 1$ is similar. This proves the estimates 3.3 and the first part of the lemma. The second part is very similar and we leave it as an exercise. □

Remark 3.8. We have $S_{1-}^m(M) \subset S^{m+}(M) := \bigcap_{\varepsilon>0} S^{m+\varepsilon}(M)$. Indeed, if $\varepsilon > 0$ and $\alpha,\beta \in \mathbb{N}^n$, then for ρ close enough to 1 so that $(1-\rho)\max(|\alpha|,|\beta|) < \varepsilon/2$, one has $\langle \xi \rangle^{m(x,\xi)-\rho|\alpha|+(1-\rho)|\beta|} \leq \langle \xi \rangle^{m(x,\xi)+\epsilon-|\alpha|}$.

Choose an operator G_m such that $\sigma(G_m)(x,\xi) = g_m(x,\xi) \mod S^{-1+}(M)$. We have $\mathrm{e}^{\pm G_m} \in \Psi^s(M)$ (see appendix A.4 for exponentiation of ΨDO's). We can now give the

Definition 3.9 (The anisotropic Sobolev space H_{G_m})**.** The anisotropic Sobolev space subordinated to m is

$$H_{G_m} = \mathrm{e}^{-G_m}\left(L^2(M)\right) \subset \mathcal{D}'(M).$$

Those spaces provide the right regularity (that is, we ask smoothness in every direction except near the unstable ones) to study the spectrum of the operator $\mathbf{P} = -i\mathcal{L}_X$ acting on $\mathcal{D}'(M, \Omega_0)$ by duality. If $H^s(M)$ and $H^u(M)$ are usual Sobolev spaces, one has

$$H^s(M) \subset H_{G_m} \subset H^u(M).$$

Indeed $s \geq m \geq u$; therefore $\langle \xi \rangle^{-s} \langle f \rangle^m \in S^{-s} S_\rho^m \subset S_\rho^{-s+m} \subset S_\rho^0$ for every $\rho < 1$ thanks to Lemma 3.7. As a consequence $(I - \Delta)^{-s/2} \, \mathrm{e}^{G_m}$ is bounded on L^2 and thus for some $R \in \Psi^{-\infty}(M)$, we have for $u \in \mathcal{C}^\infty(M)$:

$$\|\mathrm{e}^{G_m} u\|_{L^2} \leq \| \, (I - \Delta)^{s/2} \, (I - \Delta)^{-s/2} \, \mathrm{e}^{G_m} u\|_{L^2} + \|R\mathrm{e}^{G_m} u\|_{L^2} \leq C\|u\|_{H^s}.$$

3.3 Ruelle-Pollicott resonances

Let us now study the spectrum of the restriction of the closed operator $\mathbf{P} = -i\mathcal{L}_X$ to the anisotropic Sobolev space $H_{G_m}(M, \Omega_0)$ (that is, the space of currents of regularity H_{G_m}). Let

$$\mathbf{Q} = \mathrm{e}^{G_m} \mathbf{P} \mathrm{e}^{-G_m}$$

be the conjugated operator to $\mathbf{P}$, acting on $L^2(M, \Omega_0)$. The following lemma gives us the symbol of $\mathbf{Q}$:

Lemma 3.10. *The principal symbol of* $\mathbf{Q}$ *is diagonal and is given by*

$$q(x, \xi) = p(x, \xi) + i\mathbf{X}(g_m)(x, \xi) + \mathcal{O}_m(S^{-1+}(M)),$$

where p *is the symbol of* $\mathbf{P}$ *in the given chart. The notation* $\mathcal{O}_m$ *is to keep in mind that the rest depends on* m*.*

Proof. Use lemma A.11 to get

$$\mathbf{Q} = \mathbf{P} + [G_m, \mathbf{P}] + \left[G_m, \int_0^1 (\mathbf{P}_t - \mathbf{P}) \mathrm{d}t \right],$$

where $\mathbf{P}_t = \mathrm{e}^{tG_m} \mathbf{P} \mathrm{e}^{-tG_m}$. Now, using Proposition A.3 and the fact that $\mathbf{P}$ is diagonal with principal symbol H, we have $\sigma\left([G_m, \mathbf{P}]\right) = \frac{1}{i}\{\sigma(G_m), \sigma(P)\} = i\mathbf{X}(G_m)$ since $\mathbf{X}$ is the Hamiltonian vector field defined by H. Now, $\mathbf{P}_t - \mathbf{P} = \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} \mathbf{P}_\tau$ lies in $\Psi^{0+}(M)$ since it is shown in the proof of lemma A.11 that $\frac{\mathrm{d}}{\mathrm{d}\tau} \mathbf{P}_\tau = [G_m, \mathbf{P}_\tau]$ and $G_m \in \Psi^{0+}(M)$, $\mathbf{P}_\tau \in \Psi^1(M)$. Therefore, the bracket $\left[G_m, \int_0^1 (\mathbf{P}_t - \mathbf{P}) \mathrm{d}t \right]$ lies in $\Psi^{-1+}(M)$, which concludes. □

We shall now be interested in the spectrum of the operator $\mathbf{Q}$ which is a closed operator on $L^2(M)$ of domain $\mathcal{D}(\mathbf{Q}) = \{\mathbf{u} \in L^2(M, \Omega_0) | \mathbf{Qu} \in L^2(M, \Omega_0)\}$. We will write

$$\mathbf{Q} = \mathbf{Q}_1 + i\mathbf{Q}_2$$

with $\mathbf{Q}_1 = \frac{1}{2}(\mathbf{Q}+\mathbf{Q}^*)$ and $\mathbf{Q}_2 = \frac{1}{2i}(\mathbf{Q}-\mathbf{Q}^*)$. In what follows, we fix some inner product $\langle\ ,\ \rangle$ on the fibers of Ω_0, which gives an inner product, still denoted by $\langle\ ,\ \rangle$, on $L^2(M,\Omega_0)$. We have the following

Lemma 3.11. *There exists $C_0>0$ such that for every complex number λ with $\Im(\lambda)>C_0$, the resolvent $\mathbf{R}_\lambda=(\mathbf{P}-\lambda)^{-1}$ exists.*

Proof. Lemma 3.10 gives us the expression

$$\sigma(\mathbf{Q}_2)(x,\xi)=q_2(x,\xi)=\mathbf{X}(g_m)(x,\xi)+\mathcal{O}(S^0)+\mathcal{O}_m(S^{-1+}), \tag{3.4}$$

since $\sigma(\mathbf{P})(x,\xi)=H(x,\xi)$ is real. By Proposition 3.5, one has $C_1>0$ such that $\Re(q_2(x,\xi))\le C_1$ for all (x,ξ). Since $q_2\in S^{0+}\subset S^1$, the sharp Gårding inequality (Theorem A.22) gives us a constant $C_2>0$ such that for $\mathbf{u}\in\mathcal{C}^\infty(M,\Omega_0)$, $\langle(\mathbf{Q}_2-C_1)\mathbf{u}|\mathbf{u}\rangle\le C_2||\mathbf{u}||^2$ and therefore with $C_0=C_1+C_2$ we get

$$\langle(\mathbf{Q}_2-C_0)\mathbf{u}|\mathbf{u}\rangle\le 0,\quad \mathbf{u}\in\mathcal{C}^\infty(M,\Omega_0).$$

Now let $\Im(\lambda)>C_0$ and let $\varepsilon=\Im(\lambda)-C_0$. We claim that for $\mathbf{u}\in\mathcal{C}^\infty(M,\Omega_0)$, we have $||(\mathbf{Q}-\lambda)\mathbf{u}||\ge\varepsilon||\mathbf{u}||$. Indeed, one has $\Im\langle(\mathbf{Q}-\lambda)\mathbf{u}|\mathbf{u}\rangle=\langle(\mathbf{Q}_2-C_0)\mathbf{u}|\mathbf{u}\rangle-\epsilon||\mathbf{u}||^2\le-\epsilon||\mathbf{u}||^2$ so Cauchy-Schwarz inequality gives the claim.
We have proved that $\mathbf{Q}-\lambda$ is injective by density of $\mathcal{C}^\infty(M,\Omega_0)$ in $\mathcal{D}(\mathbf{Q}-\lambda)$. We can do exactly the same with the adjoint (with respect to any inner product on the fibers) to show that $\mathbf{Q}^*=\mathbf{Q}_1-i\mathbf{Q}_2$ to show that $\mathbf{Q}^*-\bar\lambda$ is also injective. Therefore, if $\mathbf{u}$ is orthogonal to $\mathrm{Ran}(\mathbf{Q}-\lambda)$, then $\mathbf{u}\in\mathrm{Ker}(\mathbf{Q}^*-\bar\lambda)=0$. We thus obtain $\overline{\mathrm{Ran}(\mathbf{Q}-\lambda)}=L^2(M,\Omega_0)$. We then use lemma B.3 to conclude that $\mathbf{Q}-\lambda$ is surjective. □

We are now in position to prove the

Proposition 3.12. *There exists a constant C independent of m such that $\mathbf{Q}$ has discrete spectrum in the region $\{\Im(\lambda)>C-C_m\}$, where C_m is the constant in Proposition 3.5.*

Proof. The idea is to construct a perturbation $\hat\chi$ such that $\mathbf{Q}-i\hat\chi-\lambda$ is invertible in the region $\Im(\lambda)>C-C_m$. First, let $\chi_0\in S^0(M)$ such that $\chi_0^2\equiv C_m>0$ on $\Gamma_0\cap\{|\xi|\ge R\}$, where R and Γ_0 are defined in the previous subsection (Proposition 3.5). We thus have $\mathbf{X}(g_m)(x,\xi)-\chi_0^2(x,\xi)\le -C_m$ for $|\xi|\ge R$. Equation (3.4) thus implies $\Re(q_2(x,\xi))-\chi_0^2(x,\xi)\le -C_m+C+\mathcal{O}_m(S^{-1+})$, where C comes from the term $\mathcal{O}(S^0)$ in (3.4). Since this symbol lies in S^μ for all $\mu>0$, we can apply sharp Gårding's inequality (Theorem A.22) to get

$$\langle(\mathbf{Q}_2-\hat\chi_0^*\hat\chi_0+C_m-C)\,\mathbf{u}|\mathbf{u}\rangle\le C_\mu||\mathbf{u}||_{H^{\frac{\mu-1}{2}}},\quad \mathbf{u}\in\mathcal{C}^\infty(M,\Omega_0),$$

where $\hat\chi_0=\mathrm{Op}(\chi_0)$ for some $C_\mu>0$; the term $\mathcal{O}_m(S^{-1+})$ has been absorbed by the term $C_\mu||\mathbf{u}||_{H^{\frac{\mu-1}{2}}}$. Writing $\chi_1(x,\xi)=C_\mu^{1/2}\langle\xi\rangle^{(\mu-1)/2}\in S^{(\mu-1)/2}(M)$, we get $C_\mu ||\mathbf{u}||_{H^{\frac{\mu-1}{2}}} = \Im(i\hat{\chi}_1^*\hat{\chi}_1\mathbf{u}|\mathbf{u})$, where $\mathrm{Op}(\hat{\chi}_1) = \chi_1$. Noting $\hat{\chi} = \hat{\chi}_0^*\hat{\chi}_0 + \hat{\chi}_1^*\hat{\chi}_1$, we get

$$\langle (\mathbf{Q}_2 - \hat{\chi} + C_m - C)\,\mathbf{u}|\mathbf{u}\rangle \leq 0, \quad \mathbf{u} \in \mathcal{C}^\infty(M, \Omega_0).$$

As in the previous lemma, we obtain that the resolvant of $(\mathbf{Q} - i\hat{\chi} - \lambda)^{-1}$ exists for $\Im(\lambda) > C - C_m$. We now prove the following

Lemma 3.13. *For $\Im(\lambda) > C - C_m$, the operator $\hat{\chi}\,(\mathbf{Q} - i\hat{\chi} - \lambda)^{-1}$ is compact.*

Proof. Denote $\mathbf{Q} - i\hat{\chi}$ by $\widetilde{\mathbf{Q}}$. Since $\sigma(\hat{\chi}) \in S^0(M)$, we have thanks to Lemma 3.10 that $\mathbf{Q}$ is elliptic of order 1 on Γ_0. Therefore $\widetilde{\mathbf{Q}} - \lambda$ is also elliptic on Γ_0 (they have the same principal symbol) and we can find (Theorem A.5) $B \in \Psi^{-1}(M)$ such that

$$(\widetilde{\mathbf{Q}} - \lambda)B = R + L,$$

with $R = \mathrm{Op}(r)$, $r \in S^0$, supp $(\chi_0) \subset$ supp (r) (we can reduce the support of χ_0 and this does not affect the ellipticity of $\widetilde{\mathbf{Q}}$ since $\chi_0 \in S^0(M)$), $r \equiv 1$ near supp (χ_0), and $L \in \Psi^{-\infty}(M)$. Write

$$\hat{\chi}\left(\widetilde{\mathbf{Q}} - \lambda\right)^{-1} = \hat{\chi}B + \hat{\chi}(1 - R)\left(\widetilde{\mathbf{Q}} - \lambda\right)^{-1} - \hat{\chi}L\left(\widetilde{\mathbf{Q}} - \lambda\right)^{-1}.$$

L is smoothing thus $\hat{\chi}L\left(\widetilde{\mathbf{Q}} - \lambda\right)^{-1}$ is compact. B lies in $\Psi^{-1}(M)$ and is thus compact on L^2, so is $\hat{\chi}B$ because χ is bounded on L^2. Finally, $\hat{\chi}(1 - R) = \hat{\chi}_0^*\hat{\chi}_0(1 - R) + \hat{\chi}_1^*\hat{\chi}_1(1 - R)$ is also compact because $\hat{\chi}_1^*\hat{\chi}_1 \in \Psi^{\mu-1}(M)$ and supp $(1 - r) \cap$ supp $(\chi_0) = \emptyset$. This concludes the proof of the lemma. □

We can now finish the proof of the Proposition. Indeed, write

$$\mathbf{Q} - \lambda = \left(1 + i\hat{\chi}(\widetilde{\mathbf{Q}} - \lambda)^{-1}\right)\left(\widetilde{\mathbf{Q}} - \lambda\right).$$

By what precedes, $1 + i\hat{\chi}(\widetilde{\mathbf{Q}} - \lambda)^{-1}$ is Fredholm of index 0. Moreover, $\left(\widetilde{\mathbf{Q}} - \lambda\right)$ is Fredholm of index 0 also since it is a bijective operator $\mathcal{D}(\mathbf{Q}) \to L^2(M, \Omega_0)$ with bounded inverse. It directly implies that $\mathbf{Q} - \lambda$ is a holomorphic family of Fredholm operators of index 0 and invertible for $\Im(\lambda) > C - C_m$. By the analytic Fredholm continuation (see Theorem B.1), $\mathbf{Q}$ has discrete spectrum on $\Im(\lambda) > C - C_m$ with finite multiplicities. □

By conjugation, we obtained that $\mathbf{P}$ has discrete spectrum on $\Im(\lambda) > C - C_m$. Let us now prove that the poles of the resolvent $\mathbf{R}_\lambda = (\mathbf{P} - \lambda)^{-1}$ are intrinsic to the vector field X. We have have showed that the resolvent $\mathbf{R}_\lambda = (\mathbf{P} - \lambda)^{-1} : H_{G_m}(M, \Omega_0) \to H_{G_m}(M, \Omega_0)$ well defined for $\Im(\lambda) > C_0$ has a meromorphic continuation to $\Im(\lambda) > C - C_m$. In fact, we have a formula for $\mathbf{R}_\lambda$:

Proposition 3.14. *For $\Im(\lambda)$ big enough (depending on m, that is, depending on s and u) we have the following formula*

$$\mathbf{R}_\lambda = i\int_0^\infty \mathrm{e}^{i\lambda t}\left(\phi^{-t}\right)^* \mathrm{d}t, \tag{3.5}$$

where $(\phi^{-t})^*$ is the pull-back $\mathcal{C}^\infty(M,\Omega_0)\to\mathcal{C}^\infty(M,\Omega_0)$ and the integral converges in $\mathcal{L}(H^s(M),H^s(M))$ and in $\mathcal{L}(H^u(M),H^u(M))$.

Proof. The derivatives of ϕ^{-t} grows at most exponentially with t so the integral at the right hand side of (3.5) converges in operator norms $H^s(M)\to H^s(M)$ and $H^u(M)\to H^u(M)$ if $\Im(\lambda)$ is big enough. Moreover, for $\mathbf{f}\in H_{G_m}(M,\Omega_0)$, we have $\mathrm{e}^{i\lambda t}(\mathbf{P}-\lambda)\left[(\phi^{-t})^*\mathbf{f}\right] = i\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{i\lambda t}(\phi^{-t})^*\mathbf{f}\right)$. Therefore,

$$(\mathbf{P}-\lambda)\left(i\int_0^\infty \mathrm{e}^{i\lambda t}\left(\phi^{-t}\right)^*\mathbf{f}\mathrm{d}t\right) = \mathbf{f},$$

which concludes. □

Denote by $\mathbf{T}_\lambda$ the operator $i\int_0^\infty \mathrm{e}^{i\lambda t}(\phi^{-t})^*\mathrm{d}t : \mathcal{C}^\infty(M,\Omega_0)\to\mathcal{D}'(M,\Omega_0)$. Fix a density vol on M, denote by $\pi_j : M\times M\to M$ the projection on the jth factor, $j=1,2$, and let $\mathbf{K}_\lambda\in\mathcal{D}'(M\times M,\pi_1^*\Omega_0\otimes\pi_2^*\Omega_0^*)$ be the Schwartz kernel of $\mathbf{T}_\lambda$ with respect to vol. Then $\mathbf{K}_\lambda$ is a holomorphic family of distributions for $\Im(\lambda)$ big enough. Moreover, since $\mathbf{T}_\lambda$ coïncide with $\mathbf{R}_\lambda|_{\mathcal{C}^\infty(M,\Omega_0)}$ for $\Im(\lambda)\gg 0$, we obtain that the family $\mathbf{K}_\lambda$ extends to a meromorphic family of distributions on $\Im(\lambda) > C - C_m$, whose poles coïncide with those of the Schwartz kernel of $\mathbf{R}_\lambda|_{\mathcal{C}^\infty(M,\Omega_0)}$. Since $\mathcal{C}^\infty(M,\Omega_0)$ is dense in $H_{G_m}(M,\Omega_0)$ we obtain that the poles of the resolvent $\mathbf{R}_\lambda$ do not depend on the choices of the escape function g_m. Letting $|u|, s\to\infty$, we obtain the

Theorem 3.15. *The family of operators $\lambda\mapsto\mathbf{R}_\lambda$ (well defined for $\Im(\lambda)\gg 0$) viewed as operators $\mathcal{C}^\infty(M,\Omega_0)\to\mathcal{D}'(M,\Omega_0)$, has a meromorphic continuation to $\mathbb{C}$. The poles of this extension are the Ruelle resonances, and we will denote the set of those by* $\mathrm{Res}(\mathbf{P})$.

Remark 3.16. If $\mathbf{u}\in\mathcal{D}'(M,\Omega_0)$ is an eigenvector of $\mathbf{P}$, one has $\mathbf{u}\in H_{G_m}$ for any order function m with large enough $|u|, s$. Since m can be chosen big enough in any direction except in the unstable one, its wavefront set is contained in the unstable direction.

4 Microlocal structure of the resolvent

We start this section by giving standard semiclassical estimates in §4.1, namely elliptic regularity and propagation of singularities. Then we prove in §4.2 some radial estimate which allows to control regularity near radial sources ; this comes from the work of Melrose [Mel94] and Vasy [Vas13]. Finally, we describe the microlocal structure of the resolvent in §4.3, following [DGRS18] and [DZ13].

4.1 Standard semiclassical estimates

In this subsection we state two standard results about semiclassical calculus. We start with an elliptic estimate which allows to control the regularity of $\mathbf{u}$ by that of $\mathbf{Pu}$ where $\mathbf{P}$ is an elliptic operator. Recall from Appendix A.3 the definitions of h-tempered distributions. In what follows, E is a vector bundle over M.

Proposition 4.1. *Let $A \in \Psi_h^0(M)$ acting diagonally on $\mathcal{D}'_h(M,E)$. Let $\mathbf{P} \in \Psi_h^k(M, \mathrm{Hom}(E))$ be elliptic on $\mathrm{WF}_h(A)$. Then for each $\mu \in \mathbb{R}$, there exists $C > 0$ such that*

$$||A\mathbf{u}||_{H_h^\mu(M,E)} \leq C||\mathbf{Pu}||_{H_h^{\mu-k}(M,E)} + \mathcal{O}(h^\infty)||\mathbf{u}||_{H_h^{-N}(M,E)},$$

for all h-tempered family of distributions $\mathbf{u} = (\mathbf{u}_h) \in \mathcal{D}'_h(M,E)$ and all N such that the right hand side is well defined.

Corollary 4.2 (Elliptic regularity)**.** *We have for $\mathbf{u}$ h-tempered and $\mathbf{P} \in \Psi_h^k(M, \mathrm{Hom}(E))$,*

$$\mathrm{WF}_h(\mathbf{u}) \cap \mathrm{ell}_h(\mathbf{P}) \subset \mathrm{WF}_h(\mathbf{Pu}).$$

Proof. Suppose (x,ξ) lies in $\mathrm{ell}_h(\mathbf{P}) \setminus \mathrm{WF}_h(\mathbf{Pu})$. Let us show $(x,\xi) \notin \mathrm{WF}_h(\mathbf{u})$. Let U be a neighborhood of (x,ξ) such that $U \Subset \mathrm{ell}_h(\mathbf{P}) \setminus \mathrm{WF}_h(\mathbf{Pu})$. Take $B \in \Psi_h^0(M)$ such that $\mathrm{WF}_h(B) \cap \mathrm{WF}_h(\mathbf{Pu}) = \emptyset$ and $U \subset \mathrm{ell}_h(B)$. One has that $B\mathbf{P}$ is elliptic on U. Moreover, the condition on the wavefronts implies $||B\mathbf{Pu}||_{H_h^{\mu-k}(M,E)} \leq \mathcal{O}(h^\infty)||\mathbf{u}||_{H_h^{-N}(M,E)}$ for big enough N and all μ. Therefore, by Proposition 4.1, one has

$$||A\mathbf{u}||_{H_h^\mu(M,E)} \leq \mathcal{O}(h^\infty)||\mathbf{u}||_{H_h^{-N}(M,E)},$$

for all $\Psi_h^0(M)$ satisfying $\mathrm{WF}_h(A) \subset U$. This concludes the proof of the corollary. $\square$

Proposition 4.1 is a direct consequence of semiclassical microlocal inversion of pseudo-differential operators :

Proof of proposition 4.1. Apply Theorem A.21 to get $\mathbf{Q} \in \Psi_h^{-k}(M, \mathrm{Hom}(E))$ such that

$$\mathbf{QP} = \widehat{\chi} + \mathbf{L},$$

where $\sigma_h(\widehat{\chi}) = \mathrm{Id}_E$ microlocally near $\mathrm{WF}_h(A)$ and $\mathbf{L} \in h^\infty \Psi_h^{-\infty}(M, \mathrm{Hom}(E))$. Now take $\mathbf{u} \in \mathcal{D}_h'(M, E)$; we have for all N big enough that $A\mathbf{u} = A(1 - \widehat{\chi})\mathbf{u} + A\widehat{\chi}\mathbf{u} = A\mathbf{QPu} + (A(1-\widehat{\chi}) - A\mathbf{L})\,\mathbf{u}$. But $A(1-\widehat{\chi}) - A\mathbf{L}$ is $\mathcal{O}_{\Psi_h^{-\infty}}(h^\infty)$ since $\mathrm{WF}_h(A) \cap \mathrm{WF}_h(1-\widehat{\chi}) = \emptyset$. Moreover we have $||A\mathbf{QPu}||_{H^\mu(M,E)} \leq C||\mathbf{Pu}||_{H^{\mu-k}(M,E)}$, which concludes. □

Theorem 4.3. *Take $\mathbf{P} \in \Psi_h^1(M, \mathrm{Hom}(E))$. Suppose that $\sigma_h(\mathbf{P})$ is diagonal and has the form*

$$\sigma_h(\mathbf{P}) = p - iq \mod hS_h^0(M),$$

where $p \in S^1(M)$ is homogeneous of degree 1 for large enough $|\xi|$, independent of h and $q \geq 0$. Denote by $\mathrm{e}^{t\mathbf{X}_p}$ be the Hamiltonian flow associated to p.
Let $A, B, B_1 \in \Psi_h^0(M)$ such that for all $(x,\xi) \in \mathrm{WF}_h(A)$, we can find $T \geq 0$ such that $\mathrm{e}^{-T\mathbf{X}_p}(x,\xi) \in \mathrm{ell}_h(B)$ and $\mathrm{e}^{-t\mathbf{X}_p}(x,\xi) \in \mathrm{ell}_h(B_1)$ for every $t \in [0,T]$. Then for each μ, one has $C > 0$ such that

$$||A\mathbf{u}||_{H_h^\mu(M,E)} \leq C||B\mathbf{u}||_{H_h^\mu(M,E)} + Ch^{-1}||B_1\mathbf{Pu}||_{H_h^\mu(M,E)} + \mathcal{O}(h^\infty)||\mathbf{u}||_{H_h^{-N}(M,E)}, \tag{4.1}$$

for all h-tempered family of distributions $\mathbf{u} = (\mathbf{u}_h) \in \mathcal{D}_h'(M,E)$ and all N such that the right hand side is well defined.

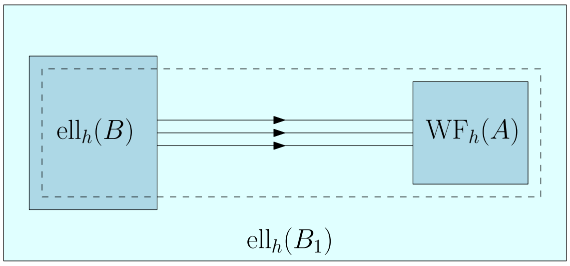

Figure 4.1. Propagation of singularities (Theorem 4.3). The dashed line is the wavefront set of the operator B_2 used in the end of the proof.

In other words, the singularities of $\mathbf{u}$ near (x,ξ) on a flow line of $\mathbf{X}_p$ can be controlled by its singularities near $\mathrm{e}^{-T\mathbf{X}_p}(x,\xi)$ and by those of $\mathbf{Pu}$ along the past flow line :

Corollary 4.4 (Propagation of semiclassical singularities)**.** *Let $\mathbf{u}$ bu an h-tempered family of distributions and $\mathbf{P}$ as in the previous proposition. Let $\gamma(t)$ be a flow line of $\mathbf{X}_p$ and $T > 0$. Assume $\gamma(-T) \notin \mathrm{WF}_h(\mathbf{u})$ and $\gamma(-t) \notin \mathrm{WF}_h(\mathbf{Pu})$ for all $t \in [0,T]$. Then $\gamma(0) \notin \mathrm{WF}_h(\mathbf{u})$.*

Proof. We take B microlocalized near $\gamma(-T)$ and B_1 microlocalized near $\gamma([-T,0])$, such that $\mathrm{WF}_h(B)\cap\mathrm{WF}_h(\mathbf{u})=\emptyset$ and $\mathrm{WF}_h(B_1)\cap\mathrm{WF}_h(\mathbf{Pu})=\emptyset$. We thus obtain that the terms $||B\mathbf{u}||_{H_h^\mu(M,E)}$ and $||B_1\mathbf{Pu}||_{H_h^\mu(M,E)}$ are $\mathcal{O}(h^\infty)||\mathbf{u}||_{H_h^{-N}}$. We conclude as in the proof of Corollary 4.2. □

The proof of the propagation of semiclassical singularities is very similar to the classical one ; it relies on a positive commutator argument induced by Garding's inequality.

Proof of Theorem 4.3. Take A,B,B_1 as in the hypothesis of the theorem. We can assume that $\mathrm{WF}_h(A)$ is contained in a small neighborhood of some $(x_0,\xi_0)\in\overline{T}^*M$. Denote by Φ^t the flow of $\mathbf{X}_p$ on $\overline{T}^*M$. Up to shrinking a little bit A, we can assume that

$$\Phi^{-T}(\mathrm{WF}_h(A))\subset\mathrm{ell}_h(B)\text{ and }\Phi^{-t}(\mathrm{WF}_h(A))\subset\mathrm{ell}_h(B_1)\text{ for all }t\in[0,T].\tag{4.2}$$

Fix an inner product on the fibers of E. This way we make $L^2(M,E)$ an Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. Set

$$\Re(\mathbf{P})=\frac{\mathbf{P}+\mathbf{P}^*}{2},\quad \Im(\mathbf{P})=\frac{\mathbf{P}-\mathbf{P}^*}{2i}.$$

We have that $\Re(\mathbf{P}),\Im(\mathbf{P})\in\Psi_h^1(M,E)$ are symmetric. We will need an escape function, given by the following

Lemma 4.5. *For all $c>0$, there exists an escape function $g\in\mathcal{C}^\infty(\overline{T}^*M)$ with* $\mathrm{supp}\, g\subset\mathrm{ell}_h(B_1)$*, $g\geq 0$ everywhere, such that*

(i) $g>0$ *near* $\mathrm{WF}_h(A)$*,*

(ii) $\mathbf{X}_p(g)\leq -cg$ *in* $\mathrm{ell}_h(B_1)\setminus\mathrm{ell}_h(B)$*.*

Proof. Take a tubular neighborhood $B(0,1)\times(-T-\delta,\delta)\subset\mathbb{R}_\theta^{2n-1}\times\mathbb{R}_\tau$ of $\{\Phi^{-t}(x_0,\xi_0),t\in[0,T]\}$ contained in $\mathrm{ell}_h(B_1)$ for small δ and $B(0,1)=\{|\theta|<1\}$, in a way so that $\mathbf{X}_p\equiv\partial_\tau$. Now take $\psi\in\mathcal{C}_c^\infty(B(0,1),[0,1])$ such that $\psi\equiv 1$ on $B(0,1/2)$ and $\chi\in\mathcal{C}_c^\infty(-T-\delta,\delta,\mathbb{R}_+)$ such that $\chi(0)>0$ and $\chi'<-c\chi$ outside $(-T-\delta,-T+\delta)$. Such a χ exists : on can take $\chi(\tau)=\chi_0(\tau)\mathrm{e}^{-c\tau}$ for some $\chi_0\in\mathcal{C}^\infty(-T-\delta,\delta)$ such that $\chi_0(0)=1$ and $\chi_0'\leq 0$ on $(-T+\delta,\delta)$. Now the function $g(\theta,\tau)=\psi(\theta)\chi(\tau)$ satisfies the requirements of the lemma. □

We now assume $B_1\mathbf{u}\in H^{\mu-1/2}(M,E)$ and $B\mathbf{Pu}\in H^\mu(M,E)$. We choose for every $\varepsilon>0$ an operator $S_\varepsilon\in\Psi_h^{\mu-\varepsilon}(M)$ with $\sigma_h(S_\varepsilon)(x,\xi)=\langle\xi\rangle^\mu\langle\varepsilon\xi\rangle^{-1}$. Now define $G_\varepsilon=S_\varepsilon G$ where $G\in\Psi_h^0(M)$ quantizes the escape function g as in the previous lemma (the constant c will be chosen later), that is $\sigma_h(G)=g$. We have of course $\sigma_h(G_\varepsilon)(x,\xi)=\langle\xi\rangle^\mu\langle\varepsilon\xi\rangle^{-1}g(x,\xi)$. Now take $\widehat{\chi}\in\Psi_h^0(M)$ such that $\chi:=\sigma_h(\widehat{\chi})\equiv 1$ on $\mathrm{WF}_h(G_\varepsilon)$ and $\chi\equiv 0$ outside $\mathrm{ell}_h(B_1)$ so that $\mathrm{WF}_h(G_\varepsilon)\subset\mathrm{ell}_h(\widehat{\chi})\subset\mathrm{WF}_h(\widehat{\chi})\subset\mathrm{ell}_h(B_1)$. Since $\mathrm{WF}_h(G_\varepsilon)\subset\mathrm{ell}_h(B_1)$ and

$B_1\mathbf{u} \in H^{\mu-1/2}$, we have $G_\varepsilon^* G_\varepsilon \mathbf{u} \in H^{-\mu+3/2}$. On the other hand $B_1\mathbf{P}\mathbf{u} \in H^\mu \subset H^{\mu-3/2}$. We can therefore compute

$$\Im\langle \widehat{\chi}\mathbf{P}\mathbf{u}, G_\varepsilon^* G_\varepsilon \mathbf{u}\rangle = \frac{i}{2}\langle [\widehat{\chi}\Re(\mathbf{P}), G_\varepsilon^* G_\varepsilon]\,\mathbf{u}, \mathbf{u}\rangle + \frac{1}{2}\langle (G_\varepsilon^* G_\varepsilon \widehat{\chi}\Im(\mathbf{P}) + \widehat{\chi}\Im(\mathbf{P}) G_\varepsilon^* G_\varepsilon)\,\mathbf{u}, \mathbf{u}\rangle. \tag{4.3}$$

Now set $\mathbf{T}_\varepsilon = \frac{i}{2h}[\widehat{\chi}\Re(\mathbf{P}), G_\varepsilon^* G_\varepsilon] \in \Psi_h^{2\mu-2}(M, \mathrm{Hom}(E))$. We have thanks to Proposition A.8 and the fact that $\chi \equiv 1$ on $\mathrm{WF}_h(G_\varepsilon)$:

$$\begin{aligned}
\sigma_h(\mathbf{T}_\varepsilon) &= \frac{1}{2}\mathbf{X}_p(g_\varepsilon^2) = g_\varepsilon \mathbf{X}_p g_\varepsilon = \langle\xi\rangle^\mu \langle\varepsilon\xi\rangle^{-1} g_\varepsilon \mathbf{X}_p g + g_\varepsilon^2 \mathbf{X}_p\left(\langle\xi\rangle^\mu\langle\varepsilon\xi\rangle^{-1}\right) \\
&= \langle\xi\rangle^\mu\langle\varepsilon\xi\rangle^{-1} g_\varepsilon \mathbf{X}_p g + g_\varepsilon^2\left(\frac{\mu}{2}\langle\xi\rangle^{-2} - \frac{\varepsilon^2}{2}\langle\varepsilon\xi\rangle^{-2}\right)\mathbf{X}_p(|\xi|^2) \\
&\leq -c g_\varepsilon^2 + C g_\varepsilon^2 + \widetilde{C}|\langle\xi\rangle^\mu b|^2,
\end{aligned} \tag{4.4}$$

where $b = \sigma_h(B)$. The last inequality stands because of the point (ii) of Lemma 4.5, and the fact that $\left(\frac{\mu}{2}\langle\xi\rangle^{-2} - \frac{\varepsilon^2}{2}\langle\varepsilon\xi\rangle^{-2}\right)\mathbf{X}_p(|\xi|^2)$ is uniformly bounded in ξ, ε by a constant C. Now take c big enough so that $c - C > c/2$ to get

$$\sigma_h(\mathbf{T}_\varepsilon) + \frac{c}{2} g_\varepsilon^2 \leq \widetilde{C}|\langle\xi\rangle^\mu b|^2.$$

Now apply the sharp Gårding's inequality to obtain

$$\left\langle \left(\mathbf{T}_\varepsilon + \frac{c}{2} G_\varepsilon^* G_\varepsilon - \widetilde{C}(S_0 B)^*(S_0 B)\right)\widehat{\chi}\mathbf{u}, \widehat{\chi}\mathbf{u}\right\rangle \leq Ch\|\widehat{\chi}\mathbf{u}\|_{H_h^{\mu-1/2}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}},$$

which reads with differents constants

$$\langle \mathbf{T}_\varepsilon \mathbf{u}, \mathbf{u}\rangle + \frac{c}{2}\|G_\varepsilon \mathbf{u}\|_{L^2}^2 \leq C\|B\mathbf{u}\|_{H_h^\mu}^2 + Ch\|B_1\mathbf{u}\|_{H^{\mu-1/2}}^2 + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}, \tag{4.5}$$

where we could remove the terms in $\widehat{\chi}$ because $\mathrm{WF}_h(\widehat{\chi}) \subset \mathrm{ell}_h(B_1)$ and $\mathrm{WF}_h(G_\varepsilon) \subset \mathrm{ell}_h(\widehat{\chi})$.
Let us now interest ourselves to the second term of the right hand side of (4.3). It reads

$$\langle \widehat{\chi}\Im(\mathbf{P}) G_\varepsilon \mathbf{u}, \mathbf{u}\rangle + \frac{1}{2}\langle (G_\varepsilon^*[G_\varepsilon, \widehat{\chi}\Im(\mathbf{P})] - [G_\varepsilon^*, \widehat{\chi}\Im(\mathbf{P})]\,G_\varepsilon)\,\mathbf{u}, \mathbf{u}\rangle.$$

We have $\sigma_h(\Im(\mathbf{P})) = -q \leq 0$, which imply $\langle \widehat{\chi}\Im(\mathbf{P}) G_\varepsilon \mathbf{u}, \mathbf{u}\rangle \leq C' h\|G_\varepsilon \mathbf{u}\|_{L^2}^2$ by Gårding's inequality (Theorem A.22). On the other hand, we have that $G_\varepsilon^*[G_\varepsilon, \widehat{\chi}\Im(\mathbf{P})] - [G_\varepsilon^*, \widehat{\chi}\Im(\mathbf{P})]\,G_\varepsilon$ lies in $h^2\Psi_h^{2\mu-1}$ uniformly in ε thanks to principal symbol calculus. As a consequence,

$$\frac{1}{2}\langle (G_\varepsilon^*[G_\varepsilon, \widehat{\chi}\Im(\mathbf{P})] - [G_\varepsilon^*, \widehat{\chi}\Im(\mathbf{P})]\,G_\varepsilon)\,\mathbf{u}, \mathbf{u}\rangle \leq C'' h^2\|B_1\mathbf{u}\|_{H_h^{\mu-1/2}}^2 + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.$$

Combining those estimates with (4.5) and letting $c > 4C'$ we get

$$\begin{aligned}\Im\langle\widehat{\chi}\mathbf{Pu}, G_\varepsilon^* G_\varepsilon\mathbf{u}\rangle \leq -\frac{c}{4}h\|G_\varepsilon\mathbf{u}\|_{L^2}^2 + Ch\|B\mathbf{u}\|_{H_h^\mu}^2 \\ + Ch^2\|B_1\mathbf{u}\|_{H_h^{\mu-1/2}}^2 + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.\end{aligned}$$

Now we have $-\Im\langle\widehat{\chi}\mathbf{Pu}, G_\varepsilon^* G_\varepsilon\mathbf{u}\rangle \leq \|G_\varepsilon\widehat{\chi}\mathbf{Pu}\|_{L^2}\|G_\varepsilon\mathbf{u}\|_{L^2} \leq C\|B_1\mathbf{Pu}\|_{H_h^\mu}\|G_\varepsilon\mathbf{u}\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{H_h^{-N}}$ uniformly in ε since G_ε is uniformly bounded in Ψ_h^μ. We thus obtain

$$\begin{aligned}\frac{c}{4}\|G_\varepsilon\mathbf{u}\|_{L^2}^2 \leq Ch^{-1}\|B_1\mathbf{Pu}\|_{H_h^\mu}\|G_\varepsilon\mathbf{u}\|_{L^2} + C\|B\mathbf{u}\|_{H_h^\mu}^2 \\ + Ch\|B_1\mathbf{u}\|_{H_h^{\mu-1/2}}^2 + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}},\end{aligned}$$

which finally imply uniformly in ε :

$$\begin{aligned}\|G_\varepsilon\mathbf{u}\|_{L^2} \leq Ch^{-1}\|B_1\mathbf{Pu}\|_{H_h^\mu} + C\|B\mathbf{u}\|_{H_h^\mu} \\ + Ch^{1/2}\|B_1\mathbf{u}\|_{H_h^{\mu-1/2}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.\end{aligned}$$

Recall that $S_\varepsilon \to S_0 \in \Psi_h^\mu \subset \Psi_h^{\mu+1/2}$. Since $B_1\mathbf{u} \in H_h^{\mu-1/2}$ and $\mathrm{WF}_h(G_\varepsilon) \subset \mathrm{ell}_h(B_1)$ we get that $G_\varepsilon\mathbf{u} \to S_0 G\mathbf{u}$ in H_h^{-1}. On the other hand, $G_\varepsilon\mathbf{u}$ is bounded in L^2 from the estimate above. Moreover, the unit ball in L^2 is compact for the weak topology ; this implies $S_0 G\mathbf{u} \in L^2$, that is, $G\mathbf{u} \in H_h^\mu$. We thus obtain, noting noting that G is elliptic on $\mathrm{WF}_h(A)$:

$$\begin{aligned}\|A\mathbf{u}\|_{H_h^\mu} \leq Ch^{-1}\|B_1\mathbf{Pu}\|_{H_h^\mu} + C\|B\mathbf{u}\|_{H_h^\mu} \\ + Ch^{1/2}\|B_1\mathbf{u}\|_{H_h^{\mu-1/2}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.\end{aligned} \tag{4.6}$$

Note that this estimate is almost what we demanded, except we have in addition the term $h^{1/2}\|B_1\mathbf{u}\|_{H_h^{\mu-1/2}}$. We claim that we can make it disappear. More precisely, let us prove that for all $\ell \in \mathbb{N}^*$, one has

$$\begin{aligned}\|A\mathbf{u}\|_{H_h^\mu} \leq C_\ell h^{-1}\|B_1\mathbf{Pu}\|_{H_h^\mu} + C_\ell\|B\mathbf{u}\|_{H_h^\mu} \\ + C^\ell h^{\ell/2}\|B_1\mathbf{u}\|_{H_h^{\mu-\ell/2}} + \mathcal{O}_\ell(h^\infty)\|\mathbf{u}\|_{H_h^{-N}},\end{aligned} \tag{4.7}$$

where

$$C_\ell = \prod_{k=1}^{\ell}\left(1 + h^{k/2}C^k\right)$$

for some C not depending on ℓ. We will use induction on ℓ, noting that for $\ell = 1$ this estimate is exactly (4.6). Assume that (4.7) holds for some $\ell \geq 1$. Take $B_2 \in \Psi_h^0$ is such that both triplets (A, B, B_2) and (B_2, B, B_1) satisfy the assumptions of the theorem. One has, with $C = \max\{C_{A,B,B_1}, C_{B_2,B,B_1}, C_{A,B,B_2}\}$, by (4.6) applied to (B_2, B, B_1) (we can assume the constants involved do not depend on μ because we can stop the process whenever $\ell > 2\mu + N$ and keep iterating with $\|B_1\mathbf{u}\|_{H_h^{-N}}$ instead of $\|B_1\mathbf{u}\|_{H_h^{\mu-\ell/2}}$ in (4.7)):

$$\begin{aligned}\|B_2\mathbf{u}\|_{H_h^{\mu-\ell/2}} \leq Ch^{-1}\|B_1\mathbf{Pu}\|_{H_h^{\mu-\ell/2}} + C\|B\mathbf{u}\|_{H_h^{\mu-\ell/2}} \\ + Ch^{1/2}\|B_1\mathbf{u}\|_{H_h^{\mu-(\ell+1)/2}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.\end{aligned} \tag{4.8}$$

On the other hand, by (4.7) applied to (A, B, B_2) :

$$\begin{aligned}\|A\mathbf{u}\|_{H_h^\mu} \leq C_\ell h^{-1}\|B_2\mathbf{Pu}\|_{H_h^\mu} + C_\ell\|B\mathbf{u}\|_{H_h^\mu} \\ + C^\ell h^{\ell/2}\|B_2\mathbf{u}\|_{H_h^{\mu-\ell/2}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.\end{aligned} \tag{4.9}$$

It suffices to combine (4.8) and (4.9) to get (4.7) for $\ell + 1$, assuming that $\|B_2\mathbf{Pu}\|_{H^\mu} \leq \|B_1\mathbf{Pu}\|_{H_h^\mu}$ for all μ, which we can do up to consider $B_2/\widetilde{C}$ for some big $\widetilde{C}$, because $\mathrm{WF}_h(B_2) \subset \mathrm{ell}_h(B_1)$.
This concludes the proof of the theorem since the constants C_ℓ are bounded with ℓ, provided h is small enough. □

4.2 Control of singularities near radial sources

Recall from subsection 2.1 the definition of radial sources and radial sinks. We will state two estimates that allows us to control the wavefront set of sufficiently regular (resp. singular) distributions near radial sources (resp. sinks).

Proposition 4.6. *Let $\mathbf{P} \in \Psi_h^1(M, \mathrm{Hom}(E))$ as in Proposition 4.3, and assume that $L \subset T^*M \setminus 0$ is a radial source for the Hamiltonian flow $\mathrm{e}^{t\mathbf{X}_p}$. Then there is a threshold $\mu_0 > 0$ such that if $B_1 \in \Psi_h^0(M)$ is elliptic on $\rho(L) \subset \partial\overline{T}^*M$, there exists some $A \in \Psi_h^0(M)$ elliptic on $\rho(L)$ satisfying that for every $\mu \geq \mu_0$, there is C such that*

$$||A\mathbf{u}||_{H_h^\mu(M,E)} \leq Ch^{-1}||B_1\mathbf{Pu}||_{H_h^\mu(M,E)} + \mathcal{O}(h^\infty)||\mathbf{u}||_{H_h^{-N}(M,E)}$$

for every h-tempered family of distributions $\mathbf{u} \in \mathcal{D}'_h(M, E)$ such that $A\mathbf{u} \in H_h^{\mu_0}(M, E)$ and all N.

In other words, if $\mathbf{Pu}$ has no wavefront near $\rho(L)$ and $\mathbf{u}$ is sufficiently regular near $\rho(L)$, then $\mathbf{u}$ has no wavefront near $\rho(L)$. As in the previous subsection, we get easily the following

Corollary 4.7. *Let μ_0 be as in the previous proposition. Assume $\mathbf{u} \in \mathcal{D}'_h(M, E)$ is h-tempered with $\mathrm{WF}_h(\mathbf{Pu}) \cap \rho(L) = \emptyset$. If $B_1 \in \Psi_h^0$ is elliptic near $\rho(L)$ and verifies $B_1\mathbf{u} \in H_h^{\mu_0}(M, E)$, then $\mathrm{WF}_h(\mathbf{u}) \cap \rho(L) = \emptyset$.*

Proof of Proposition 4.6. As in Theorem 4.3, we will make use of escape functions. Let U be a small conical neighborhood of L such that $\rho(U) \subset \mathrm{ell}_h(B_1)$.

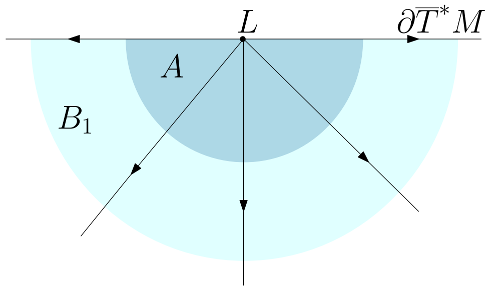

Figure 4.2. Regularity near radial sources (Proposition 4.6).

Using Lemma 3.3, we obtain a norm $f \in \mathcal{C}^\infty(T^*M \setminus 0, \mathbb{R}_{>0})$ such that $\mathbf{X}_p(f) \leq -cf$ on U. Arguing exactly as in Lemma 6.2 and Lemma 3.2, we obtain $m \in \mathcal{C}^\infty(T^*M \setminus 0, [0,1])$ such that $m \equiv 1$ near L, $\mathbf{X}_p(m) \leq 0$ and $\operatorname{supp} m \subset U$. Consider some $R > 0$ with $U \cap \{f \geq R\} \subset \operatorname{ell}_h(B_1)$ and $\chi \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ such that $\operatorname{supp} \chi \subset (R, \infty)$, $\chi \equiv 1$ on $[2R, \infty)$ and $\chi' \geq 0$ everywhere. Set

$$g = m(\chi \circ f) \in \mathcal{C}^\infty(\overline{T}^*M).$$

Take some $\tilde{f} \in \mathcal{C}^\infty(\overline{T}^*M)$ such that $f \equiv \tilde{f}$ on $\{f \geq R\}$. We choose, as in the proof of the propagation of singularities, $S_\varepsilon \in \Psi_h^{\mu-1}(M, E)$ such that

$$\sigma_h(S_\varepsilon) = \tilde{f}^m \langle \varepsilon\xi \rangle^{-1}.$$

Put $G_\varepsilon \in \Psi_h^{\mu-1}$ such that $\sigma_h(G_\varepsilon) = \sigma_h(S_\varepsilon) g$. Let $\mathbf{T}_\varepsilon = \frac{i}{2h}[\widehat{\chi}\Re(\mathbf{P}), G_\varepsilon^* G_\varepsilon]$. We have as in (4.4) :

$$\sigma_h(\mathbf{T}_\varepsilon) = g_\varepsilon \mathbf{X}_p(g_\varepsilon) = \tilde{f}\langle \varepsilon\xi\rangle^{-1} g_\varepsilon \mathbf{X}_p(g) + g_\varepsilon^2 \left(\mu \frac{\mathbf{X}_p(\tilde{f})}{\tilde{f}} - \frac{\varepsilon^2 \mathbf{X}_p(|\xi|^2)}{2} \langle \varepsilon\xi\rangle^{-2} \right).$$

Since $\mathbf{X}_p \tilde{f} \leq -c\tilde{f} < 0$ on $\operatorname{supp} g$, for all $C_0 > 0$, there exists μ_0 such that for $\mu \geq \mu_0$ we have on $\operatorname{supp} g$: $\left(\mu \frac{\mathbf{X}_p(\tilde{f})}{\tilde{f}} - \frac{\varepsilon^2 \mathbf{X}_p(|\xi|^2)}{2} \langle \varepsilon\xi\rangle^{-2} \right) \leq -C_0$. Moreover, μ_0 does not depend on $\operatorname{supp} g$ because the constant c can be taken uniform for U close enough to $\rho(L)$. Note that $\mathbf{X}_p(g) \leq 0$ (because $\chi' \geq 0$). We obtain with what precedes

$$\sigma_h(\mathbf{T}_\varepsilon) \leq -C_0 g_\varepsilon^2,$$

to get thanks to sharp Gårding's inequality (Theorem A.22), because $\mathrm{WF}_h(G_\varepsilon) \subset \operatorname{ell}_h(B_1)$:

$$\langle \mathbf{T}_\varepsilon \mathbf{u}, \mathbf{u} \rangle + C_0 \|G_\varepsilon \mathbf{u}\|_{L^2}^2 \leq Ch \|B_1 \mathbf{u}\|_{H^{\mu-1/2}}^2 + \mathcal{O}(h^\infty) \|\mathbf{u}\|_{H_h^{-N}},$$

uniformly in ε. Exactly as in the proof of Theorem 4.3, we obtain, with $A = G$,

$$\|A\mathbf{u}\|_{H_h^\mu} \leq Ch^{-1} \|B_1 \mathbf{P}\mathbf{u}\|_{H_h^\mu} + Ch^{1/2} \|B_1 \mathbf{u}\|_{H_h^{\mu-1/2}} + \mathcal{O}(h^\infty) \|\mathbf{u}\|_{H_h^{-N}}. \quad (4.10)$$

Again, we claim that we can make disappear the term $Ch^{1/2}\|B_1\mathbf{u}\|_{H_h^{\mu-1/2}}$. Indeed, because of elliptic regularity, we can assume that there is $\mathrm{WF}_h(B_1) \subset U'$ where U' is a neighborhood of $\rho(L)$ as in Definition 2.5, so that for all $(x,\xi) \in \mathrm{WF}_h(B_1)$ there is $T > 0$ such that $\mathrm{e}^{-T\mathbf{X}_p}(x,\xi) \in \mathrm{ell}_h(A)$. Now if $\widetilde{B_1}$ elliptic on $\rho(L)$ satisfies $\mathrm{WF}_h(B_1) \subset \mathrm{ell}_h(\widetilde{B_1})$ and $\mathrm{WF}_h(\widetilde{B_1}) \subset U'$, then by Theorem 4.3 we have

$$\|B_1\mathbf{u}\|_{H_h^{\mu-1/2}} \leq C\|A\mathbf{u}\|_{H_h^{\mu-1/2}} + Ch^{-1}\|\widetilde{B_1}\mathbf{P}\mathbf{u}\|_{H_h^{\mu-1/2}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.$$

This estimate combined with (4.10) imply using elliptic regularity that for every $\widetilde{B_1}$ elliptic on $\rho(L)$ there is A elliptic on $\rho(L)$ such that for all $\mu > \mu_0$:

$$\|\widetilde{B_1}\mathbf{u}\|_{H_h^\mu} \leq Ch^{-1}\|\widetilde{B_1}\mathbf{P}\mathbf{u}\|_{H_h^\mu} + Ch^{1/2}\|A\mathbf{u}\|_{H_h^{\mu-1/2}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}},$$

with a different constant C. Now we iterate this estimate to get for every $\ell \in \mathbb{N}^*$:

$$\|\widetilde{B_1}\mathbf{u}\|_{H_h^\mu} \leq Ch^{-1}\|\widetilde{B_1}\mathbf{P}\mathbf{u}\|_{H_h^\mu}\left(\sum_{k=0}^{\ell} C^k h^{k/2}\right) + C^\ell h^{\ell/2}\|A\mathbf{u}\|_{H_h^{\mu-\ell/2}} + \mathcal{O}_\ell(h^\infty)\|\mathbf{u}\|_{H_h^{-N}},$$

which concludes since the sum is bounded provided h is small enough. □

4.3 Microlocal structure of $\mathbf{R}_\lambda$

The purpose of this subsection is to show that for small $t_0 > 0$, $\mathrm{e}^{-it_0\mathbf{P}}\mathbf{R}_\lambda$ lies in a space where we can take the flat trace. This will be implied by the

Theorem 4.8 (Microlocal structure of the resolvent)**.** *Let λ_0 such that $\Im(\lambda_0) > C - C_m$ for some order function m (discussed in the previous section). For λ close to λ_0, one has*

$$\mathbf{R}_\lambda = \mathbf{Y}_\lambda - \sum_{j=1}^{J(\lambda_0)} \frac{(\mathbf{P}-\lambda_0)^{j-1}\Pi}{(\lambda-\lambda_0)^j}$$

with $\mathbf{Y}_\lambda$ holomorphic near λ_0, Π is the projection $H_{G_m}(M,\Omega) \to \ker\left((\mathbf{P}-\lambda_0)^{J(\lambda_0)}\right)$, and we have the following description of the microlocal structures :

$$\begin{aligned} \mathrm{WF}'(\mathbf{Y}_\lambda) &\subset \Delta(T^*M \setminus 0) \cup \mathrm{Char}(\mathbf{P})_+ \cup (E_u^* \times E_s^*), \\ \mathrm{WF}'(\Pi) &\subset E_u^* \times E_s^*, \end{aligned} \tag{4.11}$$

locally uniformly in λ, where $\Delta(T^*M)$ is the diagonal in $T^*M \times T^*M$ and $\mathrm{Char}(\mathbf{P})_+ = \{(\Phi^t(x,\xi),(x,\xi)) \mid t \geq 0,\ H(x,\xi) = 0\}$.

Proof. In what follows, we assume that λ is a complex number varying in compact region Z of $\mathbb{C} \setminus \mathrm{Res}(\mathbf{P})$, and we take an order function g_m as in Propostion 3.5, with $|u|, s$ to be well chosen later.
We will introduce a semiclassical parameter h in order to use Lemma A.16. Let $\mathbf{P}_{\lambda,h} = h(\mathbf{P}-\lambda)$, and taking notations of Remark 3.6 we take $G_{m,h} = \mathrm{Op}_h(g_m^\delta)$.

$$\mathbf{Q}_{\delta,h} = \mathrm{e}^{G^\delta_{m,h}}(h\mathbf{P})\mathrm{e}^{-G^\delta_{m,h}}.$$

We can compute as in Lemma 3.10 :

$$\mathbf{Q}_{\delta,h} = h\mathbf{P} + [G^\delta_{m,h}, h\mathbf{P}] + \mathcal{O}(h^2)_{\Psi_h^{-1+}}. \tag{4.12}$$

Let Γ_0 and $R>0$ be as in Propostion 3.5 and $\chi_0 \in S^0(M)$ such that :

- $\chi_0^2 \equiv C_m$ on $\Gamma_0 \cap \{|\xi| \geq R\}$,
- $\chi_0^2 \equiv C_m$ on $\{|\xi| \leq R\}$
- $\chi_0 \equiv 0$ on $\{\xi \geq 3R/2\}$ outside a small conical neighborhood of Γ_0.

Take also $\chi_1 \in \mathcal{C}_c^\infty(T^*M)$ such that $\chi_1 \equiv 1$ near 0 and $\mathrm{supp}\,\chi_1 \subset \{|\xi| \leq 3R/2\}$. and choose some $\widehat{\chi_1} \in \Psi^0(M)$ with $\sigma(\widehat{\chi_1}) = \chi_1$. For $\delta > 0$ and $j = 1, 2$ set

$$\chi_{j,\delta}(x,\xi) = \chi_j(x, \xi/\delta).$$

Choose some $\hat{\chi}_{j,\delta} \in \Psi^0(M)$ such that $\sigma(\widehat{\chi_{j,\delta}}) = \chi_{j,\delta}$, define $\hat{\chi}_{\delta,h} = h\left(\hat{\chi}_{0,\delta}\right)^* \hat{\chi}_{0,\delta} + \left(\hat{\chi}_{1,\delta}\right)^* \hat{\chi}_{1,\delta}$ and put

$$\mathbf{Q}^\chi_{\delta,h}(\lambda) = \mathbf{Q}_{\delta,h} - h\lambda - i\hat{\chi}_{\delta,h}.$$

Lemma 4.9. *For all $\varepsilon > 0$ and $\Im(\lambda) > -C_m + \varepsilon$, we have that for h small enough, $\mathbf{Q}^\chi_{\delta,h}(\lambda)$ is inversible $\mathcal{D}(\mathbf{Q}) \to L^2$ with inverse bound*

$$\left\| \left(\mathbf{Q}^\chi_{\delta,h}(\lambda)\right)^{-1} \right\|_{L^2 \to L^2} \leq C_\varepsilon / h.$$

Proof. One has, for $\mathbf{u} \in \mathcal{C}^\infty(M, \Omega_0)$,

$$\begin{aligned}\left\langle \left(\left[G^\delta_m, h\mathbf{P}\right] - h\Im(\lambda) - \hat{\chi}_{\delta,h}\right)\mathbf{u}, \mathbf{u}\right\rangle = -\|\hat{\chi}_{1,\delta}\mathbf{u}\|_{L^2} - h\left(\Im(\lambda) - C_m\right)\|\mathbf{u}\|^2_{L^2} \\ + \left\langle \left(\left[G^\delta_m, h\mathbf{P}\right] - h\hat{\chi}^*_{0,\delta}\hat{\chi}_{0,\delta} - hC_m\right)\mathbf{u}, \mathbf{u}\right\rangle.\end{aligned} \tag{4.13}$$

For all $\mu \in (0,1)$, we have in $S^\mu_h(M)/hS^{\mu-1}_h(M)$:

$$\sigma_h\left(\left[G^\delta_m, h\mathbf{P}\right]\right) = ih\mathbf{X}(g^\delta_m).$$

Thanks to properties of χ_0 and g^δ_m, we have in $S^\mu_h(M)/hS^{\mu-1}_h(M)$:

$$\sigma_h\left(\frac{1}{h}\left[G^\delta_m, h\mathbf{P}\right] - \hat{\chi}^*_{0,\delta}\hat{\chi}_{0,\delta}\right)(x,\xi) \leq -C_m, \quad (x,\xi) \in T^*M.$$

Now we apply sharp Gårding's inequality (Theorem A.22) to obtain

$$\left\langle \left(\left[G_m^\delta, h\mathbf{P} \right] - h\hat{\chi}_{0,\delta}^* \hat{\chi}_{0,\delta} - hC_m \right) \mathbf{u}, \mathbf{u} \right\rangle \leq C_\mu h^2 \|\mathbf{u}\|^2_{H_h^{(\mu-1)/2}} \leq \tilde{C}_\mu h^2 \|\mathbf{u}\|^2_{L^2}.$$

We obtain with (4.13)

$$\begin{aligned} \left\langle \left(\left[G_m^\delta, h\mathbf{P} \right] - h\Im(\lambda) - \hat{\chi}_{\delta,h} \right) \mathbf{u}, \mathbf{u} \right\rangle &\leq -h \left(\Im(\lambda) - C_m + h\tilde{C}_\mu \right) \|\mathbf{u}\|^2_{L^2} \\ &\leq -h \frac{\varepsilon}{2} \|\mathbf{u}\|^2_{L^2}, \end{aligned}$$

provided $h < \varepsilon / 2\tilde{C}_\mu$. We conclude as in Lemma 3.11, using (4.12). □

Now fix $(y, \eta) \in T^*M \cap \{|\xi| \in [R, 2R]\}$ and take $(f_h)_h$ and h-tempered family of compactly supported functions on M with $\mathrm{WF}_h(f_h)$ localized near (y, η). We set $u_h = \mathbf{R}_\lambda f_h$. We know that u_h lies in $\mathcal{D}(\mathbf{P}) \cap H_{G_m}$, so that $(\mathbf{P} - \lambda) u_h = f_h$ and $\mathbf{Q}_{\delta,h}(\lambda) \tilde{u}_h = \tilde{f}_h$ with $\tilde{u}_h = \mathrm{e}^{G_{m,h}}(u_h)$ and $\tilde{f}_h = \mathrm{e}^{G_{m,h}}(h f_h)$. We will also denote for h small enough :

$$\tilde{u}_h^\chi = \mathbf{Q}_{\delta,h}^\chi(\lambda)^{-1} \tilde{f}_h.$$

We have

$$\tilde{u}_h = \tilde{u}_h^\chi - \mathbf{Q}_{\delta,h}(\lambda)^{-1} \hat{\chi}_{\delta,h} \tilde{u}_h^\chi. \tag{4.14}$$

We will distinguish four cases according to where is located (y, η) ; for each one we will take some $(x, \xi) \in \{|\xi| \in [R, 2R]\}$ such that (x, ξ, y, η) does not lie in $\Delta(T^*M \setminus 0) \cup \mathrm{Char}(\mathbf{P})_+ \cup (E_u^* \times E_s^*)$ and find some neighborhoods U and V of (y, η) and (x, ξ) such that if $\mathrm{WF}_h(f_h) \subset U$ we have

$$\mathrm{WF}_h(u_h) \cap V = \emptyset, \qquad \text{uniformly in } \lambda. \tag{4.15}$$

Case 1 : $(y, \eta) \in \{H \neq 0\}$. We have that $\mathbf{Q}_\delta(\lambda) = (\mathbf{Q}_{\delta,h}(\lambda))_h$ is elliptic on $\{H \neq 0\}$. Thanks to Proposition 4.1, for every $(x, \xi) \in \{H \neq 0\} \setminus \{(y, \eta)\}$, there is U, V neighborhoods of (y, η) and (x, ξ) such that

$$\mathrm{WF}_h(\tilde{u}_h) \cap V = \emptyset$$

uniformly in λ, which implies (4.15). Now take $(x, \xi) \in \{H = 0\} \setminus E_u^*$ with $|\xi| \in [R, 2R]$. Take $\tilde{U}$ a small conical neighborhood of E_s^* as in definition 2.5. There exists $T > 0$ such that $\Phi^{-T}(x, \xi) \in \tilde{U}$. Let μ_0 be the threshold involved in Proposition 4.6. Taking the order function g_m so that $s > \mu_0$, we know that $u \in H_h^s(M) \subset H_h^{\mu_0}(M)$ microlocally near E_s^*. We can thus use Corollary 4.7 to get $\Phi^{-T}(x, \xi) \notin \mathrm{WF}_h(u_h)$ provided $\mathrm{WF}_h(f_h) \cap \rho(\tilde{U}) = \emptyset$ (which will be the case if $\mathrm{WF}_h(f_h)$ is close enough to (y, η)). Now take a neighborhood U of (y, η) small enough so that $\Phi^{-t}(x, \xi) \notin U$ for all $t \in [0, T]$; we can then apply Corollary 4.4 to obtain that $(x, \xi) \notin \mathrm{WF}_h(u_h)$. This is also true for neighbors of (x,ξ) ; thus we found V such that (4.15) is valid. Finally, assume $(x,\xi)\in E_u^*$ with $|\xi|\in[R,2R]$. We first assume here that $(y,\eta)\notin\Gamma_0$. We can apply propagation of singularities with the operator $\mathbf{Q}_\delta^\chi(\lambda)$ to obtain that

$$\mathrm{WF}_h(\tilde u_h^\chi)\cap V=\emptyset$$

uniformly in λ for some small neighborhood V of (x,ξ), provided δ is small enough, and $\mathrm{WF}_h(f_h)\subset U$ where U is a neighborhood of (y,η) small enough so that $U\cap\{|\xi|<\delta R\}=\emptyset$. Indeed, $\mathbf{Q}_\delta^\chi(\lambda)$ is elliptic on $\{|\xi|<\delta R\}$ and there is $T>0$ such that $\Phi^{-T}(x,\xi)$ lies in $\{|\xi|<\delta R\}$ so we can apply Proposition 4.1. To show that $\mathrm{WF}_h(u_h)\cap V=\emptyset$ uniformly in λ, it suffices thanks to (4.14) to show that

$$W:=\mathrm{WF}_h(\hat\chi_\delta)\cap\mathrm{WF}_h(\tilde u_h^\chi)=\emptyset. \tag{4.16}$$

Since $(y,\eta)\notin\mathrm{WF}_h(\hat\chi)$ and $\mathbf{Q}_\delta^\chi(\lambda)$ is elliptic on $\{H\neq0\}\cup\{|\xi|<\delta R\}$ (because of the term $-i\hat\chi_{1,\delta}^*\hat\chi_{1,\delta}$), we only need to show that

$$W\cap\{H=0\}\cap\{R\delta\le|\xi|\le 3R\delta/2\}=\emptyset$$

uniformly in λ. We have $W\cap E_u^*=\emptyset$ with what precedes ; doing exactly as in the case $(x,\xi)\in\{H=0\}\setminus E_u^*$ with $\mathbf{Q}_\delta^\chi(\lambda)$ in place of $\mathbf{Q}_\delta(\lambda)$ ($\tilde u^\chi$ and $\tilde u$ have same regularity near $\rho(E_s^*)$ because $\hat\chi$ is supported far away from E_s^*), we obtain

$$\mathrm{WF}_h(\tilde u_h^\chi)\cap(\{H=0\}\setminus E_u^*)=\emptyset$$

uniformly in λ. We thus obtained (4.16) provided $(y,\eta)\notin\Gamma_0$. Since we can take Γ_0 as close of E_0^* as desired, we obtain (4.15) for $(y,\eta)\in\{H\neq0\}\setminus E_0^*$ and $(x,\xi)\in E_u^*$.

Case 2 : $(y,\eta)\in\{H=0\}\setminus(E_u^*\cup E_s^*)$. Since $\mathbf{P}$ is elliptic on $\{H\neq0\}$ we get that for every $(x,\xi)\in\{H=0\}\cap\{|\xi|\in[R,2R]\}$, we can find a neighborhood V such that for U small enough : $\mathrm{WF}_h(u_h)\cap V=\emptyset$ uniformly in λ. Assume now $(x,\xi)\in\{H=0\}\setminus E_u^*$ with $\Phi^{-t}(x,\xi)\neq(y,\eta)$ for all $t\ge0$. Then as before, using propagation of singularities (Corollary 4.4) and control of singularities near radial sources (Corollary 4.7) we obtain that there exists a neighborhood V of (x,ξ) such that $\mathrm{WF}_h(u_h)\cap V=\emptyset$ uniformly in λ provided U is small enough. Finally, assume that (x,ξ) lies in E_u^*. Taking a neighborhood U of (y,η) small enough, one can easily show using 2.1 that there exists $\delta>0$ such that

$$(z,\zeta)\in\{H=0\}\cap\{|\xi|\le 3R\delta/2\},\ (z,\zeta)\notin E_u^*\quad\Longrightarrow\quad\forall t\ge0,\ \Phi^{-t}(z,\zeta)\notin U.$$

We can thus apply exactly the same proof as in the case 1 to get

$$\mathrm{WF}_h(\tilde u_h^\chi)\cap V=\emptyset,\quad \mathrm{WF}_h(\hat\chi_\delta)\cap\mathrm{WF}_h(u_{\delta,h}^\chi)=\emptyset,$$

for some small neighborhood V of (x,ξ) uniformly in λ which implies with (4.14) that (4.15) is valid.

Case 3 : $(y,\eta) \in E_s^*$. As before, the case $(x,\xi) \in \{H \neq 0\}$ is solved using ellipticity of $\mathbf{P}$. If (x,ξ) lies in $\{H = 0\} \setminus E_u^*$, then we can again apply propagation of singularities and regularity near radial sources to obtain that if $\Phi^{-t}(x,\xi) \neq (y,\eta)$ for all $t \geq 0$, we have a neighborhood V near (x,ξ) such that (4.15) is valid provided $\mathrm{WF}_h(f_h)$ is close enough to (y,η).

Case 4 : $(y,\eta) \in E_u^*$. If (x,ξ) does not lie in E_u^*, then dealing as in case 3 we can find V a neighborhood of (x,ξ) such that (4.15) is valid provided $\mathrm{WF}_h(f_h)$ is close enough enough to (y,η). Now assume $(x,\xi) \in E_u^*$ satisfies $\Phi^{-t}(x,\xi) \neq (y,\eta)$ for all $t \geq 0$. Again, we can deal exactly as in the case 1 (using propagation of singularities with $\mathbf{Q}_\delta^\chi(\lambda)$) to obtain V such that (4.15) is valid provided $\mathrm{WF}_h(f_h)$ is close enough to (y,η). Compiling those four cases and applying Lemma A.16, we obtain

$$\mathrm{WF}'(\mathbf{R}_\lambda) \subset \Delta(T^*M \setminus 0) \cup \mathrm{Char}(\mathbf{P})_+ \cup (E_u^* \times E_s^*) \cup (E_u^* \times E_0^*). \tag{4.17}$$

Note that we made the arguments with $\mathcal{C}^\infty(M)$ and not $\mathcal{C}^\infty(M, \Omega_0)$; but we can do exactly same (with an adapted version of Lemma A.16), so that (4.17) is also valid for the resolvent acting on forms.

To remove the term $(E_u^* \times E_0^*)$, we proceed as follows. Fix a volume form ω on M and identify $\mathcal{C}^\infty(M, \Omega_0^k)'$ with $\mathcal{D}'(M, \Omega_0^{n-1-k})$ via the pairing

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \Psi(\alpha \wedge \bar{\beta})\omega, \quad \alpha \in \mathcal{C}^\infty(M, \Omega_0^k), \quad \beta \in \mathcal{C}^\infty(M, \Omega_0^{n-1-k}),$$

where $\Psi : \mathcal{C}^\infty(M, \Omega_0^{n-1}) \to \mathcal{C}^\infty(M)$ is the map defined by

$$\Psi(\nu)\iota_X\omega = \nu, \quad \nu \in \mathcal{C}^\infty(M, \Omega_0^{n-1}).$$

A computation and (3.5) give, for $\Im(\lambda)$ large enough,

$$\langle\langle \mathbf{R}_{\lambda,k}\alpha, \beta \rangle\rangle = \langle\langle \alpha, -\tilde{\mathbf{R}}_{-\bar{\lambda},n-1-k}\beta \rangle\rangle, \quad \alpha \in \mathcal{C}^\infty(M, \Omega_0^k), \quad \beta \in \mathcal{C}^\infty(M, \Omega_0^{n-1-k}),$$

where $\tilde{\mathbf{R}}_{\mu,j}$ is the resolvent $(-\mathbf{P} - \mu)^{-1}|_{\mathcal{C}^\infty(M,\Omega_0^j)}$ of $-\mathbf{P}$. This writes

$$\mathbf{R}_{\lambda,k}^* = -\tilde{\mathbf{R}}_{-\bar{\lambda},n-1-k}, \tag{4.18}$$

where the adjoint is taken with respect to the pairing $\langle\langle\ ,\ \rangle\rangle$, and this is true for every complex number λ by analytic continuation. But now (4.17) applied to $-\mathbf{P}$ imply (we reverse the time)

$$\mathrm{WF}'(\tilde{\mathbf{R}}_\mu) \subset \Delta(T^*M \setminus 0) \cup \mathrm{Char}(-\mathbf{P})_+ \cup (E_s^* \times E_u^*) \cup (E_s^* \times E_0^*).$$

Therefore with (4.18) we finally obtain

$$\mathrm{WF}'(\mathbf{R}_\lambda) \subset \Delta(T^*M \setminus 0) \cup \mathrm{Char}(\mathbf{P})_+ \cup (E_u^* \times E_s^*). \tag{4.19}$$

Now note that

$$\mathbf{Y}_\lambda = \sum_{m\geq 0} A_m(\lambda-\lambda_0)^m, \quad A_m = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{\mathbf{R}_\lambda}{(\lambda-\lambda_0)^{m+1}}\mathrm{d}\lambda,$$

where $D_\varepsilon = \{\lambda,\ |\lambda-\lambda_0|\leq \varepsilon\}$ and $\varepsilon > 0$ is small enough. Thus with (4.19) we obtain the first part of (4.11). Therefore it remains to see that $\mathrm{WF}'(\Pi)\subset E_u^*\times E_s^*$. A simple verification gives the following

Lemma 4.10. *Suppose A, $\hat{\chi}$ are operators such that A and $B := A - i\hat{\chi}$ are invertible. Then*

$$A^{-1} = B^{-1} - iB^{-1}\hat{\chi}B^{-1} - B^{-1}\hat{\chi}A^{-1}\hat{\chi}B^{-1}.$$

This lemma applied with $A = \mathbf{Q}_{\delta,h} - h\lambda$ and $\hat{\chi}_{\delta,h}$ gives for h small enough

$$\begin{aligned}(\mathbf{Q}_{\delta,h}-h\lambda)^{-1} = {} & \mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1} - \mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1}\hat{\chi}_{\delta,h}\mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1}\\ & - \mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1}\hat{\chi}_{\delta,h}(\mathbf{Q}_{\delta,h}-h\lambda)^{-1}\hat{\chi}_{\delta,h}\mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1}.\end{aligned}$$

This reads

$$\begin{aligned}\mathbf{R}_\lambda = {} & h\mathrm{e}^{-G^\delta_{m,h}}\left(\mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1} - \mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1}\hat{\chi}_{\delta,h}\mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1}\right)\mathrm{e}^{G^\delta_{m,h}}\\ & - \mathrm{e}^{-G^\delta_{m,h}}\mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1}\hat{\chi}_{\delta,h}\mathrm{e}^{G^\delta_{m,h}}\mathbf{R}_\lambda\mathrm{e}^{-G^\delta_{m,h}}\hat{\chi}_{\delta,h}\mathbf{Q}^\chi_{\delta,h}(\lambda)^{-1}\mathrm{e}^{G^\delta_{m,h}}.\end{aligned}$$

Therefore we get for every $\delta > 0$ and h small enough

$$\Pi = -\mathrm{e}^{-G^\delta_{m,h}}\mathbf{Q}^\chi_{\delta,h}(\lambda_0)^{-1}\hat{\chi}_{\delta,h}\mathrm{e}^{G^\delta_{m,h}}\Pi\mathrm{e}^{-G^\delta_{m,h}}\hat{\chi}_{\delta,h}\mathbf{Q}^\chi_{\delta,h}(\lambda_0)^{-1}\mathrm{e}^{G^\delta_{m,h}}. \tag{4.20}$$

Now note that our proof of (4.17) actually shows that

$$\mathrm{WF}_h(\mathbf{Q}^\chi_{\delta,h})\cap(T^*M\setminus 0)\subset \Delta(T^*M\setminus 0)\cup \mathrm{Char}(\mathbf{P})_+.$$

Combining this with (4.20), we have using (A.4) that $\mathrm{WF}'(\Pi)\subset \Upsilon_\delta$ for every $\delta > 0$, where

$$\Upsilon_\delta = \{(\rho',\rho),\ \exists t,s\geq 0,\ \Phi^t(\rho)\in \mathrm{WF}_h(\hat{\chi}_\delta),\ \Phi^{-s}(\rho')\in\mathrm{WF}_h(\hat{\chi}_\delta)\}.$$

We have $\bigcap_{\delta>0}\Upsilon_\delta \subset (E_u^*\times E_s^*)\cup(\Gamma_0^+\times\Gamma_0^-)\cup(E_u^*\times\Gamma_0^-)\cup(\Gamma_0^+\times E_s^*)$, where Γ_0 is the cone in Proposition 3.5 and

$$\Gamma_0^\pm = \{(x,\xi)\in T^*M\setminus 0,\ \exists t\geq 0,\ \Phi^{\pm t}(x,\xi)\in\Gamma_0\}.$$

As a consequence, we have using $\Pi = -\frac{1}{2\pi i}\int_{\partial D_\varepsilon}\mathbf{R}_\lambda \mathrm{d}\lambda$ and (4.17) we obtain

$$\mathrm{WF}'(\Pi)\subset(E_u^*\times E_s^*)\cup\Big((\Gamma_0^+\times\Gamma_0^-)\cap(\Delta(T^*M))\Big).$$

This shows the second part of (4.11) since $(\mathbf{P}-\lambda_0)^{J(\lambda_0)}\Pi = 0$ and $\mathbf{P}$ is elliptic on $\{H\neq 0\}$. $\square$

5 Meromorphic continuation of the Ruelle zeta function

The purpose of this section is to prove the

Theorem 5.1 ([DZ13, GLP13]). *The Ruelle zêta function of an Anosov flow on a compact manifold with orientable stable bundle has a meromorphic continuation to the whole complex plane.*

We first state some results that will help us take limits into flat traces ; as a consequence we will obtain a formula (Lemma 5.5 below) that links the flat trace of the resolvent and the logarithmic derivatives of the zeta function.

5.1 Limit of flat traces

Recall from section 2.3 the definition of the operator $\mathbf{M}_\chi$ for $\chi \in \mathcal{C}_c^\infty(\mathbb{R}_{>0})$. We still denote its Schwartz kernel by $\mathbf{K}_\chi$. Denote by $\mathbf{M}_{\chi,\lambda}$ the operator $\mathbf{M}_{\chi e^{i\lambda\cdot}}$ and $\mathbf{K}_{\chi,\lambda}$ its kernel. We have the following lemma :

Lemma 5.2. *Let $t_0 > 0$. There exists $\Gamma \subset T^*M \times T^*M$ a closed conic set not intersecting the conormal to the diagonal, such that for all $N \in \mathbb{N}$, $\widetilde{\Gamma} \subset T^*M \times T^*M$ with $\widetilde{\Gamma} \cap \Gamma = \emptyset$, $\widetilde{\Gamma}$ is a subset of $T^*(U \times U)$ for some open set $U \subset M$, $\Psi \in \mathcal{C}_c^\infty(\pi_{M\times M}(\widetilde{\Gamma}))$ and $\Im(\lambda)$ big enough (depending on N), there is $C_{N,\lambda}$ such that for all $\chi \in \mathcal{C}^\infty(t_0,\infty)$:*

$$\left|\widehat{\Psi K_{k,\chi,\lambda}}(\xi,\eta)\right| \le \frac{C_{N,\lambda}\|\chi\|_{\mathcal{C}^N(\mathbb{R})}}{(1+|\xi|+|\eta|)^N}, \quad (\xi,\eta) \in \widetilde{\Gamma}, \quad 0 \le k \le n-1, \tag{5.1}$$

where $K_{k,\chi,\lambda}$ is the kernel of $S_{\mathbf{M}_{k,\chi,\lambda}}$ (with the notations of subsection 2.3).

Proof. Thanks to (2.9) we only need to prove the lemma for $k = 0$. Take U some open of trivialization of T^*M. We shall identify $T^*M|_U$ with its image $V \times \mathbb{R}^n$ for some $V \subset \mathbb{R}^n$. Let $\Psi \in \mathcal{C}^\infty(V \times V)$. For $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, we have by definition of $K_{\chi,\lambda}$:

$$\widehat{\Psi K_{\chi,\lambda}}(-\xi,\eta) = \int_V \int_{\mathbb{R}_{>0}} e^{i\lambda t}\chi(t)\Psi(\phi^{-t}(x),x)e^{-i\left(-\xi\cdot\phi^{-t}(x)+\eta\cdot x\right)}\mathrm{d}x\mathrm{d}t.$$

We will assume that $(-\xi,\eta)$ is near the diagonal, that is

$$\left|\frac{\xi}{|\xi|}+\frac{\eta}{|\eta|}\right| < \varepsilon \quad \text{and} \quad 1-\varepsilon < \frac{|\xi|}{|\eta|} < 1+\varepsilon, \tag{5.2}$$

for some parameter ε we will choose later. For simplicity we define $g(t,x) = -i\left(-\xi\cdot\phi^{-t}(x)+\eta\cdot x\right)$. Let $\delta,\nu > 0$ satisfying that for any $t \ge t_0$, $x \in V$ and ξ such that $|\xi\cdot X_x| < 2\delta|\xi|$ we have

$$|(I - {}^T\,\mathrm{d}\phi_x^{-t})\cdot\xi| \ge \nu|\xi|. \tag{5.3}$$

The existence of such constants comes from (2.1). Up to shrinking V one can assume we are in the following two cases :

Case 1 : $|\xi \cdot X_y| \geq \delta|\xi|$ for any $y \in V$. Then we get the estimate (5.1) by integrating by part in the variable t. Indeed, we have $\partial_t \mathrm{e}^g = (\partial_t g)\mathrm{e}^g$ with $\partial_t g(t,x) = -i\xi \cdot X_{\phi^{-t}(x)}$. Take L to be the operator $\frac{1}{\partial_t g}\partial_t$; then $L(\mathrm{e}^g) = \mathrm{e}^g$ and integrating by parts we get for all N :

$$\widehat{\Psi K_{\chi,\lambda}}(-\xi,\eta) = \int_V \int_{\mathbb{R}_{>0}} ({}^T L)^N \left\{ \mathrm{e}^{i\lambda t}\chi(t)\Psi(\phi^{-t}(x),x) \right\} \mathrm{e}^{g(t,x)} \mathrm{d}t\mathrm{d}x,$$

where ${}^T L$ is the operator defined by ${}^T L(\varphi) = -\partial_t \left(\varphi/\partial_t g\right)$. Now one can easily show by induction on N that if $\varphi(t,x) = \mathrm{e}^{i\lambda t}\chi(t)\Psi(\phi^{-t}(x),x)$ we have :

$$\left(({}^T L)^N \varphi \right)(t,x) = \frac{\varphi_{N,\lambda}(t,x,\xi)}{(-i\xi \cdot X_{\phi^{-t}(x)})^N},$$

where $|\varphi_{N,\lambda}(t,x,\xi)| \leq C_{N,\lambda}\|\chi\|_{\mathcal{C}^N}$ and $\varphi_{N,\lambda}$ is homogeneous of degree 0 in ξ. We have, thanks to (5.2), $|-i\xi \cdot X_{\phi^{-t}(x)}| \geq \delta|\xi| \geq \frac{\delta(1-\varepsilon)}{2}(|\xi|+|\eta|)$. This shows (5.1).

Case 2 : $|\xi \cdot X_y| < 2\delta|\xi|$ for any $y \in V$. With the same g as before, one has $\widetilde{L}(\mathrm{e}^g) = \mathrm{e}^g$ where $\widetilde{L} = \frac{\nabla g \cdot \nabla}{\|\nabla g\|^2}$ and ∇ is the usual gradient on $\mathbb{R}^n$. Note that $\widetilde{L}$ is well defined thanks to (5.3). Then once again we have for all N

$$\widehat{\Psi K_{\chi,\lambda}}(-\xi,\eta) = \int_{\mathbb{R}_{>0}} e^{i\lambda t}\chi(t) \int_V ({}^T\widetilde{L})^N \left\{ \Psi(\phi^{-t}(x),x) \right\} \mathrm{e}^{g(t,x)} \mathrm{d}t\mathrm{d}x,$$

where ${}^T\widetilde{L}$ is the operator defined by ${}^T\widetilde{L}(\varphi) = -\sum_{i=1}^n \partial_i \left\{ \varphi(\partial_i g)/|\nabla g|^2 \right\}$. Now one can easily show by induction on N, because $\nabla g(t,x) = -i(\eta + {}^T \mathrm{d}\phi{-}t_x \cdot \xi)$, that if $\varphi(t,x) = \Psi(\phi^{-t}(x),x)$ we have :

$$\left(({}^T\widetilde{L})^N \varphi \right)(t,x) = \frac{\varphi_N(t,x,\xi,\eta)}{(\eta + {}^T \mathrm{d}\phi_x^{-t} \cdot \xi)^N},$$

where $|\varphi_N(t,x,\xi,\eta)| \leq C_N \mathrm{e}^{C_N t}$ and $\varphi_{N,\lambda}$ is homogeneous of degree 0 in (ξ,η). The term $\mathrm{e}^{C_N t}$ comes from the derivatives of ϕ^{-t} which cannot grow more than exponentially fast. Now using (5.2) and (5.3) we have $|\eta + {}^T \mathrm{d}\phi{-}t_x \cdot \xi| \geq |(I - {}^T \mathrm{d}\phi_x^{-t}) \cdot \xi| - |\eta + \xi| \geq \nu|\xi| - \varepsilon(|\eta| + |\xi|) \geq \frac{\nu}{3}(|\xi| + |\eta|)$ provided ε is small enough. We therefore get (5.1) for $\Im(\lambda)$ big enough (depending on N) so that the integral converges. □

This fact combined with the following lemma will be useful to take the limit into flat traces.

Lemma 5.3. *Let Γ be a closed conical subset of $T^*M \times T^*M$ not intersecting the conormal to the diagonal. Assume $(K_T)_T$ is a family of distributions of* *order* 0 *in* $\mathcal{D}'_\Gamma(M \times M)$ *(not necessarily bounded) satisfying for all* $\widetilde{\Gamma}$ *and* Ψ *as in the previous lemma :*

$$\left|\widehat{\Psi K_T}(\xi, \eta)\right| \leq \frac{C}{(1 + |\xi| + |\eta|)^{2n+1}}, \quad (\xi, \eta) \in \widetilde{\Gamma}, \tag{5.4}$$

for some C *independant of* T*. Assume also that* $K_T \to K$ *in* $\mathcal{D}'(M \times M)$ *for some* $K \in \mathcal{D}'_\Gamma(M \times M)$ *as* $T \to \infty$*. Then*

$$\lim_{T \to \infty} \mathrm{tr}^\flat K_T = \mathrm{tr}^\flat K.$$

Proof. We first note that if $i : M \to M \times M,\ x \mapsto (x, x)$, one has for any local chart U of M, $\varphi \in \mathcal{C}^\infty(U)$ and $\psi \in \mathcal{C}^\infty(U)$ such that $\operatorname{supp} \varphi \Subset \operatorname{supp} \psi$,

$$\langle i^* K_T, \varphi\rangle = \langle i^* K_T, \varphi\psi\rangle = \frac{1}{(2\pi)^{2n}} \int_U \int_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{\Psi K_T}(\xi, \eta) \mathrm{e}^{ix(\xi+\eta)} \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}x,$$

where $\Psi = \varphi \otimes \psi$. Indeed, this formula is obviously true for smooth functions thanks to the Fourier inversion formula. Thus it is also true for K_T since (5.4) shows that the integral over $\mathbb{R}^n \times \mathbb{R}^n$ is well defined near $\{-\xi = \eta\}$; away from it $|\eta + \xi|$ is big so we can use the non stationary phase method to get enough decreasing in ξ, η. Since the constant in (5.4) is independent of T and $\widehat{\Psi K_T} \to \widehat{\Psi K}$ pointwise, we get thanks to the dominated convergence theorem $\langle i^* K_T, \varphi\rangle \to \langle i^* K, \varphi\rangle$. We conclude using a partition of unity. □

5.2 Proof of the meromorphic continuation of the Ruelle zeta function

Take the notations of subsections 2.2 and 2.3. For a periodic orbit γ and its linearized Poincaré map P_γ we have $\det(I - P_\gamma) = \sum_{k=0}^{n-1} (-1)^k \mathrm{tr}(\bigwedge^k P_\gamma)$. This fact combined to $\exp\left(-\sum_{l\geq 1} \frac{1}{l} \mathrm{e}^{zl}\right) = 1 - \mathrm{e}^z$ for $\Re(z) < 0$ gives, thanks to (2.4),

$$\begin{aligned}
\zeta(s) &= \prod_{\gamma^\#} \left(1 - \mathrm{e}^{-s\ell(\gamma^\#)}\right) = \prod_{\gamma^\#} \exp\left(-\sum_{l\geq 1} \frac{1}{l} (\mathrm{e}^{-s\ell(\gamma^\#)})^l\right) \\
&= \exp\left(-\sum_{\gamma^\#} \sum_{l\geq 1} \frac{1}{l} \mathrm{e}^{-sl\ell(\gamma^\#)}\right) = \exp\left(-\sum_\gamma \frac{\ell(\gamma^\#)}{\ell(\gamma)} \mathrm{e}^{-s\ell(\gamma)}\right) \\
&= \prod_{k=0}^{n-1} \exp\left(-\sum_\gamma \frac{\ell(\gamma^\#) \mathrm{e}^{-s\ell(\gamma)} \mathrm{tr} \bigwedge^k P_\gamma}{\ell(\gamma) |\det(I - P_\gamma)|}\right)^{(-1)^{k+q}},
\end{aligned} \tag{5.5}$$

where $q = \dim E_s$ comes from (2.5). To show the meromorphic continuation of ζ, it is thus sufficient to show that for $k = 0, \ldots, n-1$, the function

$$f_k(s) = \zeta_k'(s)/\zeta_k(s) = \sum_\gamma \frac{\ell(\gamma^\#) \mathrm{e}^{-s\ell(\gamma)} \mathrm{tr} \bigwedge^k P_\gamma}{|\det(I - P_\gamma)|},$$

where $\zeta_k(s) = \exp\left(-\sum_\gamma \frac{\ell(\gamma^\#)\mathrm{e}^{-s\ell(\gamma)}\mathrm{tr}\bigwedge^k P_\gamma}{\ell(\gamma)|\det(I-P_\gamma)|}\right)$, extends meromorphically to the complex plane with simple poles and residues that are integer. Indeed, we have the elementary

Lemma 5.4. *Let h be a meromorphic function on a simply connected domain $D \subset \mathbb{C}$, with simple poles and residues that are integer. Then there exists a meromorphic function g on D such that $h = g'/g$. Moreover if the residues are nonnegative, then g can be chosen holomorphic.*

Therefore, if f_k extends meromorphically with the desired properties, we have $\zeta_k'/\zeta_k = g_k'/g_k$ on $\Re(s) \gg 0$ for some meromorphic function g_k on $\mathbb{C}$. This gives $\zeta_k = \lambda_k g_k$ for some λ_k, and ζ_k extend meromorphically on $\mathbb{C}$, so do $\zeta = \prod_k \zeta_k^{(-1)^{k+q}}$.

Fix some $t_0 > 0$ such that $t_0 < \ell(\gamma)$ for all periodic orbit γ. The following lemma is central for the proof of the meromorphic continuation :

Lemma 5.5. *For $\Re(s) \gg 0$, we have*

$$f_k(s) = -i\mathrm{tr}^\flat\left(\mathrm{e}^{-t_0 s}\mathrm{e}^{-it_0\mathbf{P}}\mathbf{R}_{is}|_{\mathcal{C}^\infty(M,\Omega_0^k)}\right). \tag{5.6}$$

Proof. For $T > 0$, define $\chi_T \in \mathcal{C}_c^\infty(t_0 - 1/T, T + 1/T)$ such that $\chi_T \equiv 1$ on $[t_0, T]$. Let $\chi_{T,s}(t) = \chi_T(t)\mathrm{e}^{-st}$ for $s \in \mathbb{C}$. We have according to Theorem 2.10 :

$$\langle \mathrm{tr}^\flat \mathbf{T}_k, \chi_{T,s}\rangle = \sum_\gamma \frac{\ell(\gamma^\#)\mathrm{e}^{-s\ell(\gamma)}\mathrm{tr}\bigwedge^k P_\gamma}{|\det(I-P_\gamma)|}\chi_T(\ell(\gamma)).$$

Therefore, for $\Re(s) \gg 0$, one has $\left\langle \mathrm{tr}^\flat \mathbf{T}_k, \chi_{T,s}\right\rangle \underset{T\to+\infty}{\longrightarrow} f_k(s)$. We have, according to (2.10), $\left\langle \mathrm{tr}^\flat \mathbf{T}_k, \chi_{T,s}\right\rangle = \mathrm{tr}^\flat(\mathbf{M}_{k,\chi_{T,s}})$. Therefore, letting $T \to \infty$, we get thanks to Lemma 5.2 and Lemma 5.3 that for $\Re(s) \gg 0$:

$$f_k(s) = \mathrm{tr}^\flat\left(\int_{t_0}^\infty \mathrm{e}^{-st}\left(\phi^{-t}\right)^* \mathrm{d}t|_{\mathcal{C}^\infty(M,\Omega_0^k)}\right). \tag{5.7}$$

Using (3.5), we conclude. □

We shall make the change of variable $\lambda = is$ and set

$$\widetilde{f_k}(\lambda) = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\zeta_k(-i\lambda)\right)/\zeta_k(-i\lambda) = -if_k(-i\lambda),$$

which also writes $f_k(s) = i\widetilde{f_k}(is)$. Then f_k has simple poles with integer residues iff it is the case for $\widetilde{f_k}$ (and in this case we have $\mathrm{Res}_{s_0}(f_k) = \mathrm{Res}_{is_0}(\widetilde{f_k})$ for any pole s_0 of f_k). We use Proposition 4.8 and the continuity of the flat trace on $\mathcal{D}'_\Gamma(M \times M, \pi_1^*\Omega_0 \otimes \pi_2^*\Omega_0^*)$ (see [Hör90, Theorem 8.2.4] or §B.3) for any Γ not intersecting the conormal of the diagonal in $T^*M \times T^*M$, to obtain thanks to the Cauchy formula that $\widetilde{f_k}$ is holomorphic in the region $\mathbb{C} \setminus \mathrm{Res}(\mathbf{P})$. To show that $\widetilde{f_k}$ is meromorphic, take $\lambda_0 \in \mathrm{Res}(\mathbf{P})$. Choose an order function m with $C - C_m < \Im(\lambda_0)$ so that $\mathbf{P} : \mathcal{D}(\mathbf{P}) \to H_{G_m}(M, \Omega_0)$ satisfies the conditions of Theorem B.2 to get

$$\mathbf{R}_\lambda = \mathbf{Y}_\lambda - \sum_{j=1}^{J(\lambda_0)} \frac{(\mathbf{P} - \lambda_0)^{j-1}\Pi}{(\lambda - \lambda_0)^j}, \quad \Pi = -\frac{1}{2\pi i}\int_{\partial D_\varepsilon} \mathbf{R}_\lambda \mathrm{d}\lambda, \tag{5.8}$$

where $D_\varepsilon = \{\lambda \in \mathbb{C}, |\lambda - \lambda_0| < \varepsilon\}$ for some $0 < \varepsilon \ll 1$, $\mathbf{Y}_\lambda$ is holomorphic near λ_0, $[\Pi, \mathbf{P}] = 0$, $\Pi^2 = \Pi$ and $(\mathbf{P} - \lambda_0)^{J(\lambda_0)}\Pi = 0$. Now adapting the proof of Proposition 4.8 we show that the Schwartz kernel of $\mathrm{e}^{-it_0\mathbf{P}_k}\mathbf{R}_\lambda|_{\mathcal{C}^\infty(M,\Omega_0^k)}$ lies in $\mathcal{D}'_\Gamma(M \times M, \pi_1^*\Omega_0^k \otimes \pi_2^*\Omega_0^{k*})$ uniformly in λ near λ_0 for some Γ not intersecting the conormal to the diagonal, and the formula (5.8) gives also that the Schartz kernel of $\mathrm{e}^{-t_0\mathbf{P}_k}\Pi|_{\mathcal{C}^\infty(M,\Omega_0^k)}$ lies in $\mathcal{D}'_\Gamma(M \times M, \pi_1^*\Omega_0^k \otimes \pi_2^*\Omega_0^{k*})$. We then apply the continuity of the flat trace to get that $\widetilde{f_k}$ is meromorphic near λ_0.

It remains to see that f_k has simple poles with integer residues. Let $\Pi_k = \Pi|_{\mathcal{C}^\infty(M,\Omega_0^k)}$ and $\mathbf{P}_k = \mathbf{P}|_{\mathcal{C}^\infty(M,\Omega_0^k)}$. We have according to (5.6) :

$$\begin{aligned}
\widetilde{f_k}(\lambda) &= -\mathrm{tr}^\flat\left(\mathrm{e}^{it_0\lambda}\mathrm{e}^{-it_0\mathbf{P}_k}\mathbf{R}_\lambda|_{\mathcal{C}^\infty(M,\Omega_0^k)}\right) \\
&= -\mathrm{tr}^\flat\left\{\mathrm{e}^{it_0\lambda}\mathrm{e}^{-it_0\mathbf{P}_k}\left(\mathbf{Y}_{\lambda,k} - \sum_{j=1}^{J(\lambda_0)} \frac{(\mathbf{P}_k - \lambda_0)^{j-1}\Pi_k}{(\lambda - \lambda_0)^j}\right)\right\} \\
&= -\mathrm{e}^{it_0\lambda}\mathrm{tr}^\flat\left(\mathrm{e}^{-it_0\mathbf{P}_k}\mathbf{Y}_{\lambda,k}\right) \\
&\qquad\qquad + \mathrm{e}^{it_0\lambda}\mathrm{tr}_{H_{G_m}(M,\Omega_0^k)}\left(\mathrm{e}^{-it_0\mathbf{P}_k}\sum_{j=1}^{J(\lambda_0)} \frac{(\mathbf{P}_k - \lambda_0)^{j-1}\Pi_k}{(\lambda - \lambda_0)^j}\right),
\end{aligned}$$

since the flat trace and the usual trace agree on operators of finite rank, which Π_k is (note that Π_k also acts on H_{G_m} thanks to (5.8)). Let $A = \mathbf{P}_k\Pi_k|_{\ker(\mathbf{P}_k-\lambda_0)^{J(\lambda_0)}|_{H_{G_m}}}$. The last term on the right hand side reads

$$\mathrm{e}^{it_0\lambda}\mathrm{tr}_{\ker(\mathbf{P}_k-\lambda_0)^{J(\lambda_0)}|_{H_{G_m}}}\left(\mathrm{e}^{-it_0A}\sum_j \frac{(A - \lambda_0)^{j-1}}{(\lambda - \lambda_0)^j}\right), \tag{5.9}$$

where e^{-it_0A} is given by the power series extension of the exponential map at $\lambda = \lambda_0$ (which is finite since $(A - \lambda_0)^{J(\lambda_0)} = 0$). But for $l \geq 1$ one has $\mathrm{tr}\,(A - \lambda_0)^l = 0$. Therefore, the term (5.9) is just $-\mathrm{e}^{it_0\lambda}\mathrm{tr}\,(-\alpha_0\mathrm{Id}/(\lambda - \lambda_0))$, where $\mathrm{e}^{-it_0\lambda} = \sum_l \alpha_l(\lambda - \lambda_0)^l$ near λ_0. Since $\mathrm{tr}\,(\alpha_0\mathrm{Id}) = m_k(\lambda_0)\mathrm{e}^{-it_0\lambda_0}$ with $m_k(\lambda_0) = \dim\left(\ker(\mathbf{P}_k - \lambda_0)^{J(\lambda_0)}\right)$, we get (because $\mathrm{e}^{it_0\lambda}\mathrm{tr}^\flat\left(\mathrm{e}^{-it_0\mathbf{P}_k}\mathbf{Y}_{\lambda,k}\right)$ is holomorphic near λ_0) :

$$\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)\widetilde{f_k}(\lambda) = m_k(\lambda_0), \tag{5.10}$$

which concludes the proof of Theorem 5.1.

Remark 5.6. As a consequence of (5.10) together with Lemma 5.4, we have

$$\zeta(s) = \prod_{k=0}^{n-1} \zeta_k(s)^{(-1)^{k+q}},$$

where the ζ_k are holomorphic functions. The order of ζ at a resonance $\lambda_0 \in \mathrm{Res}(\mathbf{P})$ is $\sum_{k=0}^{n-1}(-1)^{k+q} m_k(\lambda_0)$ where $m_k(\lambda_0) = \dim\ker\left((\mathbf{P}_k - \lambda_0)^{J(\lambda_0)}|_{H_{G_m}(M,\Omega_0^k)}\right) = \dim\mathrm{Ran}(\Pi_k)$.

6 The Ruelle zeta function at zero for surfaces

In this section, (Σ, g) will denote a negatively curved orientable closed surface. Denote by ϕ^t the geodesic flow on the unitary cotangent bundle $M = S^*\Sigma = \{(x,\xi) \in T^*M, |\xi|_g = 1\}$. Recall that since Σ is negatively curved, ϕ^t is an Anosov contact flow on M. In fact, it is a contact flow associated to the contact form $\alpha = j^*p$ (see [GHL04]) where $j : S^*\Sigma \to T^*\Sigma$ is the inclusion and p is the one form defined by $p_{(x,\xi)}(v) = \xi\left(\mathrm{d}\pi_{(x,\xi)} \cdot v\right)$ where $\pi : T^*\Sigma \to \Sigma$ is the natural projection. We have

$$\iota_X \alpha = 1, \quad \iota_X \mathrm{d}\alpha = 0. \tag{6.1}$$

Also recall that the form $\mathrm{vol} = \alpha \wedge \mathrm{d}\alpha$ is a volume form on $S^*\Sigma$ with $\mathcal{L}_X \mathrm{vol} = 0$. The stable and unstable bundles of ϕ^t are orientable (see [GLP13], Lemma B.1) and thus by Theorem 5.1 the associated Ruelle Zeta function ζ has a meromorphic continuation to the whole complex plane. The purpose of this section is to prove the

Theorem 6.1 ([DZ17b])**.** *The Ruelle Zeta function ζ of (Σ, g) is holomorphic at zero and vanishes with order $|\chi(\Sigma)|$ where $\chi(\Sigma) = 2 - 2g$ is the Euler characteristic of Σ.*

The starting point is Remark 5.6 which gives with (2.5), because $\dim E_s^* = 1$,

$$\zeta(s) = \frac{\zeta_1(s)}{\zeta_0(s)\zeta_2(s)}.$$

The order of ζ at zero is thus

$$-m_0(0) + m_1(0) - m_2(0), \tag{6.2}$$

where $m_k(0) = \dim \mathrm{Ran}(\Pi_k)$. To compute the numbers $m_k(0)$, we show that the spaces $\mathrm{Ran}(\Pi_k)$ actually correspond to the space of generalized resonant states (that is, there is no Jordan bloc) using Lemma 6.2 below ; on the other hand, we can compute directly those spaces (Propositions 6.4 and 6.5) using the fact that smoothness of Pu implies smoothness of u under a certain nonpositive quantum flux condition (Lemma 6.3), which allows us to use hyperbolicity of the flow to conclude.

6.1 Preliminaries

Let $\mathbf{P} = -i\mathcal{L}_X$ where X is the geodesic vector field and $\lambda_0 \in \mathrm{Res}(\mathbf{P})$, and Φ^t is the flow of X lifted to T^*M. Define the space of the generalized resonant states at λ_0,

$$\mathcal{S}_k(\lambda_0) = \left\{\mathbf{u} \in \mathcal{D}'_{E_u^*}(M, \Omega_0^k), \ \mathbf{P}\mathbf{u} = \lambda_0 \mathbf{u}\right\}.$$

The following lemma shows that under a semisimplicity condition, the geometric multiplicity of λ_0 coincides with its algebraic one.

Lemma 6.2. *Assume the following condition :*

$$\forall \mathbf{u} \in \mathcal{D}'_{E_u^*}(M, \Omega_0^k), \quad (\mathbf{P} - \lambda_0)^2 \mathbf{u} = 0 \;\Rightarrow\; (\mathbf{P} - \lambda_0)\mathbf{u} = 0. \tag{6.3}$$

Then $\dim \mathcal{S}_k(\lambda_0) = m_k(\lambda_0)$*, where* $m_k(\lambda_0)$ *is defined in Remark 5.6.*

Proof. We first show that we have $\mathcal{S}_k(\lambda_0) \subset \mathrm{Ran}(\Pi_k)$. Indeed, take $\mathbf{u} \in \mathcal{S}_k(\lambda_0)$. Take $\mu \gg 0$ so that $\mathbf{u} \in H^{-\mu}(M, \Omega_0^k)$. For an appropriate choice of order function (namely, $|u| > \mu$), we have $\mathbf{u} \in H_{G_m}(M, \Omega_0^k)$ because $m \equiv u$ near E_u^*. Since $\mathbf{u} \in \mathcal{S}_k(\lambda_0)$ we have $\mathbf{u} \in \mathcal{D}(\mathbf{P})$ and we can write $\mathbf{R}_\lambda \mathbf{u} = (\lambda - \lambda_0)^{-1}\mathbf{u}$. Using the Laurent series expansion (5.8) we obtain $\Pi_k \mathbf{u} = \mathbf{u}$.
Now assume $\mathbf{u} \in \mathrm{Ran}(\Pi_k)$. We have $\Pi_k \mathbf{u} = \mathbf{u}$ and $\mathbf{u} \in \mathcal{D}'_{E_u^*}(M)$ thanks to Theorem 4.8. Therefore, $(\mathbf{P} - \lambda_0)^{J(\lambda_0)}\mathbf{u} = 0$ since $(\mathbf{P} - \lambda_0)^{J(\lambda_0)}\Pi = 0$. Now iterating (6.3), we conclude that $\mathbf{u} \in \mathcal{S}_k(\lambda_0)$. Thus $\mathrm{Ran}(\Pi_k) \subset \mathcal{S}_k(\lambda_0)$, which concludes. □

In what follows, we will consider the pairing $\langle \cdot, \cdot \rangle$ on $L^2(M)$ associated with the volume form vol. Let $P = \mathbf{P}_0 = \mathbf{P}|_{\mathcal{C}^\infty(M)} = -iX$. The following lemma shows that smoothness of Pu imply smoothness of u provided a sign condition on the quantum flux of u is satisfied (see the discussion in [DZ17b] preceding Lemma 2.3).

Lemma 6.3. *Assume that* $u \in \mathcal{D}'_{E_u^*}(M)$ *satisfies* $Pu \in \mathcal{C}^\infty(M)$ *and* $\Im\langle Pu, u\rangle \geq 0$*. Then* $u \in \mathcal{C}^\infty(M)$*.*

Proof. Take $u \in \mathcal{D}'_{E_u^*}(M)$ and $N > 0$ such that $u \in H^{-N}(M)$. Thanks to (A.4), it suffices to show that for every $(x, \xi) \in T^*M \setminus 0$ and all $A \in \Psi_h^0(M)$ microlocalized near (x, ξ), one has $\|Au\|_{L^2} = \mathcal{O}(h^\infty)\|u\|_{H_h^{-N}}$. As in Lemma 4.5 ($E_u^*$ is a radial source for ϕ^{-t}), we can find an escape function $g \in \mathcal{C}_c^\infty(T^*M, [0, 1])$ such that

(i) $g \equiv 1$ near 0,

(ii) $\mathbf{X}(g) \leq 0$ near E_u^*,

(iii) $\mathbf{X}(g) < 0$ on $E_u^* \cap \mathrm{WF}_h(A)$.

We then choose some symmetric operator $G \in \Psi_h^0(M)$ such that $\sigma_h(G) = g$, $\mathrm{WF}_h(G) \subset \mathrm{supp}\; g$ and $G = I$ microlocally near 0. Because $\mathbf{X}(g) < 0$ on $\mathrm{WF}_h(A) \cap E_u^*$, we can find $C > 0$ and a compactly microlocalized $A_1 \in \Psi_h^0(M)$ with $\mathrm{WF}_h(A_1) \cap E_u^* = \emptyset$ such that

$$|\sigma_h(A)|^2 \leq -\frac{C}{2}\mathbf{X}(g) + C|\sigma_h(A_1)|^2. \tag{6.4}$$

Moreover, thanks to Proposition A.8 and the fact that $\sigma_h(hP) = H$ where $H(x,\xi) = \xi \cdot X_x$, we have $-\sigma_h([P,G]) = \frac{1}{h} ih\mathbf{X}(g) = i\mathbf{X}(g)$ which reads

$$\sigma_h\left(\frac{1}{2i}[P,G]\right) = -\frac{1}{2}\mathbf{X}(g). \tag{6.5}$$

Combining this fact with (6.4) and (6.5) we get using sharp Gårding's inequality (Theorem A.22) applied to the operator $-\frac{C}{2i}[P,G] + CA_1^*A_1 - A^*A$:

$$\left\langle \left(-\frac{C}{2i}[P,G] + CA_1^*A_1 - A^*A\right)\widehat{\chi}u, \widehat{\chi}u\right\rangle \geq -Ch\|\widehat{\chi}u\|_{L^2},$$

where $\widehat{\chi}$ lies in $\Psi_h^0(M)$ and satisfies $\mathrm{WF}_h(\widehat{\chi}) \subset T^*M \setminus 0$ and

$$\widehat{\chi} = I \text{ microlocally near } \mathrm{WF}_h([P,G]) \cup \mathrm{WF}_h(A_1) \cup \mathrm{WF}_h(A).$$

We therefore obtain

$$\|A\widehat{\chi}u\|_{L^2}^2 \leq C\|A_1\widehat{\chi}u\|_{L^2}^2 + \frac{C}{2i}\langle\widehat{\chi}^*[P,G]\widehat{\chi}u, u\rangle + Ch\|\widehat{\chi}u\|_{L^2}. \tag{6.6}$$

Note that since $\mathcal{L}_X \operatorname{vol} = 0$, P is symmetric on $L^2(M)$. Because G is also symmetric,

$$\Im\langle GPu, u\rangle = -\frac{1}{2i}\langle [P,G]u, u\rangle\,, \tag{6.7}$$

and we know that $\mathrm{WF}_h(Pu) \cap (\overline{T}^*M \setminus 0) = \emptyset$ thanks to Remark A.14. As a consequence, since $GP = (G-I)P + P$ and $I = G$ microlocally near 0 we obtain with $\Im\langle Pu, u\rangle \geq 0$:

$$\Im\langle GPu, u\rangle \leq \mathcal{O}(h^\infty)\|u\|_{H_h^{-N}}. \tag{6.8}$$

Thanks to the restrictions of the wavefront set of $\widehat{\chi}$, we obtain that $A_1\widehat{\chi}u = A_1u + \mathcal{O}(h^\infty)_{\mathcal{C}^\infty}$ (and idem for A) and $\widehat{\chi}^*[P,G]\widehat{\chi}u = [P,G]u + \mathcal{O}(h^\infty)_{\mathcal{C}^\infty}$. Moreover, $\mathrm{WF}_h(A_1) \cap E_u^* = \emptyset$ which implies since $u \in \mathcal{D}'_{E_u^*}$ that $A_1u = \mathcal{O}(h^\infty)_{\mathcal{C}^\infty}$. Those remarks together with (6.7), (6.8) and (6.6) leads to

$$\|Au\|_{L^2} \leq \tilde{C}h^{1/2}\|\widehat{\chi}u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{H^{-N}}.$$

Iterating this estimate we obtain $\|Au\|_{L^2} = \mathcal{O}(h^\infty)\|u\|_{H^{-N}}$, which concludes. □

6.2 Calculus of the spaces of generalized resonant states

In this subsection we prove Theorem 6.1. It suffices, thanks to the discussion following the Theorem, to calculate the numbers $m_k(0)$ for $k = 0,1,2$. To do so, we will calculate the dimensions of the spaces $\mathcal{S}_k(0)$ and show that the conditions of Lemma 6.2 are fulfilled. We first deal with the cases $k = 0,2$.

Proposition 6.4. *We have*

$$\mathcal{S}_0(0) = \{c,\ c \in \mathbb{C}\}, \quad \mathcal{S}_2(0) = \{c\mathrm{d}\alpha,\ c \in \mathbb{C}\}.$$

Proof. We start by showing that $\mathcal{S}_0(0)$ is the space of constant function on M. Take $u \in \mathcal{S}_0(0)$. We have $Pu = 0$, and we can apply Lemma 6.3 to get $u \in \mathcal{C}^\infty(M)$. Therefore, $u \circ \Phi^t = u$ for all t, which implies in particular that

$$\mathrm{d}u_x \cdot v = \mathrm{d}u_{\phi^t(x)} \left(\mathrm{d}\phi_x^t \cdot v\right), \quad (x, v) \in TM.$$

Taking $v_s \in E_s(x)$ and letting $t \to +\infty$, we obtain $\mathrm{d}u_x \cdot v_s = 0$ thanks to (2.1). Similarly (letting $t \to -\infty$) we have $\mathrm{d}u_x \cdot v_u = 0$ for $v_u \in E_u(x)$. Therefore $\mathrm{d}u|_{E_s \oplus E_u} = 0$. Recall that we are in the case where X is a geodesic flow, which is a contact flow (see beginning of section 6). As a consequence, $E_u \oplus E_s = \ker \alpha$ and we can write (two linear forms with same kernels are colinear)

$$\mathrm{d}u = \varphi\alpha$$

where $\varphi \in \mathcal{C}^\infty(M)$. But now we have

$$\mathrm{d}u \wedge \mathrm{d}\alpha = \varphi\alpha \wedge \mathrm{d}\alpha = \alpha \wedge \mathrm{d}(\varphi\alpha) = 0,$$

which implies that $\varphi = 0$ and thus $\mathrm{d}u = 0$ which concludes the case $k = 0$. Now take $\mathbf{u} \in \mathcal{S}_2(0)$. We have $\iota_X \mathbf{u} = 0$ which implies that we can write (since $\iota_X \mathrm{d}\alpha = 0$)

$$\mathbf{u} = u\mathrm{d}\alpha,$$

for some $u \in \mathcal{D}'_{E_u^*}(M)$. We have

$$0 = \mathcal{L}_X \mathbf{u} = \mathrm{d}\iota_X \mathbf{u} + \iota_X \mathrm{d}\mathbf{u} = \iota_X \mathrm{d}(u\mathrm{d}\alpha) = (Xu)\mathrm{d}\alpha,$$

so $Xu = 0$ and we can apply the case $k = 0$ to obtain that u is constant, which concludes. □

We show in the next proposition that $\dim \mathcal{S}_1(0) = b_1(M)$ where $b_1(M) = \dim_{\mathbb{C}} H^1(M, \mathbb{C})$ is the first Betti number of M.

Proposition 6.5. *There exists an isomorphism of vector spaces*

$$\mathcal{S}_1(0) \cong H^1(M, \mathbb{C}).$$

Proof. First note that if $\mathbf{u} \in \mathcal{S}_1(0)$, then $\mathrm{d}\mathbf{u} = 0$. Indeed, we have $0 = \mathcal{L}_X \mathbf{u} = \iota_X \mathrm{d}\mathbf{u}$, which implies that $\mathrm{d}\mathbf{u} \in \mathcal{S}_2(0)$, because $\mathcal{L}_X \mathrm{d}\mathbf{u} = \mathrm{d}\iota_X \mathrm{d}\mathbf{u} = 0$. Thanks to Proposition 6.4, we have $\mathrm{d}\mathbf{u} = c\mathrm{d}\alpha$ for some $c \in \mathbb{C}$. Moreover, we have the formula

$$\mathbf{u} \wedge \mathrm{d}\alpha = (\iota_X \mathbf{u}) \operatorname{vol}. \tag{6.9}$$

To see this, write $\mathbf{u}=\varphi\alpha+\iota_X\mathbf{w}$ for some $\mathbf{w}\in\mathcal{D}'(M,\Omega^1)$; then $\iota_X\mathbf{u}=\varphi$ which gives the formula. Since $\iota_X\mathbf{u}=0$, we have $\mathbf{u}\wedge \mathrm{d}\alpha=0$. But now

$$c\,\mathrm{vol}(M)=c\int_M\alpha\wedge\mathrm{d}\alpha=\int_M\alpha\wedge\mathrm{d}\mathbf{u}=\int_M\mathbf{u}\wedge\mathrm{d}\alpha=0$$

thanks to Stokes's theorem. Thus $c=0$ and $\mathrm{d}\mathbf{u}=0$. Take $\mathbf{u}\in\mathcal{S}_1(0)$; since $\mathrm{d}\mathbf{u}=0$ we have thanks to Lemma B.6 the existence of $\varphi\in\mathcal{D}'_{E_u^*}(M)$ such that $\mathbf{u}-\mathrm{d}\varphi$ is smooth. If $\tilde\varphi\in\mathcal{D}'_{E_u^*}(M)$ also satisfies that $\mathbf{u}-\mathrm{d}\tilde\varphi$ is smooth, then $\mathrm{d}(\tilde\varphi-\varphi)$ is smooth and again by Lemma B.6, we obtain that $\varphi-\tilde\varphi$ is smooth, which implies that $\mathbf{u}-\mathrm{d}\varphi=\mathbf{u}-\mathrm{d}\tilde\varphi+\mathrm{d}\psi$, where $\psi=\varphi-\tilde\varphi$ is smooth. As a consequence, we have a well defined morphism

$$\Psi:\mathcal{S}_1(0)\to H^1(M,\mathbb{C}),\quad \mathbf{u}\mapsto[\mathbf{u}-\mathrm{d}\varphi].$$

Let $\mathbf{u}\in\mathcal{S}_1(0)$ such that $\Psi(\mathbf{u})=0$. Then there is $\varphi\in\mathcal{D}'_{E_u^*}$ and $\psi\in\mathcal{C}^\infty(M)$ such that $\mathbf{u}-\mathrm{d}\varphi=\mathrm{d}\psi$, which reads $\mathbf{u}=\mathrm{d}\tilde\varphi$ with $\tilde\varphi=\varphi+\psi$. One has $0=\mathcal{L}_X\mathbf{u}=\mathcal{L}_X\mathrm{d}\tilde\varphi=\iota_X\mathrm{d}\tilde\varphi=X\tilde\varphi$. By Proposition 6.4, $\tilde\varphi$ is constant and $\mathbf{u}=0$, which proves that Ψ is injective. Now take a smooth closed one form $\mathbf{v}$. We know thanks to Proposition 6.4 and (5.8) that

$$\mathbf{R}_{\lambda,0}=\mathbf{Y}_{\lambda,0}-\frac{\Pi_0}{\lambda},$$

where $\mathbf{Y}_{\lambda,0}$ is holomorphic near 0. Set $f=-\iota_X\mathbf{v}$ and $\varphi=-i\mathbf{Y}_{0,0}f\in H_{G_m}(M)$. We know thanks to Theorem 4.8 that $\varphi\in\mathcal{D}'_{E_u^*}(M)$. We have

$$X\varphi=iP\varphi=P\mathbf{Y}_{0,0}f=f+\Pi_0f,\tag{6.10}$$

since $(P-\lambda)\mathbf{R}_{\lambda,0}=I$ on $H_{G_m}(M)$ and $P\Pi_0=0$. Now note that thanks to (3.5), we have :

$$(\mathbf{R}_{\lambda,0})^*=-\tilde{\mathbf{R}}_{-\bar\lambda,0}$$

with respect to the vol-pairing on $L^2(M)$, where $\tilde{\mathbf{R}}_{\lambda,0}$ is the resolvent of $-\mathbf{P}_0$. Now Proposition 6.4 applied for $-X$ instead of X gives that the range of Π_0^* also consists of constant functions. Therefore for $g\in\mathcal{C}^\infty(M)$, since $\Pi_0^2=\Pi_0$:

$$\langle\Pi_0g,1\rangle=\langle g,\Pi_0^*1\rangle=\langle g,1\rangle.$$

As a consequence, using (6.1),

$$-\Pi_0f=-\int_M f\,\mathrm{vol}=\int_M\iota_X\mathbf{v}\,\mathrm{vol}=\int_M\mathbf{v}\wedge\iota_X\,\mathrm{vol}=\int_M\mathbf{v}\wedge\mathrm{d}\alpha=\int_M\mathrm{d}\mathbf{v}\wedge\alpha=0.$$

Therefore, (6.10) gives $-\iota_X\mathbf{v}=X\varphi$. Let $\mathbf{u}=\mathbf{v}+\mathrm{d}\varphi$. We have $\mathrm{d}\mathbf{u}=0$ and

$$\iota_X\mathbf{u}=\iota_X\mathbf{v}+\iota_X\mathrm{d}\varphi=\iota_X\mathbf{v}+X\varphi=0,$$

which proves that $\mathbf{u}\in\mathcal{S}_1(0)$ and $\Psi(\mathbf{u})=[\mathbf{v}]\in H^1(M,\mathbb{C})$. Thus Ψ is surjective. $\square$

We are now in position to compute the vanishing order of ζ at zero.

Proof of Theorem 6.1. We first show that the condition of Lemma 6.3 is satisfied, that is for every $\mathbf{u} \in \mathcal{D}'_{E_u^*}(M, \Omega_0^k)$ such that $\mathbf{P}^2\mathbf{u} = 0$, then $\mathbf{Pu} = 0$. We first deal with the case $k = 0$. Take $u \in \mathcal{D}'_{E_u^*}(M)$ such that $P^2 u = 0$, that is $\iota_X \mathrm{d}\iota_X \mathrm{d}u = 0$. We want to show that $\iota_X \mathrm{d}u = 0$. Since $\iota_X \mathrm{d}u \in \mathcal{S}_0(0)$, we have thanks to Proposition 6.4 that $\iota_X \mathrm{d}u$ is constant. On the other hand, integrating by parts,

$$c\,\mathrm{vol}(M) = \int_M \iota_X \mathrm{d}u = 0.$$

Thus $c = 0$ and $\iota_X \mathrm{d}u = 0$. The case $k = 2$ is similar ; if $\iota_X \mathrm{d}\mathbf{u} \in \mathcal{S}_2(0)$, then $\iota_X \mathrm{d}\mathbf{u} = c\mathrm{d}\alpha$ for some $c \in \mathbb{C}$ thanks to Proposition 6.4. Since $\mathcal{L}_X \alpha = 0$ we have

$$c\,\mathrm{vol}(M) = \int_M \alpha \wedge \iota_X \mathrm{d}\mathbf{u} = -\int_M \mathcal{L}_X(\alpha \wedge \mathbf{u}) = 0,$$

which concludes the case $k = 2$. Now we deal with the case $k = 1$, which is more difficult. Take $\mathbf{u} \in \mathcal{D}'_{E_u^*}(M, \Omega_0^1)$ such that $\iota_X \mathrm{d}\mathbf{u} \in \mathcal{S}_1(0)$. We want to show that $\iota_X \mathrm{d}\mathbf{u} = 0$. Take $u \in \mathcal{D}'_{E_u^*}$ such that $\alpha \wedge \mathrm{d}\mathbf{u} = u\,\mathrm{vol}$. We have thanks to (6.9) :

$$\langle u, \mathrm{vol}\rangle = \langle \alpha \wedge \mathrm{d}\mathbf{u}, 1\rangle = \langle \mathrm{d}\alpha \wedge \mathbf{u}, 1\rangle = \langle \iota_X \mathbf{u}, \mathrm{vol}\rangle = 0.$$

Letting $\mathbf{v} = \iota_X \mathrm{d}\mathbf{u}$, we have $\mathrm{d}\mathbf{v} = 0$ thanks to the proof of Proposition 6.5 because $\mathbf{v} \in \mathcal{S}_1(0)$. Since $\mathcal{L}_X(\alpha) = 0$ and $\mathcal{L}_X(\mathrm{d}\alpha) = 0$, we have

$$(Xu)\,\mathrm{vol} = \mathcal{L}_X(\alpha \wedge \mathrm{d}\mathbf{u}) = \alpha \wedge \mathrm{d}\mathbf{v} = 0.$$

Thus $Xu = 0$ and Proposition 6.4 implies that u is a constant function, but since $\langle u, \mathrm{vol}\rangle = 0$ we have $u = 0$. This implies $\alpha \wedge \mathrm{d}\mathbf{u} = 0$. Thus, one has $0 = \iota_X(\alpha \wedge \mathrm{d}\mathbf{u}) = (\iota_X \alpha) \wedge \mathrm{d}\mathbf{u} - \alpha \wedge \iota_X \mathrm{d}\mathbf{u}$, which reads since $\iota_X \alpha = 1$:

$$\mathrm{d}\mathbf{u} = \alpha \wedge \iota_X \mathrm{d}\mathbf{u} = \alpha \wedge \mathbf{v}. \tag{6.11}$$

Apply (the proof of) Proposition 6.5 to obtain $\varphi \in \mathcal{D}'_{E_u^*}(M)$, $\mathbf{w} \in \mathcal{C}^\infty(M, \Omega^1)$ such that $\mathrm{d}\mathbf{w} = 0$ and

$$\mathbf{v} = \mathbf{w} + \mathrm{d}\varphi.$$

Moreover, since $\mathbf{v} \in \mathcal{S}_0(1)$, one has $\iota_X \mathbf{v} = 0$ which leads to

$$\iota_X \mathbf{w} + X\varphi = 0. \tag{6.12}$$

We have $\mathrm{d}\mathbf{u} \wedge \bar{\mathbf{w}} = \mathrm{d}(\mathbf{u} \wedge \bar{\mathbf{w}})$; thus with (6.11) we have

$$0 = \langle \mathrm{d}\mathbf{u} \wedge \bar{\mathbf{w}}, 1\rangle = \langle \alpha \wedge \mathrm{d}\varphi \wedge \bar{\mathbf{w}}, 1\rangle + \langle \alpha \wedge \mathbf{w} \wedge \bar{\mathbf{w}}, 1\rangle$$

We have $\langle \alpha \wedge \mathbf{w} \wedge \bar{\mathbf{w}}, 1\rangle \in i\mathbb{R}$. Therefore, taking real part and using (6.9) and (6.12):

$$0 = \Re\langle \varphi \bar{\mathbf{w}} \wedge \mathrm{d}\alpha, 1\rangle = \Re\langle \varphi \iota_X \bar{\mathbf{w}}, \mathrm{vol}\rangle = -\Re\langle X\varphi, \varphi\rangle.$$

Noting that $P = -iX$, we can use Lemma 6.3 to obtain that φ is smooth, and thus $\mathbf{v}$ is also smooth. Since $\mathcal{L}_X \mathbf{v} = 0$, we have $(\phi^t)^* \mathbf{v} = \mathbf{v}$ for all t. In particular,

$$\left\langle \mathbf{v}_{\phi^t(x)}, \mathrm{d}\phi^t_x \cdot v \right\rangle = \langle \mathbf{v}_x, v \rangle \quad (x,v) \in TM, \quad t \in \mathbb{R}.$$

Letting $t \to +\infty$, we obtain $\langle \mathbf{v}_x, v_s \rangle = 0$ for $v_s \in E_s(x)$. Letting $t \to -\infty$, we obtain $\langle \mathbf{v}_x, v_u \rangle = 0$ for $v_u \in E_u(x)$. Therefore $\mathbf{v}|_{E_s \oplus E_u} = 0$; since $\iota_X \mathbf{v} = 0$, we obtain $\mathbf{v} = 0$, which concludes the case $k = 1$. We have thus proved that the order at zero of ζ is $m_1(0) - m_0(0) - m_2(0) = b_1(M) - 2$. To conclude it thus suffices to show that $b_1(M) = b_1(\Sigma)$. Let $\tilde{\pi} : S^*\Sigma \to \Sigma$ be the natural projection. Recall that we have the Gysin exact sequence ([Hat02, p. 438]):

$$0 \longrightarrow H^1(\Sigma, \mathbb{C}) \xrightarrow{\tilde{\pi}^*} H^1(S^*\Sigma, \mathbb{C}) \longrightarrow H^0(\Sigma, \mathbb{C}) \xrightarrow{\smile \mathrm{e}(\Sigma)} H^2(\Sigma, \mathbb{C}), \tag{6.13}$$

where $\smile$ is the cup product and $\mathrm{e}(\Sigma) \in H^2(\Sigma, \mathbb{C})$ is the Euler class of Σ, characterized by

$$\langle \mathrm{e}(\Sigma), [\Sigma] \rangle = \chi(\Sigma),$$

where $[\Sigma] \in H_2(\Sigma, \mathbb{C})$ is the fundamental class of Σ. Here we have $\chi(\Sigma) = 2 - 2g < 0$ thanks to Gauss-Bonnet theorem. As a consequence, the last arrow in (6.13) is an isomorphism because $\mathrm{e}(\Sigma) \neq 0$ and $H^0(\Sigma, \mathbb{C}) \cong H^2(\Sigma, \mathbb{C}) = \mathbb{C}$, and thus $\tilde{\pi}^* : H^1(\Sigma, \mathbb{C}) \to H^1(S^*\Sigma, \mathbb{C})$ is also an isomorphism. Thus $b_1(M) = b_1(\Sigma)$, which concludes. $\square$

A Microlocal and semiclassical calculus

A.1 Pseudo-differential operators on $\mathbb{R}^n$

We refer to [Ler17] or [Hör94] for a complete description of pseudo-differential operators.

Definition A.1. Let $m \in \mathbb{R}$ and $\rho \in (0,1]$. We will say that a smooth function $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ lies in S^m_ρ if for all multi-indices $\alpha, \beta \in \mathbb{N}^n$, there exists $C_{\alpha\beta}$ such that

$$\forall x, \xi \in \mathbb{R}^n, \quad |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-\rho|\alpha|+(1-\rho)|\beta|},$$

where $\langle \xi \rangle = \sqrt{1+|\xi|^2}$. We will denote S^m_1 by S^m, and $a \in S^m$ will be called a symbol of order m, and $S^{-\infty}_\rho = \cap_{m \in \mathbb{R}} S^m_\rho$.

Remark A.2. One can also define the class S^m_ρ if $m \in S^0$ is an order function, replacing $\langle \xi \rangle^{m-\rho|\alpha|+(1-\rho)|\beta|}$ by $\langle \xi \rangle^{m(x,\xi)-\rho|\alpha|+(1-\rho)|\beta|}$ in the previous estimate.

A symbol $a \in S^m$ induces a *pseudo-differential operator* $A = \hat{a} = \mathrm{Op}(a) = a(x,D) : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is the Schwarz space on $\mathbb{R}^n$, by

$$(Au)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{ix\xi} a(x,\xi) \hat{u}(\xi) d\xi, \quad u \in \mathcal{S}, \quad x \in \mathbb{R}^n. \tag{A.1}$$

One can extend A to the Sobolev spaces H^s so that $A : H^s \to H^{s-m}$ is continuous for all $s \in \mathbb{R}$. Here is a little summary of the properties of pseudo-differential operators.

Proposition A.3 (Basic properties of ΨDO's)**.** *Let $m, m' \in \mathbb{R}$, $\rho \in (1/2, 1]$, $a \in S^m_\rho$ and $b \in S^{m'}_\rho$. We have the following properties:*

- $ab \in S^{m+m'}$ *and* $\mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}(a \diamond b)$ *with* $a \diamond b = ab \mod S^{m+m'-(2\rho-1)}_\rho$.
- $\mathrm{Op}(a)^* = \mathrm{Op}(a^*)$ *where* $a^* \in S^m_\rho$ *and* $a^* = \overline{a} \mod S^{m-(2\rho-1)}_\rho$.
- $a \diamond b - b \diamond a = \frac{1}{i}\{a,b\} \mod S^{m+m'-2(2\rho-1)}_\rho$, *where* $\{\cdot,\cdot\}$ *denotes the usual Poisson bracket on* $\mathbb{R}^n$.

Remark A.4. These properties also works for classes with variable order $m \in S^0$.

The following result [Ler17, Theorem 8.6] allows us to (almost) invert microlocally the ΨDO's.

Theorem A.5 (Microlocal inversion)**.** *Let $\chi \in S^0$ and $a \in S^m$ such that a is elliptic on* $\mathrm{supp}\, \chi$, *namely* $\inf_{(x,\xi) \in \mathrm{supp}\, \chi} |a(x,\xi)| \langle \xi \rangle^{-m} > 0$. *Let $\psi \in S^0$ such that* $\mathrm{supp}\, \psi \subset \mathrm{int}\{\chi = 1\}$. *Then there exists $b \in S^{-m}$ such that*

$$b \diamond a = \psi + l,$$

where l lies in $S^{-\infty}$.

A.2 Pseudo-differential operators on Manifolds

In the following, M will denote a smooth compact manifold of dimension n, and N another compact manifold.

Definition A.6. Let $A : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ be an operator. We'll say that A is a pseudo-differential operator or order m if for any coordinate chart $\kappa : U \subset M \to V \subset \mathbb{R}^n$, one has for any $\psi, \phi \in \mathcal{C}_c^\infty(U)$ and $u \in \mathcal{C}^\infty(M)$:

$$\psi A \phi u = \psi \kappa^* a_\kappa(x, D) \kappa_*(\phi u),$$

for some $a_\kappa \in S^m$. We'll denote by $\Psi^m(M)$ the set of such operators.

We also define $S^m_\rho(M)$ the class of symbols $a \in \mathcal{C}^\infty(T^*M)$ such that there exists $C_{K\alpha\beta}$ such that

$$\forall (x,\xi) \in K \times \mathbb{R}^n, \quad |\partial_\xi^\alpha \partial_x^\beta (\kappa_* a)(x,\xi)| \leq C_{K\alpha\beta} \langle \xi \rangle^{m - \rho|\alpha| + (1-\rho)|\beta|}, \tag{A.2}$$

for any trivialization chart $\kappa : U \to V$ and any compact subset $K \subset V$. For $a \in S^m(M)$, we can quantize a by the following process. Let (U_κ, κ) be some atlas of M and ψ_κ be a subordinate partition of unity. For $u \in \mathcal{C}^\infty(M)$, let

$$\mathrm{Op}(a)u := \sum_\kappa \chi_\kappa \kappa^* A_\kappa \chi_\kappa \kappa_* u,$$

where $A_\kappa = (\kappa_* a)(x, D)$. This quantization depends of the coordinate charts. However, one can show that using a different atlas, we obtain the same operator modulo $\Psi^{m-1}(M)$. Thus, we have a well defined bijection

$$\Psi^m(M)/\Psi^{m-1}(M) \cong S^m(M)/S^{m-1}(M);$$

the image of an operator A under this map is called its principal symbol and will be denoted by $\sigma(A)$. It carries a geometrical meaning. Note that every point of A.3 also works for pseudo-differential operators on manifolds, with principal symbols.

One can define pseudo-differential operators acting on a vector bundle E as follows : we say that $\mathbf{A} : \mathcal{C}^\infty(M, E) \to \mathcal{D}'(M, E)$ lies in $\Psi^k(M, \mathrm{Hom}(E))$ if in every local frame $\mathbf{e}_1, \cdots \mathbf{e}_r$ of E over an open set $U \subset M$ one has

$$\mathbf{A}(f \mathbf{e}_l) = \sum_{j=1}^r (A_{jl} f) \mathbf{e}_j, \quad f \in \mathcal{C}_c^\infty(U), \quad l \in \{1, \cdots, r\},$$

where $A_{jl} \in \Psi^k(U)$. In this case the principal symbol $\sigma(\mathbf{A})$ of $\mathbf{A}$ lies in the class $S^k(M, \mathrm{Hom}(E))/S^{k-1}(M, \mathrm{Hom}(E))$.

A.3 Semiclassical calculus

Let us now introduce semiclassical pseudodifferential operators. We refer here to [Zwo12] or [DZ17a] for more details.

Definition A.7. Let $A = (A_h)_{h\in(0,1)}$, $A_h : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ be a family of operators. We'll say that A is a semiclassical ΨDO of order m if for any coordinate chart $\kappa : U \subset M \to V \subset \mathbb{R}^n$, one has for any $\psi, \phi \in \mathcal{C}_c^\infty(U)$ and $u \in \mathcal{C}^\infty(M)$:

$$\psi A_h \phi u = \psi \kappa^* a_{\kappa,h}(x, hD)\kappa_*(\phi u),$$

for some $a_{\kappa,h} \in S^m$. We will use the notation $\mathrm{Op}_h(a) := a(x, hD)$ for a in S^m and we demand that the semi norms of $a_{\kappa,h}$ in S^m are bounded uniformly in h. The dependance of h of the symbols will not usually be explicit. We'll denote by $\Psi_h^m(M)$ the set of such operators.

Note that every $A \in \Psi^k(M)$ define an operator $(A_h) \in \Psi_h^k(M)$ with $A_h = \mathrm{Op}_h(A)$. The semiclassical principal symbol of a semiclassical operator A will be denoted by $\sigma_h(A)$ and lies in $S_h^m(M)/hS_h^{m-1}(M)$, where $S_h^m(M)$ is the set of families $a = (a_h)$ with $a_h \in S^m(M)$ uniformly in h. Quantizing a symbol $a \in S_h^m$ as in the previous subsection gives us a bijection (with $(\kappa_* a)(x, D)$ replaced by $(\kappa_* a_h)(x, hD)$):

$$\Psi_h^m(M)/\Psi_h^{m-1}(M) \cong S^m(M)/hS_h^{m-1}(M).$$

We have the following version of A.3 for semiclassical ΨDO's (we removed the ρ's for simplicity):

Proposition A.8 (Basic properties of semiclassical ΨDO's)**.** *Let $A \in \Psi_h^m(M)$ and $B \in \Psi_h^{m'}(M)$. Then*

- *$AB \in \Psi_h^{m+m'}(M)$ and $\sigma_h(AB) = \sigma_h(A)\sigma_h(B) \mod hS_h^{m+m'-1}(M)$,*
- *$\sigma_h(A^*) = \overline{\sigma_h(A)} \mod hS_h^{m-1}(M)$,*
- *$\sigma_h([A, B]) = \frac{h}{i}\{\sigma_h(A), \sigma_h(B)\} \mod h^2 S_h^{m+m'-2}(M)$, where $\{\cdot,\cdot\}$ is the canonical Poisson bracket on T^*M.*

The semiclassical norms $||\cdot||_{H_h^\mu}$ are defined locally as follows: for u supported in a coordinate patch let $||u||_{H_h^\mu} := (2\pi)^{-n}||\langle h\xi\rangle^\mu \hat{u}(\xi)||_{L^2} = (2\pi h)^{-n}||\mathcal{F}_h(u)||_{H^\mu}$, where $\mathcal{F}_h$ is the semiclassical Fourier transform, namely $\mathcal{F}_h(u)(\xi) = \hat{u}(\xi/h)$. Semiclassical ΨDO's act between semiclassical Sobolev spaces H_h^μ, which means that if $A \in \Psi_h^k$, one has $||A_h u||_{H_h^{\mu+k}} \leq C||u||_{H_h^\mu}$ for $u \in H^\mu$ and C independent of h. The norms on $H_h^\mu(M)$ are defined globally using partitions of unity, mainly let $(U_\kappa,\ \kappa : U_\kappa \to \mathbb{R}^n)$ is a finite atlas on M and (ψ_κ) is a partition of unity subordinated to (U_κ), then

$$\|u\|_{H_h^\mu}^2 := \sum_\kappa \|\kappa_*(\psi_\kappa u)\|_{H_h^\mu}^2.$$

For a different choice of atlas the norm will be equivalent.

Definition A.9. We will call a family of distributions $u = (u_h)_{h\in(0,1)}$ to be h-tempered if there is $\mu \in \mathbb{R}$, $N \geq 0$ such that

$$||u_h||_{H_h^\mu} = \mathcal{O}(h^{-N})$$

as $h \to 0$. We will denote by $\mathcal{D}_h(M)$ the set of h-tempered distributions (or $\mathcal{D}'_h(M, E)$ for those with values in a vector bundle E).

Similarly, we introduce h-tempered families of operators which are families $B = (B_h)_h$, $B_h : \mathcal{C}^\infty(M) \to \mathcal{D}'(M)$ such that there is $N \geq 0$ satisfying

$$||B_h||_{H_h^\mu \to H_h^{\mu-N}} = \mathcal{O}(h^{-N})$$

as $h \to 0$.

Distributions independent of h are h-tempered. We allow h-tempered distributions to become singular as $h \to 0$, but if so it is at a controlled rate. As we have seen before, if $A \in \Psi_h^k$ and u is h-tempered, then $Au = (A_h u_h)_h$ is also h-tempered.

One can define semiclassical pseudo-differential operators acting on a vector bundle E as follows : we say that $\mathbf{A} = (\mathbf{A}_h)_h : \mathcal{C}^\infty(M, E) \to \mathcal{D}'(M, E)$ lies in $\Psi_h^k(M, \mathrm{Hom}(E))$ if in every local frame $\mathbf{e}_1, \cdots \mathbf{e}_r$ of E over an open set $U \subset M$ one has

$$\mathbf{A}(f\mathbf{e}_l) = \sum_{j=1}^r (A_{jl} f)\mathbf{e}_j, \quad f \in \mathcal{C}_c^\infty(U), \quad l \in \{1, \cdots, r\},$$

where $A_{jl} \in \Psi_h^k(U)$. In this case the semiclassical principal symbol $\sigma_h(\mathbf{A})$ of $\mathbf{A}$ lies in the class $S_h^k(M, \mathrm{Hom}(E))/S_h^{k-1}(M, \mathrm{Hom}(E))$.

A.4 Exponentiation of pseudo-differential operators

Adapting the proof of [Zwo12, Theorem 8.6] for classes of symbols S^m we have the following

Theorem A.10. *Let g be an escape function for an order function m as in section 3, and $G \in \Psi_h^{0+}(M)$ such that $\sigma_h(G) = g$. Then there is a unique family of pseudo differential operators $B(t) : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that*

$$\partial_t B(t) = GB(t), \quad B(0) = I.$$

$B(t)$ is a ΨDO with symbol $\sigma_h(B(t)) = b_t \in S_h^{tm+}$. We will denote $B(t)$ by $\exp(tG)$*.*

This exponentiation will be useful to construct anisotropic Sobolev spaces and to conjugate operators. We have the following

Lemma A.11. *Let $P \in \Psi_h^1(M)$, and $G \in \Psi_h^{0+}(M)$ as in the previous theorem. Let $P_t = \exp(tG)P\exp(-tG)$.*

$$P_t = P + t\,[G, P] + \left[G, \int_0^t (P_s - P)\mathrm{d}s\right], \quad t \in [0,1].$$

Proof. Note that e^{tG} and G commute. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}P_t &= G\mathrm{e}^{tG}P\mathrm{e}^{-tG} - \mathrm{e}^{tG}PG\mathrm{e}^{-tG}\\ &= [G, P_t]\\ &= [G, P] + [G, P_t - P].\end{aligned}$$

Integrating between 0 and t, we get the lemma. □

A.5 Classical and semiclassical wavefront set

The wavefront set of a distribution is a very useful tool to describe its singularities in the phase space. Let $\pi : T^*M \to M$ denote the canonical projection.

Definition A.12. For $u \in \mathcal{D}'(\mathbb{R}^n)$, we define its wavefront set $\mathrm{WF}(u)$ as follows : $(x,\xi) \in T^*\mathbb{R}^n \setminus 0$ does **not** lie in $\mathrm{WF}(u)$ if there exists a conical neighborhood $U \times \Gamma \subset T^*\mathbb{R}^n \setminus 0$ of (x,ξ) and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ supported in U with $\chi(x) \neq 0$ such that for all $N \geq 0$ there exists $C_N > 0$ satisfying

$$\widehat{\chi u}(\eta) \leq C_N \langle\eta\rangle^{-N}, \quad \eta \in \Gamma. \tag{A.3}$$

The wavefront set is a closed conical subset of $T^*\mathbb{R}^n \setminus 0$. If u is a distribution on M instead, we define $\mathrm{WF}(u) \subset T^*M \setminus 0$ as follows : let (U_κ, κ) be an atlas of trivialization of T^*M, with χ_κ a subordinate partition of unity and set $\mathrm{WF}(u) := \bigcup_\kappa \kappa^*\mathrm{WF}(\kappa_*(\chi_\kappa u))$. More generally, if $\mathbf{u} \in \mathcal{D}'(M, E)$ is distribution with values in a vector bundle E, we say that $(x,\xi) \in \mathrm{WF}(\mathbf{u})$ if for some local basis $\mathbf{e}_1, \ldots, \mathbf{e}_r$ of E near x, writing $\mathbf{u}|_U = \sum_i u_i\mathbf{e}_i$, one has $(x,\xi) \in \mathrm{WF}(u_j)$ for some j. For a closed conical subset Γ of T^*M, we define as in [Hör90, Definition 8.2.2] :

$$\mathcal{D}'_\Gamma(M) = \{u \in \mathcal{D}'(M), \mathrm{WF}(u) \subset \Gamma\}.$$

Recall that the topology of $\mathcal{D}'_\Gamma$ is defined as follows : we say $u_m \to u$ in $\mathcal{D}'_\Gamma$ if $u_j \to u$ in $\mathcal{D}'$ and the constants in (A.3) are uniform in m.

For a ΨDO $A \in \Psi^k(M)$, we define the wavefront set $\mathrm{WF}(A)$ as follows : $(x,\xi) \in T^*M \setminus 0$ does not lie in $\mathrm{WF}(A)$ iff one can find a conical neighborhood Γ of (x,ξ) such that $a(x,\xi') = \mathcal{O}(\langle\xi'\rangle^{-\infty})$ in Γ, where $A = \mathrm{Op}(a)$ with $a \in S^k(M)$.

If $\mathbf{A} \in \Psi^k(M, \mathrm{Hom}(E))$ acts on sections of a vector bundle E, we say that $(x,\xi) \notin \mathrm{WF}(\mathbf{A})$ iff $\mathbf{a}(x,\xi') = \mathcal{O}(\langle \xi' \rangle)$ for ξ' in some conical neighborhood of (x,ξ), where $\mathrm{Op}(\mathbf{a}) = \mathbf{A}$ and $\mathbf{a} \in S^k(M, \mathrm{Hom}(E))$.

For an operator $B : \mathcal{C}^\infty(M) \to \mathcal{D}'(N)$, fixing a non-vanishing density μ on M, one gets the Schwartz Kernel $K_B \in \mathcal{D}'(N \times M)$ of B with respect to μ defined by the relation $\langle K_B, (\phi\mu) \otimes \nu \rangle = \langle B\phi, \nu \rangle$ for $\phi \in \mathcal{C}^\infty(M)$ and $\nu \in \mathcal{C}^\infty(N, D)$, where D is the line bundle of densities. We show that $\mathrm{WF}(K_B)$ does not depend of the choice of the density μ and we can thus define $\mathrm{WF}'(B) := \{(y,\eta,x,-\xi) | (y,\eta,x,\xi) \in \mathrm{WF}(K_B)\}$. More generally, if $\mathbf{B} : \mathcal{C}^\infty(M,E) \to \mathcal{D}'(N,F)$ works with vector bundles E and F over M and N (that is, in a local frame, $\mathbf{B}\left(\sum_i u_i \mathbf{e}_i\right) = \sum_{jl} B_{jl} u_j \mathbf{f}_l$ for some operators B_{jl}), its Schwartz kernel $K_{\mathbf{B}}$ lies in $\mathcal{D}'(N \times M, \pi_N^* F \otimes \pi_M^* E^*)$ and we can define its wavefront set $\mathrm{WF}(K_{\mathbf{B}})$; as before we get $\mathrm{WF}'(\mathbf{B})$.

To introduce the semiclassical wavefront set, let us consider the fiber-radially compactified cotangent bundle $\overline{T}^* M$ modeled by the ball $B^*M = \{(x,\xi) \in T^*M \text{ s.t. } |\xi| \leq 1\}$ for some smooth norm on T^*M. Then we have an embedding $\iota : T^*M \to \mathrm{int}(\overline{T}^* M)$ defined by $\iota(x,\xi) = (x, \frac{\xi}{\langle \xi \rangle})$, so that $\overline{T}^* M$ is a manifold with boundary $S^*M = (T^*M \setminus 0)/\mathbb{R}_{>0}$ and with interior T^*M.

Definition A.13. The semiclassical wavefront set $\mathrm{WF}_h(u)$ of an h-tempered family of distributions $(u_h) \subset \mathcal{D}'(M)$ is defined as follows : we'll say that $(x,\xi) \in \overline{T}^* M$ does **not** lie in $\mathrm{WF}_h(u)$ if there exists $\chi \in \mathcal{C}^\infty(M)$ supported in a trivialization chart, $\chi(x) \neq 0$, and a neighborhood U of (x,ξ) in $\overline{T}^* M$, and $h_0 > 0$ such that for all $N \geq 0$, there is $C_N > 0$ satisfying

$$\widehat{\chi u}(\eta/h) \leq C_N h^N \langle \eta \rangle^{-N}, \quad \eta \in U \cap T^*M, \quad 0 < h < h_0,$$

where we identified U with its image in $\mathbb{R}^n \times \mathbb{R}^n$.

Remark A.14. The semiclassical wavefront set away from the fibers infinity does not tell any information about the smoothness of the distributions; however, it captures oscillations in h. For an h-independent distribution u, we have

$$\mathrm{WF}(u) = \mathrm{WF}_h(u) \cap T^*M \setminus 0. \tag{A.4}$$

Moreover, if u is smooth, then $\mathrm{WF}_h(u) \cap (\overline{T}^* M \setminus 0) = \emptyset$.

We define as before the wavefront set of a semiclassical ΨDO $A \in \Psi_h^k(M)$, replacing $\mathcal{O}(\langle \xi' \rangle^{-\infty})$ by $\mathcal{O}(h^\infty \langle \xi' \rangle^{-\infty})$.

Definition A.15. If $A \in \Psi_h^k(M)$ satisfies $\mathrm{WF}_h(A) \cap \partial \overline{T}^* M = \emptyset$, we'll say that A is compactly microlocalized. Note that every compactly microlocalized semiclassical pseudodifferential operator lies in $\cap_k \Psi_h^k(M)$.

Using Schwartz kernels, we can similarly define the wave front set of h-tempered family of operators $B_h : \mathcal{C}^\infty(M) \to \mathcal{D}'(N)$, which is a subset of $\overline{T}^*(N \times M)$.

We have the following useful characterization of the classical wavefront set :

Lemma A.16. *Let $(K_\lambda)_\lambda$ be a bounded family in $\mathcal{D}'(M \times M)$, and $\mathcal{K}_\lambda$ its operator associated on M. Let Γ is a closed conic set in $T^*M \times T^*M$ and $R > 0$. Assume that for each $(y, \eta, x, -\xi) \in T^*(M \times M) \setminus \Gamma$ with $|\eta|, |\xi| \in [R, 2R]$ we can find relatively compact neighborhoods U of (x, ξ) and V of (y, η) in T^*M such that for all h-tempered family of functions $f_h \in \mathcal{C}_c^\infty(\pi_M(U))$,*

$$\mathrm{WF}_h(f) \subset U \Longrightarrow \mathrm{WF}_h(\mathcal{K}_\lambda f) \cap V = \emptyset. \tag{A.5}$$

uniformly in λ, that is, for every N and $\chi \in \mathcal{C}_c^\infty(\pi_M(V))$, there exists $C_{N,\chi,f}$ independant of λ such that $|\mathcal{F}_h(\chi \mathcal{K}_\lambda f_h)(\zeta)| \le C_{N,\chi,f} h^N$ for all $\zeta \in V$ and λ. Then $(K_\lambda)_\lambda$ is a bounded family in $\mathcal{D}'_\Gamma(M \times M)$.

Proof. Take $(y, \eta, x, -\xi) \notin \Gamma$ with $|\xi|, |\eta| \in [R, 2R]$. Take neighborhoods U and V of (x, ξ) and $(y, -\eta)$ such that (A.5) is valid. Up to shrinking a little bit U and V we may assume that $\pi_M(U)$ and $\pi_M(V)$ are supported in a trivialization patch and identifying their image under the trivialization we have $U, V \subset \mathbb{R}^n \times \mathbb{R}^n$. Take $\chi_x \in \mathcal{C}_c^\infty(\pi_M(U))$ and $\chi_y \in \mathcal{C}^\infty(\pi_M(V))$. Put $f_h(x', \xi') = \chi_x(x') \mathrm{e}^{ix' \cdot \xi'/h}$ for $(x', \xi') \in \pi_M(U) \times \mathbb{R}^n$. One has

$$\widehat{\chi_x \chi_y K_\lambda}(\eta'/h, -\xi'/h) = \widehat{\chi_y \mathcal{K}_\lambda f_h}(\eta'/h).$$

Now we have $\mathrm{WF}_h(f) \subset U$ (this is an exercise) and it follows from (A.5) that $\mathrm{WF}_h(\mathcal{K}_\lambda f) \cap V = \emptyset$. Therefore for all N there is C_N such that $\widehat{\chi_y \mathcal{K}_\lambda f_h}(\eta'/h) \le C_N h^N$ for all $(\eta'/h, \xi'/h) \in V \times U$, which implies that $\widehat{\chi_x \chi_y K_\lambda}(\eta', -\xi') \le \widetilde{C_N} \langle \xi' \rangle^{-N} \langle \eta' \rangle^{-N}$ for every (η', ξ') in some conical neighborhood of (η, ξ). This concludes. □

A.6 Microlocal and semiclassical inversion

We recall here some facts about ellipticity and microlocal inversion that will be helpful. See [DZ17a, Appendix E] for more details.

Definition A.17. Let $A \in \Psi^k(M)$ and $(x, \xi) \in T^*M$. We'll say that $(x, \xi) \in \mathrm{ell}(A)$ if there exists a conical neighborhood U of (x, ξ) in T^*M and $c > 0$ such that $|\sigma(A)|(y, \eta) \ge c\langle \eta \rangle^k$ for all $(y, \eta) \in U$. If $\mathbf{A} \in \Psi^k(M, \mathrm{Hom}(E))$ acts on vector bundles, we ask that $|\det \sigma(\mathbf{A})|(y, \eta) \ge c\langle \eta \rangle^k$ instead.

Theorem A.18. *Let $\mathbf{A} \in \Psi^k(M, \mathrm{Hom}(E))$ and $\widehat{\chi} \in \Psi^0(M, \mathrm{Hom}(E))$ such that $\mathrm{WF}(\widehat{\chi}) \subset \mathrm{ell}(\mathbf{A})$. Then there exists $\mathbf{B} \in \Psi^{-k}(M, \mathrm{Hom}(E))$ such that $\mathbf{AB} = \widehat{\chi} + \mathbf{L}$ where $\mathbf{L}$ lies in $\Psi^{-\infty}(M, \mathrm{Hom}(E))$.*

Definition A.19. Let $A \in \Psi_h^k(M)$ and $(x,\xi) \in \overline{T}^*M$. We'll say that $(x,\xi) \in \mathrm{ell}_h(A)$ if there exists a neighborhood U of (x,ξ) in $\overline{T}^*M$ and $c > 0$ independant of h such that $|\sigma_h(A)|(y,\eta) \geq c\langle\eta\rangle^k$ for all $(y,\eta) \in U \cap T^*M$ and h small enough. If $\mathbf{A} \in \Psi^h(M, \mathrm{Hom}(E))$ acts on vector bundles, we ask that $|\det \sigma_h(\mathbf{A})|(y,\eta) \geq c\langle\eta\rangle^k$ instead.

Remark A.20. Note that as in the definition of the wavefront set, the semi-classical set of elliptic points need not be conical, contrarily of the classical one.

Theorem A.21. *Let $\mathbf{A} \in \Psi_h^k(M, \mathrm{Hom}(E))$ and $\widehat{\chi} \in \Psi_h^0(M, \mathrm{Hom}(E))$ such that $\mathrm{WF}_h(\widehat{\chi}) \subset \mathrm{ell}_h(\mathbf{A})$. Then there exists $\mathbf{B} \in \Psi_h^{-k}(M, \mathrm{Hom}(E))$ such that $\mathbf{AB} = \widehat{\chi} + \mathbf{L}$ where $\mathbf{L}$ lies in $h^\infty\Psi_h^{-\infty}(M, \mathrm{Hom}(E))$.*

A.7 Gårding's inequality

The Gårding's inequality is a result of positivity which will be crucial in the developements of certains of our estimates.

Theorem A.22 (Sharp Gårding's inequality, [Zwo12] Theorem 9.11)**.** *Let $A \in \Psi_h^k(M)$ such that $\Re(\sigma_h(A)) \geq 0$ everywhere. Then there is C such that*

$$\Re\langle Au, u\rangle \geq -Ch\|u\|_{H_h^{\frac{k-1}{2}}(M)}, \quad u \in \mathcal{C}^\infty(M).$$

B A few results in operator theory

B.1 Fredholm theory

Theorem B.1 (Fredholm analytic continuation)**.** *Let X and Y be Banach spaces, $\Omega \subset \mathbb{C}$ a connected open set, and $P : \Omega \to \mathcal{L}(X,Y), \lambda \mapsto P_\lambda$ be a holomorphic family of Fredholm operators. Suppose that there exists $\mu \in \Omega$ such that P_λ is invertible. Then there exists a discrete subset $S \subset \Omega$ such that P_λ is invertible for all $\lambda \notin S$, and $\lambda \mapsto P_\lambda^{-1}$ extends to a meromorphic function on Ω, that is, for every pole $\lambda_0 \in S$, there exists $k > 0$ and a neighborhood U of z_0 such that for all $z \in U$, we have*

$$P_\lambda^{-1} = Y_\lambda + \sum_{j=1}^{J(\lambda_0)} \frac{A_j}{(\lambda-\lambda_0)^j},$$

where Y_λ is holomorphic near λ_0. Moreover, the A_j are of finite rank.

In fact, if P_λ is of the form $P - \lambda$, we have a more precise statement :

Theorem B.2. *Suppose $X = Y$ and P satisfies the conditions of the previous theorem. Assume P is of the form $P_\lambda = P - \lambda$. Then near a pole λ_0, the resolvent satisfies*

$$(P-\lambda)^{-1} = Y_\lambda - \sum_{j=1}^{J(\lambda_0)} \frac{(P-\lambda_0)^{j-1}\Pi}{(\lambda-\lambda_0)^j},$$

where

$$\Pi = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} (\lambda - P)^{-1} \mathrm{d}\lambda$$

with $D_\varepsilon = \{\lambda \in \mathbb{C}, |\lambda - \lambda_0| < \varepsilon\}$ for small enough ε. Moreover, $\Pi^2 = \Pi$, $[P,\Pi] = 0$ and $(P-\lambda)^{J(\lambda_0)}\Pi = 0$.

Proof. We set $R_\lambda = (P-\lambda)^{-1}$. We write thanks to the previous theorem $\mathbf{R}_\lambda = Y_\lambda + \sum_{j=1}^{J(\lambda_0)} \frac{A_j}{(\lambda-\lambda_0)^j}$ near a pole λ_0 of P. The equation $R_\lambda(P-\lambda) = \mathrm{Id}_X$ shows that if $\lambda, \mu \in U \setminus S$,

$$R_\lambda R_\mu = \frac{R_\lambda - R_\mu}{\lambda - \mu}.$$

Therefore, if $\gamma_j(t) = \lambda + r_j \mathrm{e}^{it}$, $t \in [0, 2\pi]$, $j = 1, 2$, are two circles with $0 < r_1 < r_2 < \varepsilon$, one has

$$\begin{aligned}\Pi^2 &= \frac{1}{(2\pi i)^2}\int_{\gamma_2}\int_{\gamma_1} R_{z_1}R_{z_2}\mathrm{d}z_1\mathrm{d}z_2 \\ &= \frac{1}{(2\pi i)^2}\int_{\gamma_2}\int_{\gamma_1} \frac{R_{z_1}-R_{z_2}}{z_1 - z_2}\mathrm{d}z_1\mathrm{d}z_2.\end{aligned}$$

Now, since $r_1 < r_2$, we have $\int_{\gamma_1} R_{z_2}/(z_1 - z_2)\mathrm{d}z_1 = 0$ thanks to Cauchy's theorem. Therefore,

$$\Pi^2 = \frac{1}{(2\pi i)^2}\int_{\gamma_1} R_{z_1}\left(\int_{\gamma_2}\frac{\mathrm{d}z_2}{z_1 - z_2}\right)\mathrm{d}z_1 = \Pi.$$

Writing $A_n = \frac{1}{2\pi i}\int_\gamma (\lambda - \lambda_0)^{n-1}R_\lambda \mathrm{d}\lambda$ and noting that $PR_\lambda = \mathrm{Id} + \lambda R_\lambda$ we get

$$(P - \lambda)A_n = A_{n+1},$$

which concludes. □

B.2 On closed operators

Let H be an Hilbert space and suppose $T : H \to H$ is an unbounded closed operator with domain $\mathcal{D}(T)$ dense in H.

Lemma B.3. *Assume* $\overline{\mathrm{Ran}(T)} = H$ *and there exists* $\varepsilon > 0$ *such that*

$$\|Tu\| \geq \varepsilon\|u\|, \quad u \in \mathcal{D}(T).$$

Then $T : \mathcal{D}(T) \to H$ *is surjective.*

Proof. Take $v \in H$. Since $\overline{\mathrm{Ran}(T)} = H$, we can take a sequence u_n in $\mathcal{D}(T)$ such that $Tu_n \to v$ as $n \to \infty$. Then $\|T(u_p - u_q)\| \geq \varepsilon\|u_p - u_q\|$ so u_n is also a Cauchy sequence in H, and there is $u \in H$ such that $u_n \to u$ as $n \to \infty$. But now T is closed so $u \in \mathcal{D}(T)$ and $v = Tu$, which concludes. □

B.3 The flat trace

Given two compact manifolds M and an operator $B : \mathcal{C}^\infty(M) \to \mathcal{D}'(M)$ satisfying

$$\mathrm{WF}'(B) \cap \Delta(T^*M) = \emptyset, \tag{B.1}$$

where $\Delta(T^*M)$ is the diagonal in $T^*M \times T^*M$, we define its flat trace $\mathrm{tr}^\flat B$ as follows. First, fix a density vol on M and let $K_B \in \mathcal{D}'(M \times M)$ be the Schwartz kernel of B with respect to vol. The condition on the wavefront set of B means that $\mathrm{WF}(K_B)$ does not intersect the conormal bundle to the diagonal in $M \times M$. Let $\iota : M \to M \times M, x \mapsto (x, x)$ be the inclusion in the diagonal. According to [Hör90, Theorem 8.2.4], one can define the pullback $\iota^* K_B \in \mathcal{D}'(M)$. In fact, if $\Gamma \subset T^*M \times T^*M$ is a closed conical subset not intersecting the conormal to the diagonal, then ι^* is continuous $\mathcal{D}'_\Gamma \to \mathcal{D}'_{\iota^*\Gamma}$.

Definition B.4. The quantity

$$\mathrm{tr}^\flat B := \langle \iota^* K_B, \mathrm{vol}\rangle$$

does not depend of the density vol and is called the flat trace of B.

Now, let $T : \mathcal{C}^\infty(M) \to \mathcal{D}'(\mathbb{R}_{>0} \times M)$ be an operator satisfying that its Schwartz kernel $K_T \in \mathcal{D}'(\mathbb{R}_{>0} \times M \times M)$ with respect to a density vol satisfies

$$\mathrm{WF}(K_T) \cap \{(t, 0, x, \xi, x, -\xi) | t > 0,\ (x, \xi) \in T^*M \setminus 0\} = \emptyset. \tag{B.2}$$

Then again one can define its pullback $j^*K_T \in \mathbb{D}'(\mathbb{R}_{>0} \times M)$ where $j : (t, x) \mapsto (t, x, x)$.

Definition B.5. The distribution $\mathrm{tr}^\flat T \in \mathcal{D}'(\mathbb{R}_{>0})$ defined by

$$\langle \mathrm{tr}^\flat T, \chi\rangle := \langle j^*K_T, (\pi_1^*\chi)\,\mathrm{vol}\rangle, \quad \chi \in \mathcal{C}^\infty(\mathbb{R}_{>0}),$$

where $\pi_1 : \mathbb{R}_{>0} \times M \to \mathbb{R}_{>0}$ is the projection over the first factor, does not depend of the choice of the density vol and is called the flat trace of T.

More generally if $\mathbf{T} : \mathcal{C}^\infty(M, E) \to \mathcal{D}'(M, E)$ acts on sections of a vector bundle E, then we can define an operator $S_{\mathbf{T}}$ defined locally as $S_{\mathbf{T}} = \sum_i T_{ii}$, where the operators T_{kl} are defined in a local frame $(\mathbf{e}_i)$ by $\mathbf{T}\left(\sum_j u_j \mathbf{e}_j\right) = \sum_{kl} T_{kl} u_k \mathbf{e}_l$. The definition of $S_{\mathbf{T}}$ does not depend of the local frame, and we write $\mathrm{tr}^\flat \mathbf{T} := \mathrm{tr}^\flat S_{\mathbf{T}}$.

B.4 Cohomology with distributions

Let M be a smooth manifold. Recall the definition of the cohomology groups

$$H^k(M, \mathbb{C}) = \frac{\{\mathbf{u} \in \mathcal{C}^\infty(M, \Omega^k), \mathrm{d}\mathbf{u} = 0\}}{\{\mathrm{d}\mathbf{v}, \mathbf{v} \in \mathcal{C}^\infty(M, \Omega^{k-1})\}}.$$

The following lemma shows that the classes $\mathcal{D}'_\Gamma$ are also useful to deal with cohomology.

Lemma B.6. *Let* $\Gamma \subset T^*M \setminus 0$ *a closed conic set. Take* $\mathbf{u} \in \mathcal{D}'_\Gamma(M, \Omega^k)$ *such that* $\mathrm{d}\mathbf{u}$ *is smooth. Then there exists* $\mathbf{v} \in \mathcal{C}^\infty(M, \Omega^k)$ *and* $\mathbf{w} \in \mathcal{D}'_\Gamma(M, \Omega^{k-1})$ *such that*

$$\mathbf{u} = \mathbf{v} + \mathrm{d}\mathbf{w}.$$

Proof. Take a Riemannian metric g on M ; Hodge theory gives us the Hodge Laplacian $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} : \mathcal{C}^\infty(M, \Omega) \to \mathcal{C}^\infty(M, \Omega)$, $\Delta \in \Psi^2(M, \Omega)$, with a Green operator $G : \mathcal{C}^\infty(M, \Omega) \to \mathcal{C}^\infty(M, \Omega)$, $G \in \Psi^{-2}(M, \Omega)$, such that $G\Delta - I$ and $\Delta G - I$ are smoothing. Set $\mathbf{w} = \mathrm{d}^* G\mathbf{u} \in \mathcal{D}'_\Gamma(M, \Omega^{k-1})$. We have

$$\mathrm{d}\mathbf{w} = \mathrm{d}\mathrm{d}^* G\mathbf{u} = \Delta G\mathbf{u} - \mathrm{d}^*\mathrm{d}G\mathbf{u} = \mathbf{u} + (\Delta G - I)\,\mathbf{u} - \mathrm{d}^*\mathrm{d}G\mathbf{u}.$$

Since $\Delta G - I$ is smoothing, it suffices to show that $\mathrm{d}^*\mathrm{d}G\mathbf{u}$ is smooth. But one has $\Delta(\mathrm{d}G\mathbf{u}) = \mathrm{d}\Delta G\mathbf{u} = \mathrm{d}\,(\Delta G - I)\,\mathbf{u} + \mathrm{d}\mathbf{u}$, which implies that $\Delta(\mathrm{d}G\mathbf{u})$ is smooth because $\mathrm{d}\mathbf{u}$ is, and thus $\mathrm{d}G\mathbf{u}$ is smooth thanks to elliptic regularity because the principal symbol of Δ

$$\sigma(\Delta)(x, \xi) = |\xi|_g \mathrm{Id}_\Omega$$

is elliptic. Therefore $\mathrm{d}^*\mathrm{d}G\mathbf{u}$ is also smooth, which concludes. $\square$

C Recurrence estimates

We shall prove in this section Lemma 2.2. To do so, we will first need the following lemma which will guarantee that two different closed trajectories of nearby periods cannot be too close. In the following, X is an Anosov flow on a compact manifold M, and we denote by ϕ^t its flow. We fix some metric on M.

Lemma C.1. *Fix $t_0 > 0$. There exists $C, L, \delta, \varepsilon_0 > 0$ such that for every $t, s \geq t_0$ and $x, y \in M$ with*

$$|t-s| < \delta \ \text{ and } \ d(x,y) \leq \delta \mathrm{e}^{-Lt},$$

we have for every $\varepsilon < \varepsilon_0$

$$d(x,\phi^t(x)) \leq \varepsilon, d(y,\phi^s(y)) \leq \varepsilon \Longrightarrow |t-s| \leq C\varepsilon, \exists \tau \in (-1,1), d(x,\phi^\tau(y)) \leq C\varepsilon.$$

Proof. First note that since ϕ^t is a one-parameter group, one has for some $C, L > 0$

$$\|\phi^t\|_{\mathrm{Diff}^2(M)} \leq C\mathrm{e}^{L|t|}, \quad t \in \mathbb{R}. \tag{C.1}$$

where $\|\cdot\|_{\mathrm{Diff}^2(M)}$ is some norm on the space of diffeomorphisms on M with $\mathcal{C}^2$ regularity. In particular, one has

$$d(\phi^t(x),\phi^t(y)) \leq C_0\mathrm{e}^{L|t|}d(x,y), \quad x,y \in M, \quad t \in \mathbb{R}. \tag{C.2}$$

Take x in M and $\delta > 0$ small enough so that there is a coordinate chart U near x with $\{z \in M,\ d(z,x) \leq 2C_0\delta\} \subset U$. Now note that thanks to (C.2), $d(\phi^t(x),\phi^t(z)) \leq C_0\delta$ whenever $d(x,z) \leq \delta\mathrm{e}^{-Lt}$ with $t \geq t_0$. As a consequence, if $\varepsilon < C_0\delta/2$, then $d(x,\phi^t(x)) \leq \varepsilon$ implies $\phi^t(z) \in U$ for every z such that $d(x,z) \leq \delta\mathrm{e}^{-Lt}$. We identify U with its image in $\mathbb{R}^n$, and take $y \in B(x,\delta\mathrm{e}^{-Lt})$. The constants C appearing in the following considerations might evolve. We have thanks to (C.1) and the preceding remarks :

$$\partial_{x_i}\partial_{x_j}\phi^t(z) \leq C\mathrm{e}^{Lt}, \quad z \in B(x,\delta\mathrm{e}^{-Lt}).$$

This implies with the Taylor expansion of $z \mapsto \phi^t(z)$:

$$|\phi^t(y) - \phi^t(x) - (\mathrm{d}\phi^t)_x \cdot (y-x)| \leq C\mathrm{e}^{Lt}|y-x|^2.$$

On the other hand, $\partial_t^2\phi^t(y)$ is bounded in t, y (this is the direction of the flow) which implies with the Taylor expansion of $t \mapsto \phi^t(y)$:

$$|\phi^s(y) - \phi^t(y) - (s-t)X_{\phi^t(y)}| \leq C|s-t|^2.$$

We therefore obtain

$$|\phi^s(y) - \phi^t(x) - (\mathrm{d}\phi^t)_x \cdot (y-x) - (s-t)X_{\phi^t(y)}| \leq C\left(\mathrm{e}^{Lt}|y-x|^2 + |s-t|^2\right),$$

which gives

$$\left|\left((\mathrm{d}\phi^t)_x - I\right)\cdot(y-x) + X_{\phi^t(y)}(s-t)\right| \le C\left(\mathrm{e}^{Lt}|y-x|^2 + |s-t|^2\right) + |\phi^s(y)-y| + |\phi^t(x)-x|. \tag{C.3}$$

Now we will use the following

Lemma C.2. *If δ is small enough there exists a continuous family of invertible linear transformations $T_{x,z} : T_xM \to T_zM$, $z \in B(x, 2C_0\delta)$, such that $T_{x,x} = I$ and*

$$T_{x,z}\left(E_\bullet(x)\right) = E_\bullet(z), \quad \bullet = u, s, 0.$$

Moreover, one has $C, \varepsilon_0 > 0$ such that

$$\left|\left((\mathrm{d}\phi^t)_x - T_{x,\phi^t(x)}\right)\cdot v\right| \ge C^{-1}|v|, \quad d(x,\phi^t(x)) < \varepsilon_0, \quad t \ge t_0. \tag{C.4}$$

Admitting the lemma, one obtains since $y \mapsto T_{x,y}$ is continuous :

$$\left|\left(I - T_{x,\phi^t(x)}\right)\cdot(y-x)\right| \le C|\phi^t(x)-x||y-x|.$$

Moreover $\left|X_{\phi^t(y)} - X_{\phi^t(x)}\right| \le C|\phi^t(y)-\phi^t(x)|$. Those remarks together with (C.3) give :

$$\begin{aligned}\left|\left((\mathrm{d}\phi^t)_x - T_{x,\phi^t(x)}\right)\cdot(y-x) + (s-t)X_{\phi^t(x)}\right| \le C\left(\mathrm{e}^{Lt}|y-x|^2 + |s-t|^2\right) + |\phi^s(y)-y| \\ + |\phi^t(x)-x| + C|\phi^t(x)-\phi^t(y)||s-t| \\ + C|\phi^t(x)-x||y-x|.\end{aligned}$$

Moreover $d(x,y) < \delta\mathrm{e}^{-Lt}$, thus we have if $\varepsilon < \varepsilon_0$, $d(x,\phi^t(x)) < \varepsilon$ and $d(y,\phi^t(y)) < \varepsilon$:

$$\left|\left((\mathrm{d}\phi^t)_x - T_{x,\phi^t(x)}\right)\cdot(y-x) + (s-t)X_{\phi^t(x)}\right| \le C\delta\left(|y-x| + |s-t|\right) + C\varepsilon. \tag{C.5}$$

Now since $E_s(x) \oplus E_u(x)$ is transverse to X, we have (if δ is small enough) $\tau \in (-1,1)$ such that $\phi^\tau(y) - x \in E_s(x) \oplus E_u(x)$. As a consequence, using that X is non vanishing and the fact that $T_{x,z}$ preserves the distribution $E_s \oplus E_u \oplus E_0$, we have with (C.4) :

$$\left|\left((\mathrm{d}\phi^t)_x - T_{x,\phi^t(x)}\right)\cdot(\phi^\tau(y)-x) + (s-t)X_{\phi^t(x)}\right| \ge C^{-1}\left(|s-t| + |\phi^\tau(y)-x|\right).$$

Combining this with (C.5) applied to $\phi^\tau(y)$ instead of y this gives the desired result.

It remains to prove Lemma C.2. If $2C_0\delta$ is smaller than the injectivity radius of the exponential map at x, we can define

$$T_{x,z} = \left(\Pi_s(z)\circ\gamma_z\circ\Pi_s(x)\right) \oplus \left(\Pi_u(z)\circ\gamma_z\circ\Pi_u(x)\right) \oplus \left(\Pi_0(z)\circ\gamma_z\circ\Pi_0(x)\right)$$

where $\Pi_\bullet(z)$ are the projections on $E_\bullet(z)$ with respect to the distribution $E_s(z) \oplus E_u(z) \oplus E_0(z)$, and γ_z is the parallel transport from x to z. Since $T_{x,x} = I$ we have that $T_{x,z}$ is invertible if z is close enough to x. Now for sufficiently large t, one has by the Anosov property :

$$|(\mathrm{d}\phi^t)_x v_s| \leq \frac{1}{2}|v_s|, \quad |v_u| \leq \frac{1}{2}|(\mathrm{d}\phi^t)_x v_u|, \quad v_s \in E_s(x), \quad v_u \in E_u(x).$$

We have $2|v_u| \leq |(\mathrm{d}\phi^t)_x v_u - T_{x,\phi^t(x)}v_u| + |T_{x,\phi^t(x)}v_u|$ and thus $|v_u| \leq C|(\mathrm{d}\phi^t)_x v_u - T_{x,\phi^t(x)}v_u|$ if ε_0 is small enough so that $\|T_{x,\phi^t(x)}\|$ and $\|T_{x,\phi^t(x)}\|^{-1}$ are close to 1. Similarly $|v_s| \leq C|(\mathrm{d}\phi^t)_x v_s - T_{x,\phi^t(x)}v_s|$. This implies

$$|v| \leq |v_u| + |v_s| \leq C|(\mathrm{d}\phi^t)_x v_u - T_{x,\phi^t(x)}v_u| + C|(\mathrm{d}\phi^t)_x v_s - T_{x,\phi^t(x)}v_s|$$

if ε_0 is small enough. Since $T_{x,\phi^t(x)}$ sends $E_\bullet(x)$ on $E_\bullet(\phi^t(x))$ for $\bullet = s, u$, we obtain (C.4) for t big enoug, say for $t \geq Nt_0$. Now let $t \geq t_0$. Then for $\tilde{\varepsilon}_0$ small enough, we have $d(x, \phi^t(x)) < \tilde{\varepsilon}_0 \implies d(x, \phi^{Nt}(x)) < \varepsilon_0$. Moreover, we have $|\left((\mathrm{d}\phi^{Nt})_x - I\right) v| \leq C|\left((\mathrm{d}\phi^t)_x - I\right) v|$, for $v \in E_u \oplus E_s$. As a consequence, we have with (C.4) applied to ϕ^{Nt} :

$$|v| \leq C\left|(\mathrm{d}\phi^t)_x - I)v\right| + C\left(\left|(T_{x,\phi^{Nt}(x)} - I)v\right| + \left|(T_{x,\phi^t(x)} - I)v\right|\right).$$

This shows (C.4) for all $t \geq t_0$, provided $\tilde{\varepsilon}_0$ is small enough so that we have $C\left(\|(T_{x,\phi^t(x)} - I)\| + \|T_{x,\phi^{Nt}(x)} - I\|\right) < 1/2$. This completes the proof. $\square$

We are now in position to prove the bound on the number of closed orbits.

Proof of Lemma 2.2. Let μ be the Riemannian volume on M and define a measure on $M \times \mathbb{R}$ by $\nu = \mu \otimes \mathrm{d}t$. Let $t_0 > 0$ and set

$$A^\varepsilon = \{(x, t) \in M \times \mathbb{R}_+,\ d(x, \phi^t(x)) < \varepsilon,\ t \geq t_0\}.$$

Take $\delta > 0$ be the constant of Lemma C.1 and fix $T > 0$. Let $x_1, \cdots, x_N$ be a maximal set of points in M (with N depending on T) such that $d(x_i, x_j) \geq \delta \mathrm{e}^{-LT}/2$, that is for every $x \in M$, one has $d(x, x_j) \leq \delta \mathrm{e}^{-LT}/2$ for some j. We have

$$A^\varepsilon \subset \bigcup_{j=1}^N \bigcup_{k=1}^M A^\varepsilon_{j,k}, \quad A^\varepsilon_{j,k} = A^\varepsilon \cap \left(B(x_j, \delta \mathrm{e}^{-LT}/2) \times [\tau_k - \delta/2, \tau_k + \delta/2]\right),$$

where $\tau_1 < \cdots < \tau_M$ are such that $[t_0, T] \subset \cup_k [\tau_k - \delta/2, \tau_k + \delta/2]$, with M depending on T and $M \leq CT$. If $A^\varepsilon_{j,k}$ is nonempty, fix $(x, t) \in A^\varepsilon_{j,k}$. Then for all $(y, s) \in A^\varepsilon_{j,k}$, one has $|t - s| < \delta$ and $d(x, y) \leq \delta \mathrm{e}^{-LT}$. Then Lemma C.1 gives that for ε small enough :

$$A^\varepsilon_{j,k} \subset \mathcal{V}^{C\varepsilon}\left(\{(\phi^\tau(x), t),\ |\tau| < 1\}\right),$$

where $\mathcal{V}^\varepsilon(B) = \{\rho,\ d(\rho, B) < \varepsilon\}$ is the ε tubular neighborhood of a set B. As a consequence, we obtain $\nu(A^\varepsilon_{j,k}) \leq \tilde{C}\varepsilon^n$. Now note that $N \leq C\mathrm{e}^{nLT}$ because

$$\mathrm{vol}(M) \geq \mathrm{vol}\left(\bigcup_{j=1}^{N} B(x_j, \delta\mathrm{e}^{-LT}/4)\right) = \sum_{j=1}^{N} \mathrm{vol}\left(B(x_j, \delta\mathrm{e}^{-LT}/4)\right) \geq C'\mathrm{e}^{-nLT}N.$$

We finally obtain for ε small enough

$$\nu(A^\varepsilon) \leq CT\mathrm{e}^{nLT}\varepsilon^n \leq C\mathrm{e}^{n\tilde{L}T}\varepsilon^n. \tag{C.6}$$

Now take $\gamma(t) = \phi^t(x_0)$ a closed trajectory of period τ no more that T. We know by (C.2) that for $\varepsilon > 0$:

$$|t - \tau| \leq \varepsilon,\ \ \exists s:\ d(\gamma(s), x) \leq \varepsilon\mathrm{e}^{-L\tau} \implies d(x, \phi^t(x)) \leq C\varepsilon.$$

Now for ε small enough depending on T, the tubular neighborhood defined in the left hand side do not intersect any such tubular neighborhood of a different closed orbit of period less than T. Moreover, this tubular neighborhood has a volume in (x, t) bounded from below by $C^{-1}\varepsilon^n\mathrm{e}^{-(n-1)L\tau}$. As a consequence, for ε small enough, we get with (C.6) :

$$N(T)C^{-1}\varepsilon^n\mathrm{e}^{-(n-1)LT} \leq \mathrm{vol}(A^{C\varepsilon}) \leq C\mathrm{e}^{n\tilde{L}T}(C\varepsilon)^n,$$

which concludes. $\square$

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